

A SHARPENING OF A PROBLEM ON BERNSTEIN POLYNOMIALS AND CONVEX FUNCTIONS

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Abstract. We present an elementary proof of a conjecture by I. Rașa which is an inequality involving Bernstein basis polynomials and convex functions. It was affirmed in positive very recently by the use of stochastic convex orderings.

1. Introduction

The classical Bernstein polynomials, defined for $f \in C[0, 1]$ by

$$(B_n f)(x) = \sum_{v=0}^n p_{n,v}(x) f\left(\frac{v}{n}\right) \quad (x \in [0, 1]),$$

with the basis polynomials

$$p_{n,v}(x) = \binom{n}{v} x^v (1-x)^{n-v} \quad (v = 0, 1, 2, \dots),$$

are the most prominent positive linear approximation operators (see [9]). If $f \in C[0, 1]$ is convex the inequality

$$\sum_{i=0}^n \sum_{j=0}^n [p_{n,i}(x) p_{n,j}(x) + p_{n,i}(y) p_{n,j}(y) - 2p_{n,i}(x) p_{n,j}(y)] f\left(\frac{i+j}{2n}\right) \geq 0 \quad (1)$$

is valid, for $x, y \in [0, 1]$.

This inequality involving Bernstein basis polynomials and convex functions was stated as an open problem 25 years ago by Ioan Rașa. During the Conference on Ulam's Type Stability (Rytro, Poland, 2014), Rașa [11] recalled his problem.

Inequalities of type (1) have important applications. They are useful when studying whether the Bernstein-Schnabl operators preserve convexity (see [3, 4, 5]).

Very recently, J. Mrowiec, T. Rajba and S. Wařowicz [10] affirmed the conjecture in positive. Their proof makes heavy use of probability theory. As a tool they applied stochastic convex orderings (which they proved for binomial distributions) as well as

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the so-called concentration inequality. After that one of the authors gave a short elementary proof [1] of inequality (1). The other author remarked in [12] that (1) is equivalent to

$$(B_{2n}f)(x) + (B_{2n}f)(y) \geq 2 \sum_{i=0}^n \sum_{j=0}^n p_{n,i}(x) p_{n,j}(y) f\left(\frac{i+j}{2n}\right). \tag{2}$$

If f is convex on $[0, 1]$ also $B_{2n}f$ is convex on $[0, 1]$ (see [2, Corollary 6.3.8]). Therefore, we have

$$(B_{2n}f)(x) + (B_{2n}f)(y) \geq 2(B_{2n}f)\left(\frac{x+y}{2}\right). \tag{3}$$

Thus the following problem seems to be a natural one: Prove that

$$(B_{2n}f)\left(\frac{x+y}{2}\right) \geq \sum_{i=0}^n \sum_{j=0}^n p_{n,i}(x) p_{n,j}(y) f\left(\frac{i+j}{2n}\right), \tag{4}$$

for all convex $f \in C[0, 1]$ and $x, y \in [0, 1]$.

If (4) is valid, then (2) – and hence (1) – is a consequence of (3) and (4). Starting from these remarks, the second author presented the inequality (4) as an open problem in [12]. A probabilistic solution was found by A. KomisarSKI and T. Rajba [7] using the methods developed in [10] and [8].

The purpose of this short note is to give an analytic proof of the following theorem.

THEOREM 1. *Let $n, m \in \mathbb{N}$. If $f \in C[0, 1]$ is a convex function, then the inequality*

$$(B_{mn}f)\left(\frac{1}{m} \sum_{v=1}^m x_v\right) \geq \sum_{i_1=0}^n \cdots \sum_{i_m=0}^n \left(\prod_{v=1}^m p_{n,i_v}(x_v)\right) f\left(\frac{1}{mn} \sum_{v=1}^m i_v\right),$$

is valid for all $x_1, \dots, x_m \in [0, 1]$.

Obviously, the special case $m = 2$ is inequality (4).

2. An elementary proof of Theorem 1

Using the obvious identity

$$p_{n,i}(x) = \frac{1}{i!} \left(\frac{\partial}{\partial z}\right)^i \left[(1+xz)^n\right] \Bigg|_{z=-1}$$

we obtain

$$\begin{aligned} & \sum_{i_1=0}^n \cdots \sum_{i_m=0}^n \left(\prod_{v=1}^m p_{n,i_v}(x_v)\right) f\left(\frac{1}{mn} \sum_{v=1}^m i_v\right) \\ &= \sum_{k=0}^{mn} f\left(\frac{k}{mn}\right) \sum_{i_1+\dots+i_m=k} \prod_{v=1}^m \left(\frac{1}{i_v!} \left(\frac{\partial}{\partial z}\right)^{i_v} (1+x_v z)^n\right) \Bigg|_{z=-1} \\ &= \sum_{k=0}^{mn} f\left(\frac{k}{mn}\right) \frac{1}{k!} \left[\left(\frac{\partial}{\partial z}\right)^k \prod_{v=1}^m (1+x_v z)^n\right] \Bigg|_{z=-1}, \end{aligned}$$

where the last equality follows by Leibniz rule for derivatives. In particular, we have

$$\begin{aligned} & \sum_{i_1=0}^n \cdots \sum_{i_m=0}^n \left(\prod_{v=1}^m p_{n,i_v}(x) \right) f \left(\frac{1}{mn} \sum_{v=1}^m i_v \right) \\ &= \sum_{k=0}^{mn} f \left(\frac{k}{mn} \right) \frac{1}{k!} \left[\left(\frac{\partial}{\partial z} \right)^k (1+xz)^{mn} \right] \Big|_{z=-1} = (B_{mn}f)(x). \end{aligned}$$

Inserting

$$(B_{mn}f) \left(\frac{1}{m} \sum_{v=1}^m x_v \right) = \sum_{k=0}^{mn} f \left(\frac{k}{mn} \right) \frac{1}{k!} \left[\left(\frac{\partial}{\partial z} \right)^k \left(\frac{1}{m} \sum_{v=1}^m (1+x_v z) \right)^{mn} \right] \Big|_{z=-1}$$

we obtain

$$\begin{aligned} & (B_{mn}f) \left(\frac{1}{m} \sum_{v=1}^m x_v \right) - \sum_{i_1=0}^n \cdots \sum_{i_m=0}^n \left(\prod_{v=1}^m p_{n,i_v}(x_v) \right) f \left(\frac{1}{mn} \sum_{v=1}^m i_v \right) \\ &= \sum_{k=0}^{mn} f \left(\frac{k}{mn} \right) \frac{1}{k!} \left(\frac{\partial}{\partial z} \right)^k \left[\left(\frac{1}{m} \sum_{v=1}^m (1+x_v z) \right)^{mn} - \prod_{v=1}^m (1+x_v z)^n \right] \Big|_{z=-1}. \end{aligned}$$

For fixed $n \in \mathbb{N}$ and $x_1, \dots, x_m \in [0, 1]$, we define

$$g(z) \equiv g_{m,n}(z; x_1, \dots, x_m) = z^{-2} \left(\left(\frac{1}{m} \sum_{v=1}^m (1+x_v z) \right)^{mn} - \prod_{v=1}^m (1+x_v z)^n \right).$$

Note that g is a polynomial in z of degree at most $mn - 2$.

LEMMA 1. Fix $x_1, \dots, x_m \in [0, 1]$. Then, the function g satisfies $g^{(k)}(-1) \geq 0$, for $k = 0, 1, \dots, mn - 2$.

Proof. For abbreviation, put $a_v = 1 + x_v z$. We have

$$\begin{aligned} & \left(\frac{1}{m} \sum_{v=1}^m (1+x_v z) \right)^{mn} - \prod_{v=1}^m (1+x_v z)^n \\ &= \left(\frac{1}{m} \sum_{v=1}^m a_v \right)^{mn} - \left(\prod_{v=1}^m a_v \right)^n \\ &= \left(\left(\frac{1}{m} \sum_{v=1}^m a_v \right)^m - \prod_{v=1}^m a_v \right) \sum_{j=0}^{n-1} \left(\frac{1}{m} \sum_{v=1}^m a_v \right)^{n-1-j} \left(\prod_{v=1}^m a_v \right)^j. \end{aligned}$$

We see that the second factor

$$\sum_{j=0}^{n-1} \left(\frac{1}{m} \sum_{v=1}^m a_v \right)^{n-1-j} \left(\prod_{v=1}^m a_v \right)^j$$

has non-negative derivatives of all orders at $z = -1$. This reduces the problem to the case $n = 1$, i.e.,

$$\left(\sum_{v=1}^m a_v\right)^m - m^m \prod_{v=1}^m a_v$$

when multiplied by m^m . For $m = 2$, we have

$$(a_1 + a_2)^2 - 4a_1a_2 = (a_1 - a_2)^2 = (x_1 - x_2)^2 z^2$$

such that $g_{2,1}(z; x_1, x_2) = (x_1 - x_2)^2 / 4$. For $m \geq 2$, Gusić [6, Theorem 1] (cf. [13, Eq. (2)]) proved the representation

$$\left(\sum_{v=1}^m a_v\right)^m - m^m \prod_{v=1}^m a_v = \sum_{1 \leq i < j \leq m} (a_i - a_j)^2 P_{i,j}(a_1, \dots, a_m),$$

for some homogeneous polynomials $P_{i,j}$ of degree $n - 2$ with all non-negative coefficients. Applying this with $a_i = 1 + x_i z$ we see that each $P_{i,j}(a_1, \dots, a_m)$ has all the derivatives at $z = -1$ non-negative and so does $(a_i - a_j)^2 / z^2$ by the $m = 2$ case. Thus $g(z)$ has all the derivatives at $z = -1$ non-negative as well. \square

The key result is the next proposition. The proof follows the lines of [1, Prop. 1].

PROPOSITION 1. Fix $x_1, \dots, x_m \in [0, 1]$. Then, for any real numbers a_0, \dots, a_{mn} , the identity

$$\begin{aligned} & \sum_{i_1=0}^n \cdots \sum_{i_m=0}^n \left[\left(\prod_{v=1}^m p_{n,i_v} \left(\frac{1}{m} \sum_{v=1}^m x_v \right) \right) - \left(\prod_{v=1}^m p_{n,i_v}(x_v) \right) \right] \cdot a_{|i|} \\ &= \sum_{k=0}^{mn-2} (\Delta^2 a_k) \frac{1}{k!} g^{(k)}(-1) \end{aligned} \tag{5}$$

is valid.

Here we put $|i| = \sum_{v=1}^m i_v$ and Δ denotes the forward difference $\Delta a_k := a_{k+1} - a_k$ such that $\Delta^2 a_k = a_{k+2} - 2a_{k+1} + a_k$.

Because g is a polynomial in z of degree at most $mn - 2$, it is obvious that $g^{(mn-1)}(-1) = g^{(mn)}(-1) = 0$.

Proof of Theorem 1. For $k = 0, 1, \dots, mn - 2$, we put

$$a_k = f\left(\frac{k}{mn}\right).$$

If $f \in C[0, 1]$ is a convex function it follows that $\Delta^2 a_k \geq 0$, for $k = 0, 1, \dots, mn - 2$. Therefore, application of Proposition 1 proves Theorem 1. \square

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