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# A MATRIX INEQUALITY FOR POSITIVE DOUBLE JOHN DECOMPOSITION

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*Abstract.* In 1998, Barthe [2] established the reversed Brascamp-Lieb inequality and its geometric version. The matrix inequality for John decomposition played a key role in the proof of the geometric Brascamp-Lieb inequality (see also Ball [1]). In this paper, we propose a new matrix inequality based on the so called "double John decomposition", which is a generalization of the results of Ball and Barthe.

## 1. Introduction

The celebrated Brascamp-Lieb inequality states: the multilinear operator on $L_{p_1}(\mathbb{R}^{n_1})\times\cdots\times L_{p_m}(\mathbb{R}^{n_m})$ defined by

$$F(f_1,\ldots,f_m)=\int_{\mathbb{R}^n}\prod_{i=1}^m f_i(B_ix)dx$$

is saturated by Gaussian functions. For details, see [3, 5, 2].

1n 1998, Barthe [2] established the reverse Brascamp-Lieb inequality which was conjectured by Ball [1]. Especially, he obtained the following well-known geometric Brascamp-Lieb inequality:

THEOREM 1. *Let $m,n$ be integers. For $i=1,\ldots,m$, let $(c_i)_{i=1}^m$ be positive real numbers, $(n_i)_{i=1}^m$ be integers, and let $B_i$ be a linear surjective map from $\mathbb{R}^n$ onto $\mathbb{R}^{n_i}$, satisfying $B_iB_i^t=I_{n_i}$ and*

$$\sum_{i=1}^m c_iB_iB_i^t=I_n.$$

*If for $i=1,\ldots,m$, $f_i$ is a non-negative integrable function on $\mathbb{R}^{n_i}$, then one has*

$$\int_{\mathbb{R}^n}\prod_{i=1}^m f_i^{c_i}(B_ix)dx\leqslant\prod_{i=1}^m\Big(\int_{\mathbb{R}^{n_i}}f_i\Big)^{c_i},$$

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*and*

$$\int_{\mathbb{R}^n}^{*} \sup\left\{\prod_{i=1}^m f_i^{c_i}(y_i) : x=\sum_{i=1}^m c_iB_i^ty_i, y_i\in\mathbb{R}^{n_i}\right\} dx \geqslant \prod_{i=1}^m\left(\int_{\mathbb{R}^{n_i}} f_i\right)^{c_i}.$$

*Here* $\int^*$ *denotes the outer integral.*

In the proof of Theorem 1, a matrix inequality plays a crucial role: if

$$B_i^tB_i = I_{n_i}, \qquad \sum_{i=1}^m c_iB_iB_i^t = I_n, \tag{1}$$

and $A_i$ are $m_i\times m_i$ positive definite matrixes, then

$$\det\left(\sum_{i=1}^m c_iB_iA_iB_i^t\right) \geqslant \prod_{i=1}^m \left(\det A_i\right)^{c_i}.$$

The decomposition of identity satisfying (1) is called a *john decomposition*. For the case $n_i = 1,\ i = 1,\dots,m,$ it was crucial in the well-known John theorem, see [1].

In 2011, Li and Leng [4] defined the positive double John basis, and established the following matrix inequality: if $(c_i)_{i=1}^m$ are positive numbers, and a sequence of pairs $\{u_i,v_i\}_{i=1}^m$ satisfies

$$I_n = \sum_{i=1}^m c_iu_i\otimes v_i, \tag{2}$$

then for $\lambda_i,\delta_i > 0,$ one has

$$\det\left(\sum_{i=1}^m c_i\lambda_iu_i\otimes u_i\right)\det\left(\sum_{i=1}^m c_i\delta_iv_i\otimes v_i\right) \geqslant \prod_{i=1}^m(\lambda_i\delta_i)^{c_i}.$$

The decomposition of identity satisfying (2) was called a *positive double John basis*. Using this matrix inequality, they proved a generalized version of Brascamp-Lieb inequality.

Their result are of dimension 1, but as far as we know, the result of Barthe [2] is a multidimensional version. In this paper we defined the multidimensional version of *positive double John decomposition* as follows.

DEFINITION 1. Suppose $m\geqslant n,\ n_i<n$ and $c_i>0,\ i=1,\cdots,m$. Let $U_i,\ V_i$ be $n_i\times n$ matrices. If $V_iU_i^t = I_{n_i}$ and

$$\sum_{i=1}^m c_iU_i^tV_i = I_n,$$

then we say that $U_i,V_i$ satisfy the *positive double John decomposition*.

Our main result is the following matrix inequality, which is the generalization of the results of both Barthe [2] and Li and Leng [4].

THEOREM 2. (Main) *For $c_i > 0$, $i = 1,\cdots,m$, let $U_i$, $V_i$ be $n_i \times n$ matrices satisfying the positive double John decomposition. Then for any $n_i \times n_i$ positive definite diagonal matrices $A_i, B_i$, we have*

$$\det\Big(\sum_{i=1}^{m} c_i U_i^t A_i U_i\Big)\det\Big(\sum_{i=1}^{m} c_i V_i^t B_i V_i\Big) \geqslant \prod_{i=1}^{m}\Big(\det A_i \cdot \det B_i\Big)^{c_i}. \tag{3}$$

## 2. Proof of Theorem 2

Now we prove our main result. The following Cauthy-Binet formula is needed.

LEMMA 1. *Let $m \geqslant n$ be positive integers and $I \subset \{1,2,\ldots,m\}$. Let $A$ be an $n \times m$ matrix and $B$ an $m \times n$ matrix. If $A_I$ denotes the square matrix obtained from $A$ by keeping only the columns with indices in $I$, and $B_I$ denotes the square matrix obtained from $B$ by keeping the rows with indices in $I$, then we have the formula*

$$\det(AB) = \sum_{|I|=n} \det(A_I)\det(B_I).$$

LEMMA 2. *Let $P_i$ be $n \times n_i$ matrix, and $Q_i$ be $n_i \times n$ matrix, for $i = 1,\ldots,m$. Let $I_i \subseteq \{1,\cdots,n_i\}$ with $|I_1| + \cdots + |I_m| = n$, and $I = (I_1,\ldots,I_m)$. Denote $|I| = |I_1| + \cdots + |I_m|$. Denote $D_I = (D_{I_1},\ldots,D_{I_m})$, where $D^t_{I_j}$ is an $n \times |I_j|$ matrix obtained from $P_j$ keeping the columns with indices in $I_j$; and denote $G_I = (G^t_{I_1},\ldots,G^t_{I_m})^t$, where $G_{I_j}$ is an $|I_j| \times n$ matrix obtained from $Q_j$ keeping the rows with indices in $I_j$. Then*

$$\det\Big(\sum_{i=1}^{m} P_i Q_i\Big) = \sum_{|I|=|I_1|+\ldots+|I_m|=n} \det(D_I)\det(G_I).$$

*Proof.* Clearly,

$$\sum_{i=1}^{m} P_i Q_i = \Big(P_1,\ldots,P_m\Big)_{n\times(n_1+\cdots+n_m)} \begin{pmatrix} Q_1 \\ \vdots \\ Q_m \end{pmatrix}_{(n_1+\cdots+n_m)\times n}.$$

By Lemma 1, we get the desired result. □

*Proof of Theorem 2.* Let $I_i \subseteq \{1,\cdots,n_i\}$ with $|I_1|+\cdots+|I_m| = n$, and $I = (I_1,\ldots,I_m)$. Denote $|I| = |I_1| + \cdots + |I_m|$. Let $D^t_I = (D^t_{I_1},\ldots,D^t_{I_m})$, where $D^t_{I_j}$ is an $n \times |I_j|$ matrix obtained from $U^t_j$ keeping the columns with indices in $I_j$; and let $G_I = (G^t_{I_1},\ldots,G^t_{I_m})^t$, where $G_{I_j}$ is an $|I_j| \times n$ matrix obtained from $V_j$ keeping the rows with indices in $I_j$.

Note that

$$\sum_{i=1}^{m} c_i U_i^t V_i = \left(c_1 U_1^t, \ldots, c_m U_m^t\right)_{n\times(n_1+\cdots+n_m)} \begin{pmatrix} V_1 \\ \vdots \\ V_m \end{pmatrix}_{(n_1+\cdots+n_m)\times n}$$

$$\doteq \left(c_1 D_1^1, \ldots, c_1 D_1^{n_1}, \ldots, c_m D_m^1, \ldots, c_m D_m^{n_m}\right)_{n\times(n_1+\cdots+n_m)} \begin{pmatrix} G_1^1 \\ \vdots \\ G_1^{n_1} \\ \vdots \\ G_m^1 \\ \vdots \\ G_m^{n_m} \end{pmatrix}_{(n_1+\cdots+n_m)\times n},$$

where $D_i^j$ is the $j$-th column of $U_i^t$ and $G_i^j$ is the $j$-th row of $V_i$.

Write $c_I = \prod_{i=1}^{m} c_i^{|I_i|}$. Substituting $P_i = c_i U_i^t$ and $Q_i = V_i$ into Lemma 2, we obtain

$$1 = \det I_n = \det\left(\sum_{i=1}^{m} c_i U_i^t V_i\right) = \sum_{|I|=|I_1|+\ldots+|I_m|=n} c_I \det(D_I)\det(G_I). \tag{4}$$

Denote $n_i \times n_i$ positive definite diagonal matrices $A_i$, $B_i$ by

$$A_i = \begin{pmatrix} a_{i1} & & \\ & \ddots & \\ & & a_{in_i} \end{pmatrix}, \qquad B_i = \begin{pmatrix} b_{i1} & & \\ & \ddots & \\ & & b_{in_i} \end{pmatrix}.$$

We see that

$$\begin{pmatrix} A_1 V_1 \\ \vdots \\ A_m V_m \end{pmatrix}_{(n_1+\cdots+n_m)\times n} = \begin{pmatrix} a_{11} G_1^1 \\ \vdots \\ a_{1n_1} G_1^{n_1} \\ \vdots \\ a_{m1} G_m^1 \\ \vdots \\ a_{mn_m} G_m^{n_m} \end{pmatrix}_{(n_1+\cdots+n_m)\times n}$$

Substituting $P_i = c_i U_i^t$ and $Q_i = A_i V_i$ into Lemma 2, we have

$$\det\left(\sum_{i=1}^{m} c_i U_i^t A_i V_i\right) = \sum_{|I|=|I_1|+\ldots+|I_m|=n} a_I c_I \det(D_I)\det(G_I),$$

where $a_I = \prod_{\substack{j\in I_i \\ i=1,\ldots,m}} a_{ij}$.

Applying the arithmetic-geometric means inequality, we have

$$\begin{aligned}
&\sum_{|I|=|I_1|+\ldots+|I_m|=n} a_I c_I \det(D_I)\det(G_I)\\
&\geqslant \prod_{|I|=|I_1|+\ldots+|I_m|=n} a_I^{c_I \det(D_I)\det(G_I)}\\
&= \prod_{i=1}^{m}\prod_{j=1}^{n_i} a_{ij}^{\sum_{|I|=n, I_i \ni j} c_I \det(D_I)\det(G_I)}.
\end{aligned}$$

Observe that $V_i U_i^t = I_{n_i}$ implies

$$G_i^j D_i^j = 1.$$

Let $u_1,\ldots,u_{n-1}$ be such that $G_i^j u_i = 0$ and $D^j, u_1,\ldots,u_{n-1}$ are linear independent, then we see

$$\begin{aligned}
\det\big((I_n - c_i D^j G^j)(D^j, u_1,\ldots,u_{n-1})\big) &= \det\big((1-c_i)D^j, u_1,\ldots,u_{n-1}\big)\\
&= (1-c_i)\det(D^j, u_1,\ldots,u_{n-1}),
\end{aligned}$$

which implies

$$\det(I_n - c_i D^j G^j) = 1 - c_i.$$

Therefore, we get

$$\begin{aligned}
&\sum_{|I|=n, I_i \ni j} c_I \det(D_I)\det(G_I)\\
&= \sum_{|I|=n} c_I \det(D_I)\det(G_I) - \sum_{|I|=n, j\notin I_i} c_I \det(D_I)\det(G_I)\\
&= \det(I_n) - \det(I_n - c_i G_i^j D_i^j)\\
&= 1-(1-c_i)\\
&= c_i.
\end{aligned}$$

Now we have shown that

$$\sum_{|I|=|I_1|+\ldots+|I_m|=n} a_I c_I \det(D_I)\det(G_I) \geqslant \prod_{i=1}^{m}\left(\prod_{j=1}^{n_i} a_{ij}\right)^{c_i}. \tag{5}$$

Similarly, for $A_i$, $B_i$, we have

$$\begin{aligned}
&\det\left(\sum_{i=1}^{m} c_i U_i^t A_i U_i\right)\det\left(\sum_{i=1}^{m} c_i V_i^t B_i V_i\right)\\
&\qquad = \sum_{|I|=n} a_I c_I \det(D_I)^2 \sum_{|I|=n} b_I c_I \det(G_I)^2,
\end{aligned}$$

which is greater than

$$\left(\sum_{|I|=n} c_I\sqrt{a_I b_I}\det(D_I)\det(G_I)\right)^2$$

employing the Cauchy-Schwartz inequality. Applying (5) we get

$$\det\left(\sum_{i=1}^{m} c_i U_i^t A_i U_i\right)\det\left(\sum_{i=1}^{m} c_i V_i^t B_i V_i\right) \geqslant \prod_{i=1}^{m}(\det A_i\cdot\det B_i)^{c_i}.$$

This completes the proof. □

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