

FUNCTIONS WITH STRICTLY CONVEX EPIGRAPH

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Abstract. The aim of this paper is to provide a full and simple characterization of functions with domain in an *arbitrary* topological real vector space whose epigraph is *strictly* convex. In order to achieve this description, we first give both affine and topological properties of strictly convex subsets of general topological real vector spaces. Then, we investigate the relationship between functions and their epigraph from the geometric and topological viewpoints.

Introduction

This paper is concerned with the relationship between strict convexity of functions defined over a domain in a topological real vector space and strict convexity of their epigraphs. A subset of a topological real vector space is said to be strictly convex if it is convex and if in addition there is no non-trivial segment in its boundary. This notion will be made more precise in Section 2.

Even though strict convexity is less studied than convexity in the literature, there are nevertheless many fields where strict convexity of a subset of a topological real vector space is used. We give here three different examples which illustrate this geometric property.

The first example, which is actually the starting point of the present work, concerns epigraphs of functions. It is a well-known result that a function has a convex epigraph if and only if it is convex. This is a bridge between geometric convexity and analytic convexity (see for example [9, Theorem 4.1, page 25]). Therefore, a natural question is to know whether the same equivalence holds when replacing “convexity” by “strict convexity”. To the best of our knowledge, nothing has been studied about this issue in the literature, even in the case when the domain of f lies in \mathbf{R}^n . This is why we propose to fill the gap in the present paper, not only in \mathbf{R}^n but in the general framework of topological vector spaces. This is done in the Main Theorem that we state in Section 1.

The second example deals with strict convexity of the unit ball in a normed real vector space (in that case, the norm itself is sometimes called strictly convex, which is unfortunate). This property is equivalent to saying that there exists a real number $p > 1$ such that the p -th power of the norm is a strictly convex function (for a proof, we may

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see Theorem 11.1 in [3, page 110]), and this is actually equivalent to the strict convexity of the epigraph of this function as the Main Theorem will show.

It is important to work with such norms since they yield interesting properties in functional analysis. For instance, given a Banach space E with strictly convex unit ball, any non-empty family of commutative non-expansive mappings from a non-empty closed convex and weakly compact subset of E into itself has a common fixed point (see for example [1]).

Nevertheless, if the unit ball of a normed vector space is not strictly convex, this may be offset in at least two different ways. Indeed, any reflexive Banach space can be endowed with an equivalent norm whose unit ball is strictly convex (see for example [8]). On the other hand, any separable Banach space can be endowed with an equivalent norm which is smooth and whose unit ball is strictly convex (see for example [6, page 33]).

The third example concerns optimization problems – more precisely, the relationship between strict convexity and uniqueness of minimizers. When dealing with an optimization problem on a topological real vector space, the search for a value of the variable where the cost function achieves a minimum is much more easier in case this function and the constraint set are both convex (see for example [4]). Moreover, if the cost function is strictly convex, such a minimizer is then unique. On the other hand, if the constraint set is both strictly convex and given by the epigraph of a function (we shall see in which case this is possible owing to the Main Theorem), and if the cost function has no minimum over the whole space, then such a minimizer is unique too.

As we may notice throughout these three examples, it is of great importance to know whether the epigraph of a function is strictly convex or not.

Of course, for a function defined over \mathbf{R}^n , the strict convexity of its epigraph is merely equivalent to being strictly convex. But what happens for a function with an *arbitrary* domain which lies in an *arbitrary* topological vector space?

In order to give a complete answer to this question and deduce some of its consequences in Section 1, we shall examine two issues in Section 2: topological aspects of epigraphs of functions defined on any topological space on the one hand, and the notion of strict convexity for sets in general topological real vector spaces (no matter which dimension they have or whether they are Hausdorff) on the other hand.

Finally, Section 3 is devoted to the proofs of all the results given in the previous sections.

1. Motivations, Main Theorem and consequences

The relationship between convexity of sets and convexity of functions is given by the following well-known result that is quite easy to prove.

PROPOSITION 1.1. *Let C be a subset of a real vector space and $f : C \rightarrow \mathbf{R}$ a function. Then we have the following equivalences:*

$$C \text{ and } f \text{ are both convex} \iff \text{Epi}(f) \text{ is convex} \iff \text{Epi}_s(f) \text{ is convex} .$$

This is a geometric way of characterizing the convexity of a function by looking at its epigraph.

Such a property naturally raises the issue of studying what happens when convexity is replaced by strict convexity whose meaning will be given in Definition 2.5 (real vector spaces being of course replaced by *arbitrary* – possibly non-Hausdorff – topological real vector spaces).

At first sight, we may believe that for a function defined over a general topological real vector space, strict convexity of its epigraph is merely equivalent for the function to be strictly convex in the usual sense. But this is *false* as we can observe in the following example.

EXAMPLE 1.1. Consider the real vector space $V := C^0(\mathbf{R}, \mathbf{R}) \cap \mathcal{L}^2(\mathbf{R}, \mathbf{R}) \subseteq \mathbf{R}^{\mathbf{R}}$ endowed with the topology \mathcal{T} of pointwise convergence (this is nothing else than the product topology, which is therefore Hausdorff), and let $f : C := V \longrightarrow \mathbf{R}$ be the function defined by $f(u) := \|u\|_2^2$.

On the one hand, f is strictly convex since for any $u \in V$ its Hessian at u with respect to the norm $\|\cdot\|_2$ on V is equal to $2\langle \cdot, \cdot \rangle$, and hence positive definite.

On the other hand, whereas $(0, 0)$ and $(0, 2)$ are in the epigraph of f , their midpoint $(0, 1)$ does not belong to $\text{ri}(\text{Epi}(f)) = \widehat{\text{Epi}(f)}$, that is, $\text{Epi}(f)$ is not a neighborhood of $(0, 1)$ for the product topology on $V \times \mathbf{R}$ as we can check with the sequence $(u_n)_{n \geq 1}$ of V defined by

$$u_n(x) := \begin{cases} 2\sqrt{nx - n^2 + 1/n} & \text{for } x \in [n - 1/n^2, n], \\ 2/\sqrt{n} & \text{for } x \in [n, 2n], \\ 2\sqrt{2n^2 - nx + 1/n} & \text{for } x \in [2n, 2n + 1/n^2], \text{ and} \\ 0 & \text{for } x \leq n - 1/n^2 \text{ or } x \geq 2n + 1/n^2, \end{cases}$$

which converges to zero with respect to \mathcal{T} but satisfies $(u_n, 1) \notin \text{Epi}(f)$ for any $n \geq 1$ since one has $f(u_n) \geq f(u_n \times \mathbf{1}_{[n, 2n]}) = 4 > 1$. This proves that $\text{Epi}(f) \subseteq V \times \mathbf{R}$ is not strictly convex.

Moreover, even in the Hausdorff finite-dimensional case, things are not as simple as they seem. Indeed, if we consider the open disc $C := \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$, the function $f : C \longrightarrow \mathbf{R}$ defined by $f(x, y) := x^2 + y^2$ is strictly convex but its epigraph is not.

On the other hand, either in the non-Hausdorff finite dimensional case or in the infinite dimensional case, convexity does not always implies continuity. In contrast and among other things, we shall see that *strict convexity* of the epigraph *does always* insure continuity of the function.

Passing by, it is to be mentioned that any n -dimensional topological real vector space is isomorphic to the Cartesian product $\mathbf{R}^k \times \mathbf{R}^{n-k}$ for some integer $0 \leq k \leq n$, where the first factor is equipped with the usual topology and the second one with the trivial topology (see for instance [7, Chapter 1, Section 7, Problem A]).

So, the question of finding a complete characterization of functions with domain in a topological real vector space whose epigraph is strictly convex does deserve our attention.

This characterization is described as follows, where rb stands for the relative boundary (see Definition 2.4).

MAIN THEOREM. *Let C be a subset of a topological real vector space and let $f : C \rightarrow \mathbf{R}$ be a function. Then we have the following equivalence:*

$$\left(\begin{array}{l} C \text{ is convex and open in } \text{Aff}(C) , \\ f \text{ is strictly convex and continuous ,} \\ \forall z_0 \in \text{rb}(C) \subseteq \bar{C}, f(z) \rightarrow +\infty \text{ as } z \rightarrow z_0 \end{array} \right) \iff \text{Epi}(f) \text{ is strictly convex .}$$

REMARK 1.1. It is to be noticed that this equivalence is still true if “continuous” is changed into “locally bounded above”.

The proof, that we postpone until Section 3, splits into the direct implication and its converse.

The direct implication is the consequence of three main facts. The first one is the convexity of $\text{Epi}(f)$ given by the convexity of f . The second one is the closeness of the epigraph of f in $\text{Aff}(C) \times \mathbf{R}$ due to both the continuity of f and its behavior near the boundary of C , which insures that any segment whose end points are in the boundary of $\text{Epi}(f)$ is contained in $\text{Epi}(f)$. The third one is the property that any open segment whose endpoints are in the boundary of $\text{Epi}(f)$ actually lies inside the interior of $\text{Epi}(f)$ in $\text{Aff}(C) \times \mathbf{R}$ as a result of the strict convexity of f and the two previous facts.

As for the converse implication, there are four main things to be used. The first one is the convexity of both C and f given by the convexity of $\text{Epi}(f)$. The second one is the openness of C in $\text{Aff}(C)$ as a consequence for the epigraph not to contain vertical segments in its boundary. The third one is the fact that the interior of $\text{Epi}(f)$ in $\text{Aff}(C) \times \mathbf{R}$ lies inside the strict epigraph of f . The fourth one is the property for f to be locally bounded on some non-empty open subset of C as a result of the non-emptiness of the interior of $\text{Epi}(f)$ in $\text{Aff}(C) \times \mathbf{R}$. All these properties yield the continuity of f and give the behavior of f near the boundary of C .

Before giving some consequences of this result, all of whose will also be proved in Section 3, let us just show on a simple example how it may be usefull for checking strict convexity of the epigraph of a function.

EXAMPLE 1.2. Consider the open convex subset $C := (-1, 1) \times (-1, 1)$ of the topological real vector space $V := \mathbf{R}^2$, and let $f : C \rightarrow \mathbf{R}$ be the smooth function defined by $f(x, y) := 1/[(1 - x^2)(1 - y^2)]$.

For any $(x, y) \in C$, we then compute $\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{2(1 + 3x^2)}{(1 - x^2)(1 - y^2)} > 0$, and the Hessian matrix of f at (x, y) has a determinant which is equal to

$$4(5x^2y^2 + 3y^2 + 3x^2 + 1)/[(1 - x^2)(1 - y^2)]^4 > 0 .$$

The function f is therefore strictly convex and hence the Main Theorem insures that its epigraph is strictly convex since we have $\text{Aff}(C) = \mathbf{R}^2$ and $f(x, y) \rightarrow +\infty$ as (x, y) converges to any point $(x_0, y_0) \in \partial C$.

The first consequence of the Main Theorem is obtained by taking $C := V$.

PROPOSITION 1.2. *Given a strictly convex function $f : V \rightarrow \mathbf{R}$ defined on a topological real vector space V , we have the equivalence*

$$\text{Epi}(f) \text{ is strictly convex} \iff \text{Epi}(f) \text{ has a non-empty interior in } V \times \mathbf{R}.$$

REMARK 1.2. This result is not true if C is not equal to V as we can see with the strictly convex function $f : (0, 1) \rightarrow \mathbf{R}$ defined by $f(x) := x^2$.

Another use of the Main Theorem is related to the property for a subset C of a real vector space V to be convex if and only if all its intersections with the straight lines of V are convex.

Indeed, let us recall the following easy-to-prove result about convex functions.

PROPOSITION 1.3. *For any subset C of a real vector space V and any function $f : C \rightarrow \mathbf{R}$, we have (1) \iff (2) \iff (3) with*

- (1) $\text{Epi}(f)$ is convex,
- (2) $\text{Epi}(f|_{C \cap G})$ is convex for any affine subspace G of V , and
- (3) $\text{Epi}(f|_{C \cap L})$ is convex for any straight line L of V .

Then, a natural question is to know whether these equivalences are still true when replacing convexity by strict convexity.

Here is the answer.

PROPOSITION 1.4. *For any subset C of a topological real vector space V and any function $f : C \rightarrow \mathbf{R}$, we have (1) \iff (2) \implies (3) with*

- (1) $\text{Epi}(f)$ is strictly convex,
- (2) $\text{Epi}(f|_{C \cap G})$ is strictly convex for any affine subspace G of V , and
- (3) $\text{Epi}(f|_{C \cap L})$ is strictly convex for any straight line L in V .

It is to be noticed that the implication (3) \implies (1) in Proposition 1.4 is *not* true. Indeed, the function $f : V \rightarrow \mathbf{R}$ that we considered in Example 1.1 has an epigraph which is not strictly convex whereas for any vectors $u_0, w \in V$ with $w \neq 0$ the function $\varphi : \mathbf{R} \rightarrow \mathbf{R}$ defined by $\varphi(t) := f(u_0 + tw) = \|w\|_2^2 t^2 + 2\langle u_0, w \rangle t + \|u_0\|_2^2$ is obviously strictly convex and continuous. Therefore, $\text{Epi}(f|_{u_0 + \mathbf{R}w})$ is strictly convex according to the Main Theorem and since the map $\gamma : \mathbf{R} \rightarrow u_0 + \mathbf{R}w$ defined by $\gamma(t) := u_0 + tw$ is a homeomorphism. This last point is a consequence of Theorem 2 in [2, Chapitre I,

page 14] since $u_0 + \mathbf{R}w$ is a finite dimensional affine space whose subspace topology is Hausdorff (as is the topology on V).

Nevertheless, in case V is equal to the canonical topological real vector space \mathbf{R}^n , this implication is true as a consequence of the Main Theorem.

PROPOSITION 1.5. *Given a subset C of \mathbf{R}^n and a function $f : C \rightarrow \mathbf{R}$, the two following properties are equivalent:*

- (1) $\text{Epi}(f)$ is strictly convex,
- (2) $\text{Epi}(f|_{C \cap L})$ is strictly convex for any straight line L in \mathbf{R}^n .

As a straightforward consequence of Proposition 1.5, we obtain in particular the following classical result.

COROLLARY 1.1. *Any function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ satisfies the equivalence*

$$\text{Epi}(f) \text{ is strictly convex} \quad \iff \quad f \text{ is strictly convex.}$$

2. Preliminaries

In order to make precise the terms used in the previous section, and before proving in Section 3 the Main Theorem and its consequences, we have to give here some definitions and properties about the epigraph of a function and the notion of strict convexity.

2.1. Epigraphs

We begin by recalling the definitions of the epigraph and the strict epigraph for a general function.

DEFINITION 2.1. Given a set X and a function $f : X \rightarrow \mathbf{R}$, the *epigraph* of f is defined by

$$\text{Epi}(f) := \{(x, r) \in X \times \mathbf{R} \mid f(x) \leq r\},$$

whereas its *strict epigraph* is defined by

$$\text{Epi}_s(f) := \{(x, r) \in X \times \mathbf{R} \mid f(x) < r\}.$$

REMARK 2.1. It is straightforward that these two sets satisfy the relation

$$\text{Epi}_s(f) = (X \times \mathbf{R}) \setminus \sigma(\text{Epi}(-f)),$$

where σ denotes the involution of $X \times \mathbf{R}$ defined by $\sigma(x, r) := (x, -r)$.

From now on, X will be a *topological* space and we shall give a list of useful properties of the epigraph of a function with domain in X (Proposition 2.1 to Proposition 2.5) that we will need in the sequel (we may refer to [5, pages 34 and 123]).

PROPOSITION 2.1. *Given a subset S of a topological space X and a function $f : S \rightarrow \mathbf{R}$ such that $\text{Epi}(f)$ has a non-empty interior in $X \times \mathbf{R}$, there exists a non-empty open subset U of X with $U \subseteq S$ on which f is bounded above.*

Proof. Given $(x_0, t_0) \in \overset{\circ}{\text{Epi}}(f)$, there exists an open subset U of X such that one has $x_0 \in U$ and $U \times \{t_0\} \subseteq \text{Epi}(f)$ since $\text{Epi}(f)$ is a neighborhood of (x_0, t_0) in $X \times \mathbf{R}$. Therefore, for any $x \in U$, we get $f(x) \leq t_0$. \square

PROPOSITION 2.2. *Given a function $f : X \rightarrow \mathbf{R}$ defined on a topological space X , a point $x_0 \in X$ and a number $r_0 \in \mathbf{R}$, the following properties are equivalent:*

- (1) $(x_0, r_0) \in \overset{\circ}{\text{Epi}}(f)$.
- (2) There exists $r \in (-\infty, r_0)$ which satisfies $x_0 \in \overset{\circ}{f^{-1}(-\infty, r)}$.
- (3) There exists $s \in (-\infty, r_0)$ which satisfies $\{x_0\} \times [s, +\infty) \subseteq \overset{\circ}{\text{Epi}}(f)$.

This is a characterization of the interior of the epigraph of a function.

Proof. Point 1 \implies Point 2. Assume we have $(x_0, r_0) \in \overset{\circ}{\text{Epi}}(f)$.

Then there exist a neighborhood V of x_0 in X and a real number $\varepsilon > 0$ that satisfy the inclusion $V \times [r_0 - 2\varepsilon, r_0 + 2\varepsilon] \subseteq \text{Epi}(f)$, from which we get $f(x) \leq r_0 - 2\varepsilon$ for any $x \in V$, or equivalently $V \subseteq f^{-1}(-\infty, r)$ with $r := r_0 - \varepsilon < r_0$.

This proves that x_0 belongs to the interior of $\overset{\circ}{f^{-1}(-\infty, r)}$ in \mathbf{R} .

Point 2 \implies Point 3. Assume that we have $x_0 \in \overset{\circ}{f^{-1}(-\infty, r)}$ for some number $r \in (-\infty, r_0)$.

Therefore, there is a neighborhood V of x_0 in X with $V \subseteq f^{-1}(-\infty, r) \subseteq f^{-1}(-\infty, r]$, which yields $V \times [r, +\infty) \subseteq \text{Epi}(f)$, and hence the inclusion $\{x_0\} \times [s, +\infty) \subseteq \overset{\circ}{\text{Epi}}(f)$ holds with $s := (r + r_0)/2$ since we have $s \in (r, r_0)$ and since $[r, +\infty)$ is a neighborhood in \mathbf{R} of any $\tau \in [s, +\infty)$.

Point 3 \implies Point 1. This is clear. \square

PROPOSITION 2.3. *Any function $f : X \rightarrow \mathbf{R}$ defined on a topological space X satisfies the following:*

- (1) $\overset{\circ}{\text{Epi}}(f) = \overset{\circ}{\text{Epi}_s}(f)$.
- (2) $\overset{\circ}{\text{Epi}}(f) \cap (\{x\} \times \mathbf{R}) \subseteq \{x\} \times (f(x), +\infty)$ for any $x \in X$.
- (3) $\overline{\text{Epi}}(f) = (X \times \mathbf{R}) \setminus \sigma(\overset{\circ}{\text{Epi}_s}(-f))$.
- (4) $\overline{\text{Epi}}(f) = \overline{\text{Epi}_s}(f)$.

This property gives a topological relationship between the epigraph and the strict epigraph of a function.

Proof. Point 1. Given $(x, r) \in \widehat{\text{Epi}}(f)$, there exists a real number $r_0 < r$ that satisfies the inclusion $\{x\} \times [r_0, +\infty) \subseteq \text{Epi}(f)$, which yields in particular $f(x) \leq r_0$, and hence $f(x) < r$. This proves $\widehat{\text{Epi}}(f) \subseteq \text{Epi}_s(f)$, and hence $\widehat{\text{Epi}}(f) \subseteq \widehat{\text{Epi}}_s(f)$ by taking the interiors in the product space $X \times \mathbf{R}$.

Conversely, the obvious inclusion $\text{Epi}_s(f) \subseteq \text{Epi}(f)$ yields $\widehat{\text{Epi}}_s(f) \subseteq \widehat{\text{Epi}}(f)$.

Point 2. By Point 1 above, we have $\widehat{\text{Epi}}(f) \subseteq \text{Epi}_s(f)$, from which one can deduce $\widehat{\text{Epi}}(f) \cap (\{x\} \times \mathbf{R}) \subseteq \text{Epi}_s(f) \cap (\{x\} \times \mathbf{R}) = \{x\} \times (f(x), +\infty)$ for any $x \in X$.

Point 3. By Remark 2.1 applied to $-f$, we have $\text{Epi}(f) = (X \times \mathbf{R}) \setminus \sigma(\text{Epi}_s(-f))$ since σ is an involution, and hence

$$\widehat{\text{Epi}}(f) = \overline{(X \times \mathbf{R}) \setminus \sigma(\text{Epi}_s(-f))} = (X \times \mathbf{R}) \setminus \sigma(\widehat{\text{Epi}}_s(-f))$$

since σ is a homeomorphism.

Now, Point 1 above gives the desired equality.

Point 4. The equality in Point 3 also writes $\widehat{\text{Epi}}(f) = \overline{(X \times \mathbf{R}) \setminus \sigma(\text{Epi}(-f))}$ since σ is a homeomorphism, which yields $\widehat{\text{Epi}}(f) = \widehat{\text{Epi}}_s(f)$ since we have the relation $\text{Epi}_s(f) = (X \times \mathbf{R}) \setminus \sigma(\text{Epi}(-f))$ by Remark 2.1. \square

PROPOSITION 2.4. *Given a function $f : X \rightarrow \mathbf{R}$ defined on a topological space X and a point $x_0 \in X$, the two following properties are equivalent:*

- (1) f is upper semi-continuous at x_0 .
- (2) $\{x_0\} \times (f(x_0), +\infty) = \widehat{\text{Epi}}(f) \cap (\{x_0\} \times \mathbf{R})$.

This is a geometric characterization of the upper semi-continuity of a function in terms of its epigraph.

Proof. Point 1 \implies Point 2. Assume that Point 1 is satisfied.

Given an arbitrary real number $\varepsilon > 0$, there exists a neighborhood V of x_0 in X that satisfies $f(x) \leq f(x_0) + \varepsilon/2$ for any $x \in V$.

Now, we have $V \times [f(x_0) + \varepsilon/2, +\infty) \subseteq \text{Epi}(f)$, and hence $(x_0, f(x_0) + \varepsilon) \in \widehat{\text{Epi}}(f)$.

So we proved the direct inclusion \subseteq .

The reverse inclusion \supseteq is straightforward by Point 2 in Proposition 2.3.

This proves Point 2.

Point 2 \implies Point 1. Assume that Point 2 is satisfied.

Conversely, given an arbitrary real number $\varepsilon > 0$, one can write

$$(x_0, f(x_0) + \varepsilon) \in \{x_0\} \times (f(x_0), +\infty) \subseteq \widehat{\text{Epi}}(f).$$

Thus, there exists a neighborhood V of x_0 in X satisfying $V \times \{f(x_0) + \varepsilon\} \subseteq \text{Epi}(f)$, which yields $f(x) \leq f(x_0) + \varepsilon$ for any $x \in V$.

This proves Point 1. \square

PROPOSITION 2.5. *Given a subset A of a topological space X , a point $x_0 \in \bar{A}$ and a function $f : A \rightarrow \mathbf{R}$, we have the equivalence*

$$(f(x) \rightarrow +\infty \text{ as } x \rightarrow x_0) \iff ((x_0, r) \notin \overline{\text{Epi}(f)} \text{ for any } r \in \mathbf{R}).$$

This is a characterization of the closure of the epigraph of a function.

Proof. * (\implies) Assume the left-hand side of \iff holds.

Given $r \in \mathbf{R}$, there exists a neighborhood V of x_0 in X such that we have $f(V) \subseteq [r + 2, +\infty)$.

Thus, we get $(V \times (-\infty, r + 1]) \cap \text{Epi}(f) = \emptyset$, which shows that (x_0, r) does not belong to $\overline{\text{Epi}(f)}$ since $V \times (-\infty, r + 1]$ is a neighborhood of (x_0, r) in $X \times \mathbf{R}$.

* (\impliedby) Assume the right-hand side of \iff holds.

Given $r \in \mathbf{R}$, there exist a neighborhood V of x_0 in X and a real number $\varepsilon > 0$ such that $V \times [r - \varepsilon, r + \varepsilon]$ does not meet $\text{Epi}(f)$.

Since we have $r \in [r - \varepsilon, r + \varepsilon]$, this yields $f(V) \cap (-\infty, r] = \emptyset$, which is equivalent to the inclusion $f(V) \subseteq (r, +\infty)$.

So, we have proved $f(x) \rightarrow +\infty$ as $x \rightarrow x_0$. \square

2.2. Strict convexity

In this subsection, we first recall the definitions of the relative interior and the relative closure of a subset of a topological real vector space since they underly strict convexity, and then we establish a couple of useful properties needed in Section 3.

We begin with two basic notions in affine geometry: the affine hull and convex sets (see for example [9] and [10]).

DEFINITION 2.2. The *affine hull* $\text{Aff}(S)$ of a subset S of a real vector space V is the smallest affine subspace of V which contains S .

Therefore, for any subsets A and B of V satisfying $A \subseteq B$, we obviously have $\text{Aff}(A) \subseteq \text{Aff}(B)$.

PROPOSITION 2.6. *Given real vector spaces V and W , any subsets $A \subseteq V$ and $B \subseteq W$ satisfy*

$$\text{Aff}(A \times B) = \text{Aff}(A) \times \text{Aff}(B) .$$

Proof. For any $x, y \in B$ with $x \neq y$, the straight line L passing through x and y lies in $\text{Aff}(B)$, and hence for any $a \in A$ the straight line $\{a\} \times L$ of $V \times W$ lies in $\text{Aff}(A \times B)$ since it contains the points $(a, x), (a, y) \in A \times B$. Therefore, we have $A \times \text{Aff}(B) \subseteq \text{Aff}(A \times B)$.

On the other hand, we also get $\text{Aff}(A) \times B \subseteq \text{Aff}(A \times B)$ by the same reasoning.

These two inclusions yield

$$\text{Aff}(A) \times \text{Aff}(B) \subseteq \text{Aff}(\text{Aff}(A) \times B) \subseteq \text{Aff}(\text{Aff}(A \times B)) = \text{Aff}(A \times B).$$

Conversely, since $\text{Aff}(A) \times \text{Aff}(B)$ is an affine subspace of $V \times W$ which contains $A \times B$, we obviously have $\text{Aff}(A \times B) \subseteq \text{Aff}(A) \times \text{Aff}(B)$. \square

DEFINITION 2.3. Given points x and y in a real vector space V , the set

$$[x, y] := \{(1-t)x + ty \mid t \in [0, 1]\}$$

is called the (*closed*) *line segment* between x and y , whereas the set

$$]x, y[:= [x, y] \setminus \{x, y\}$$

is called the *open line segment* between x and y (the latter set is therefore empty in case one has $x = y$).

A subset C of V is said to be *convex* if we have $[x, y] \subseteq C$ for all $x, y \in C$.

In other words, C is convex if and only if its intersection with any straight line L in V is an “interval” of L .

From now on and throughout the section, V will denote a *topological* real vector space.

Before we go on, let us just point out some facts.

REMARK 2.2.

- 1) Given a neighborhood U of the origin in V , the following properties hold:
 - a) For any $x \in V$, there exists a real number $\varepsilon > 0$ such that we have $[-\varepsilon, \varepsilon]x \subseteq U$.
 - b) For any $x \in V$, there exists a real number $\lambda > 0$ such that we have $x \in \lambda U$ (the subset U of V is then said to be *absorbing*).
 - c) We have $\text{Vect}(U) = V$.

Indeed, Point a is easy to prove and the implications Point a \implies Point b \implies Point c are straightforward.

- 2) Given a finite-dimensional real vector space W , there exists a *unique* topological real vector space structure on W which is Hausdorff. Endowed with this structure, W is then isomorphic to the canonical topological real vector space \mathbf{R}^n , where n denotes the dimension of W (see Theorem 2 in [2, Chapitre I, page 14]).

DEFINITION 2.4. Let S be a subset of a topological real vector space V .

- (1) The *relative interior* $\text{ri}(S)$ of S is the interior of S with respect to the relative topology of $\text{Aff}(S)$ in V .

- (2) The *relative closure* $\text{rc}(S)$ of S is the closure of S with respect to the relative topology of $\text{Aff}(S)$ in V .
- (3) The *relative boundary* $\text{rb}(S)$ of S is the boundary of S with respect to the relative topology of $\text{Aff}(S)$ in V (so we have $\text{rb}(S) = \text{rc}(S) \setminus \text{ri}(S)$).

We obviously have $\overset{\circ}{S} \subseteq \text{ri}(S)$ and $\text{rc}(S) \subseteq \bar{S}$, which yields $\text{rb}(S) \subseteq \partial S$.

PROPOSITION 2.7. *Let V be a topological real vector space.*

(1) *For any subsets A and B of V , we have the implication $A \subseteq B \implies \text{rc}(A) \subseteq \text{rc}(B)$.*

(2) *For any subset S of V , we have*

- (a) $\text{Aff}(\text{rc}(S)) = \text{Aff}(S)$, and
- (b) $\text{ri}(S) \neq \emptyset \iff (S \neq \emptyset \text{ and } \text{Aff}(\text{ri}(S)) = \text{Aff}(S))$.

(3) *For any subset S of V and any affine subspace W of V , we have*

- (a) $\text{ri}(S) \cap \text{Aff}(S \cap W) \subseteq \text{ri}(S \cap W)$, and
- (b) $\text{rb}(S \cap W) \subseteq \text{rb}(S) \cap W$.

Proof. Point 1. Given subsets A and B of V with $A \subseteq B$, we can write

$$\text{rc}(A) = \bar{A} \cap \text{Aff}(A) \subseteq \bar{B} \cap \text{Aff}(B) = \text{rc}(B)$$

since we have $\text{Aff}(A) \subseteq \text{Aff}(B)$ and $\bar{A} \subseteq \bar{B}$.

Point 2.a. Using $S \subseteq \text{rc}(S)$, we first get $\text{Aff}(S) \subseteq \text{Aff}(\text{rc}(S))$. Conversely, we have $\text{rc}(S) \subseteq \text{Aff}(S)$ by definition of $\text{rc}(S)$, and hence $\text{Aff}(\text{rc}(S)) \subseteq \text{Aff}(\text{Aff}(S)) = \text{Aff}(S)$.

Point 2.b. If the open subset $\text{ri}(S)$ of $\text{Aff}(S)$ is not empty, then we can write $\text{Aff}(\text{ri}(S)) = \text{Aff}(S)$ by Point 1.c in Remark 2.2, and S is not empty since one has $\text{ri}(S) \subseteq S$.

Conversely, if we have $S \neq \emptyset$ and $\text{Aff}(\text{ri}(S)) = \text{Aff}(S)$, then we get $\text{Aff}(\text{ri}(S)) = \text{Aff}(S) \neq \emptyset$, and hence $\text{ri}(S) \neq \emptyset$ by using the obvious equality $\text{Aff}(\emptyset) = \emptyset$.

Point 3.a. Since $\text{ri}(S)$ is open in $\text{Aff}(S)$, the intersection $\text{ri}(S) \cap \text{Aff}(S \cap W)$ is open in $\text{Aff}(S \cap W) \subseteq \text{Aff}(S)$.

On the other hand, we have $\text{ri}(S) \subseteq S$ and $\text{Aff}(S \cap W) \subseteq \text{Aff}(W) = W$, and hence the inclusion $\text{ri}(S) \cap \text{Aff}(S \cap W) \subseteq S \cap W$.

Therefore, we get $\text{ri}(S) \cap \text{Aff}(S \cap W) \subseteq \text{ri}(S \cap W)$ since $\text{ri}(S \cap W)$ is the largest open subset of $\text{Aff}(S \cap W)$ which is contained in $S \cap W$.

Point 3.b. We first have $\text{Aff}(S \cap W) \subseteq \text{Aff}(W) = W$, and hence $\text{rb}(S \cap W)$ lies in W . On the other hand, combining Point 1 and Point 3.a yields

$$\begin{aligned} \text{rb}(S \cap W) &= \text{rc}(S \cap W) \setminus \text{ri}(S \cap W) \\ &\subseteq \text{rc}(S \cap W) \setminus [\text{ri}(S) \cap \text{Aff}(S \cap W)] \\ &= [\text{rc}(S \cap W) \setminus \text{ri}(S)] \cup [\text{rc}(S \cap W) \setminus \text{Aff}(S \cap W)] \\ &= \text{rc}(S \cap W) \setminus \text{ri}(S) \\ &\quad (\text{use } \text{rc}(S \cap W) \subseteq \text{Aff}(S \cap W)) \\ &\subseteq \text{rc}(S) \setminus \text{ri}(S) = \text{rb}(S). \quad \square \end{aligned}$$

PROPOSITION 2.8. *Let X be a subset of a topological real vector space V and S a subset of $X \times \mathbf{R}$ such that they satisfy $\text{Aff}(S) = \text{Aff}(X) \times \mathbf{R}$. Then we have the following properties:*

- (1) *The interior of S in $X \times \mathbf{R}$ contains $\text{ri}(S)$.*
- (2) *The closure of S in $X \times \mathbf{R}$ lies in $\text{rc}(S)$.*

Proof. Point 1. Given $(x_0, r_0) \in \text{ri}(S)$, there exist a neighborhood U of x_0 in V and a neighborhood W of r_0 in \mathbf{R} such that we have $[U \cap \text{Aff}(X)] \times W \subseteq S$, which implies

$$(U \cap X) \times W = (X \times \mathbf{R}) \cap [U \cap \text{Aff}(X)] \times W \subseteq (X \times \mathbf{R}) \cap S = S.$$

Then, since $U \cap X$ is a neighborhood of x_0 in X , we get that (x_0, r_0) is in the interior of S with respect to $X \times \mathbf{R}$.

Point 2. From $X \times \mathbf{R} \subseteq \text{Aff}(X) \times \mathbf{R} = \text{Aff}(S)$, we deduce that the closure $\bar{S}^{X \times \mathbf{R}}$ of S in $X \times \mathbf{R}$ satisfies

$$\bar{S}^{X \times \mathbf{R}} = \bar{S} \cap (X \times \mathbf{R}) \subseteq \bar{S} \cap \text{Aff}(S) = \text{rc}(S). \quad \square$$

Now, here is the definition of a strictly convex set, which is the key notion of the present work.

DEFINITION 2.5. A subset C of a topological real vector space V is said to be *strictly convex* if for any two distinct points $x, y \in \text{rc}(C)$ we have $]x, y[\subseteq \text{ri}(C)$.

REMARK 2.3.

- 1) This definition coincides with the usual one when V is the canonical topological real vector space \mathbf{R}^n since in this case the closeness of $\text{Aff}(C)$ in V yields $\text{rc}(C) = \bar{C}$.
- 2) It is to be noticed that strict convexity is a property of *topological* real vector spaces whereas convexity is a mere affine property of real vector spaces.
- 3) A strictly convex subset of V is of course convex.

- 4) According to the common geometric intuition, saying that a subset C of V is strictly convex means that C is convex and that there is no non-trivial segment in the relative boundary of C . Owing to Proposition 16 in [2, Chapitre II, page 15], this is an easy consequence of the very definition of strict convexity.

PROPOSITION 2.9. *For any strictly convex subset C of a topological real vector space, we have the implication*

$$C \neq \emptyset \implies \text{ri}(C) \neq \emptyset.$$

Proof. Assume that C is not empty. Two cases are then considered, depending on whether $\text{rc}(C)$ is a single point or not.

If we have $\text{rc}(C) = \{x\}$ for some $x \in V$, then C also reduces to $\{x\}$ since one has $\emptyset \neq C \subseteq \text{rc}(C)$. Therefore, we obtain $\text{Aff}(C) = \{x\}$, and hence $\text{ri}(C) = \{x\} \neq \emptyset$.

On the other hand, if $\text{rc}(C)$ contains two distinct points $x, y \in V$, then the inclusion $]x, y[\subseteq \text{ri}(C)$ holds by strict convexity of C , and hence one has $\text{ri}(C) \neq \emptyset$ since $]x, y[$ is not empty. \square

REMARK 2.4. When dealing with a single strictly convex subset C of a general topological real vector space V , we will always assume in the hypotheses that C has a non-empty interior in V in order to insure $\text{Aff}(C) = V$, and this makes sense by Proposition 2.9 and Point 2.b in Proposition 2.7.

We shall now prove that strict convexity is a two-dimensional (topological) notion whereas convexity is – by its very definition – a one-dimensional (affine) notion.

PROPOSITION 2.10. *Given a topological real vector space V with $\dim(V) \geq 2$, any subset C of V whose interior is not empty satisfies the equivalence*

$$C \text{ is strictly convex} \iff C \cap P \text{ is strictly convex for any affine plane } P \text{ in } V.$$

Proof. * (\implies) Assume the left-hand side of \iff holds, and let $x, y \in \text{rc}(C \cap P)$ with $x \neq y$.

Then, we first have $x, y \in \bar{C}$ since the inclusion $\text{rc}(C \cap P) \subseteq \bar{C}$ holds according to Point 1 in Proposition 2.7, and this yields $]x, y[\subseteq \overset{\circ}{C}$ by strict convexity of C .

On the other hand, we have $x, y \in P$ from $\text{rc}(C \cap P) \subseteq \text{Aff}(C \cap P) \subseteq \text{Aff}(P) = P$, and hence we get $]x, y[\subseteq \overset{\circ}{C} \cap P$ by convexity of P .

In particular, the open subset $\overset{\circ}{C} \cap P$ of P is not empty, and hence $\text{Aff}(\overset{\circ}{C} \cap P) = P$ holds by Point 3 in Remark 2.2, which yields $P \subseteq \text{Aff}(C \cap P)$ since we have $\overset{\circ}{C} \subseteq C$.

Therefore, we get $\overset{\circ}{C} \cap P \subseteq \overset{\circ}{C} \cap \text{Aff}(C \cap P) \subseteq \text{ri}(C \cap P)$ by Point 3.a in Proposition 2.7, which implies $]x, y[\subseteq \text{ri}(C \cap P)$.

This proves that $C \cap P$ is strictly convex.

* (\Leftarrow) Assume the right-hand side of \iff holds, and let $x, y \in \bar{C}$ with $x \neq y$.

Since the dimension of $V = \text{Aff}(C)$ is greater than one, C does not lie in a line, and hence there exists a point $z \in \overset{\circ}{C}$ such that x, y, z are not collinear. Therefore, $P := \text{Aff}(\{x, y, z\})$ is an affine plane in V .

Then, Proposition 16 in [2, Chapitre II, page 15] implies that the open segments $]z, x[$ and $]z, y[$ are in $\overset{\circ}{C} \cap P$, and hence in $C \cap P$. This insures that $\text{rc}(]z, x[)$ and $\text{rc}(]z, y[)$ are in $\text{rc}(C \cap P)$.

But we have $x \in \text{rc}(]z, x[)$ since any neighborhood U of x in V satisfies $]z, x[\cap U \neq \emptyset$ by Point 1.a in Remark 2.2. And the same argument yields $y \in \text{rc}(]z, y[)$.

So, we have obtained $x, y \in \text{rc}(C \cap P)$, and hence $]x, y[\subseteq \text{ri}(C \cap P)$ since $C \cap P$ is strictly convex.

Now, if we pick a point $u \in]x, y[$ and define $v := u - z$, then there exists a real number $t > 0$ that satisfies $w := u + tv \in C \cap P$ since $u + \mathbf{R}v$ is the straight line passing through the points u and z of $C \cap P \subseteq \text{Aff}(C \cap P)$ and since $C \cap P$ is a neighborhood of $u \in \text{ri}(C \cap P)$ in $\text{Aff}(C \cap P)$.

Therefore, the segment $[z, w]$ lies in the convex set $C \cap P$, and hence in C , which yields $[z, u] \subseteq [z, w] \setminus \{w\} \subseteq \overset{\circ}{C}$ by Proposition 16 in [2, Chapitre II, page 15].

This proves that C is strictly convex. \square

REMARK 2.5. In case V is one-dimensional but its topology is not Hausdorff, the strictly convex subsets of V , unlike those of \mathbf{R} (endowed with its usual topology), do *not* coincide with its convex subsets. Indeed, if the topology of V is for example trivial, then the only strictly convex subsets of V are the empty set and V itself.

PROPOSITION 2.11. *Given a subset C of a topological real vector space and a function $f : C \rightarrow \mathbf{R}$, we have the following equivalence*

$$\text{Epi}(f) \text{ is strictly convex} \iff \text{Epi}_s(f) \text{ is strictly convex.}$$

Proof. According to Lemma 3.1, we first have $\text{Aff}(\text{Epi}(f)) = \text{Aff}(\text{Epi}_s(f))$.

Hence, Points 1 and 4 in Proposition 2.3 yield $\text{ri}(\text{Epi}(f)) = \text{ri}(\text{Epi}_s(f))$ and $\text{rc}(\text{Epi}(f)) = \text{rc}(\text{Epi}_s(f))$, and we are done. \square

Finally, in order to be complete, let us recall the definition of a (strictly) convex function.

DEFINITION 2.6. Given a convex subset C of a real vector space, a function $f : C \rightarrow \mathbf{R}$ is said to be

- (1) *convex* if we have $f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$ for any points $x, y \in C$ and any number $t \in (0, 1)$.
- (2) *strictly convex* if we have $f((1-t)x + ty) < (1-t)f(x) + tf(y)$ for any two distinct points $x, y \in C$ and any number $t \in (0, 1)$.

REMARK 2.6.

- 1) It is to be noticed that both convexity and strict convexity of functions are mere affine notions.
- 2) A strictly convex function is of course convex.
- 3) Given a convex subset C of a real vector space, a function $f : C \rightarrow \mathbf{R}$ is convex if and only if it satisfies

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i) \quad (\text{Jensen's inequality})$$

for any integer $n \geq 1$, any points $x_1, \dots, x_n \in C$, and any numbers $\lambda_1, \dots, \lambda_n \in [0, +\infty)$ with $\sum_{i=1}^n \lambda_i = 1$.

This is obtained by induction on $n \geq 1$.

- 4) Given a convex subset C of a real vector space V , a convex function $f : C \rightarrow \mathbf{R}$, a subset $S \subseteq C$ and a real number $M > 0$, we have the following equivalence:

$$(\forall x \in S, f(x) \leq M) \iff (\forall x \in \text{Conv}(S), f(x) \leq M),$$

where $\text{Conv}(S)$ stands for the convex hull of S , *i. e.*, the smallest convex subset of V which contains S .

Indeed, each $x \in \text{Conv}(S)$ writes $x = \sum_{i=1}^n \lambda_i x_i$ for some integer $n \geq 1$, some points $x_1, \dots, x_n \in S$ and some numbers $\lambda_1, \dots, \lambda_n \in [0, +\infty)$ which satisfy $\sum_{i=1}^n \lambda_i = 1$.

Therefore, if the function f is bounded above by M on S , this implies $f(x) \leq \sum_{i=1}^n \lambda_i f(x_i) \leq \sum_{i=1}^n \lambda_i M = M$ according to Point 3 above.

3. Proofs

This section is devoted to the proofs of the Main Theorem and of all its consequences that we mentioned in Section 1.

Let us begin with the following affine property.

LEMMA 3.1. *For any function $f : S \rightarrow \mathbf{R}$ defined on a subset S of a real vector space, we have $\text{Aff}(\text{Epi}(f)) = \text{Aff}(S) \times \mathbf{R} = \text{Aff}(\text{Epi}_s(f))$.*

Proof. First of all, we have $\text{Epi}_s(f) \subseteq \text{Epi}(f) \subseteq S \times \mathbf{R} \subseteq \text{Aff}(S) \times \mathbf{R}$, and hence we immediately get $\text{Aff}(\text{Epi}_s(f)) \subseteq \text{Aff}(\text{Epi}(f)) \subseteq \text{Aff}(S) \times \mathbf{R}$.

Conversely, given $x \in S$, we have $\{x\} \times (f(x), f(x) + 1] \subseteq \text{Epi}_s(f)$ by the very definition of $\text{Epi}_s(f)$, which gives $\{x\} \times \mathbf{R} = \text{Aff}(\{x\} \times (f(x), f(x) + 1]) \subseteq \text{Aff}(\text{Epi}_s(f))$ by Proposition 2.6. Therefore, we get $S \times \mathbf{R} \subseteq \text{Aff}(\text{Epi}_s(f))$, and this yields $\text{Aff}(S) \times \mathbf{R} = \text{Aff}(S \times \mathbf{R}) \subseteq \text{Aff}(\text{Epi}_s(f))$ by Proposition 2.6 once again. \square

REMARK 3.1. A straightforward consequence of the first equality in Lemma 3.1 is that any function $f : S \rightarrow \mathbf{R}$ defined on a subset S of a real vector space satisfies $\text{ri}(\text{Epi}(f)) \subseteq \text{ri}(S) \times \mathbf{R}$.

Then, let us establish two technical but useful topological properties.

LEMMA 3.2. *Given a subset S of a topological real vector space V and a function $f : S \rightarrow \mathbf{R}$, we have the implication*

$$\left(\begin{array}{l} S \text{ is open in } \text{Aff}(C) , \\ f \text{ is lower semi-continuous} , \\ \forall z_0 \in \text{rb}(S) \subseteq \bar{S}, f(z) \rightarrow +\infty \text{ as } z \rightarrow z_0 \end{array} \right) \implies \text{Epi}(f) \text{ is closed in } \text{Aff}(S) \times \mathbf{R} .$$

Proof. Assume the left-hand side of \implies holds, and let $(a, \alpha) \in \text{rc}(\text{Epi}(f))$.

First of all, notice that a is in $\text{rc}(S)$ since the projection of $V \times \mathbf{R}$ onto V is continuous and since we have $\text{Aff}(\text{Epi}(f)) = \text{Aff}(S) \times \mathbf{R}$ by Lemma 3.1.

If we had $a \notin S = \text{ri}(S)$ (remember that S is open in $\text{Aff}(S)$), then we would obtain $a \in \text{rb}(S)$, and hence $f(z) \rightarrow +\infty$ as $z \rightarrow z_0 := a$, which yields $(a, \alpha) \notin \text{rc}(\text{Epi}(f))$ by Proposition 2.5: contradiction.

Therefore, the point a is necessarily in S .

Now, assume that we have $f(a) > \alpha$, and let $\varepsilon := (f(a) - \alpha)/2 > 0$.

Since f is lower semi-continuous at a , there exists a neighborhood U of a in S which satisfies $f(U) \subseteq [f(a) - \varepsilon, +\infty) = [\alpha + \varepsilon, +\infty)$.

But S is a neighborhood of a in $\text{Aff}(S)$, and hence so is U .

Thus, combining $(a, \alpha) \in \text{rc}(\text{Epi}(f))$ and $\text{Aff}(\text{Epi}(f)) = \text{Aff}(S) \times \mathbf{R}$, we can find $z \in U$ and $\theta \in (\alpha - \varepsilon, \alpha + \varepsilon)$ with $f(z) \leq \theta$, which yields $f(z) \in (-\infty, \alpha + \varepsilon)$, contradicting the previous point.

So, we necessarily have $f(a) \leq \alpha$, or equivalently $(a, \alpha) \in \text{Epi}(f)$.

Conclusion: we proved $\text{rc}(\text{Epi}(f)) \subseteq \text{Epi}(f)$, which means that $\text{Epi}(f)$ is closed in $\text{Aff}(\text{Epi}(f)) = \text{Aff}(S) \times \mathbf{R}$. \square

LEMMA 3.3. *Given a subset S of a topological real vector space V and a function $f : S \rightarrow \mathbf{R}$, we have (1) \implies (2) \implies (3) \implies (4) with*

- (1) $\text{Epi}(f)$ is strictly convex,
- (2) $\forall s > 0, \tau_s(\text{rc}(\text{Epi}(f))) \subseteq \text{ri}(\text{Epi}(f))$,
- (3) $\text{rc}(\text{Epi}(f)) \subseteq \text{ri}(S) \times \mathbf{R}$, and
- (4) S is open in $\text{Aff}(S)$,

where for each $s \in \mathbf{R}$ the map $\tau_s : V \times \mathbf{R} \longrightarrow V \times \mathbf{R}$ is defined by $\tau_s(x, r) := (x, r + s)$.

Proof. Point 1 \implies Point 2. Assume that $\text{Epi}(f)$ is strictly convex.

Since for each $s \in \mathbf{R}$ the map τ_s is an affine homeomorphism, we have the equality $\tau_s(\text{rc}(\text{Epi}(f))) = \text{rc}(\tau_s(\text{Epi}(f)))$, and hence $\tau_s(\text{rc}(\text{Epi}(f))) \subseteq \text{rc}(\text{Epi}(f))$ for $s \geq 0$.

Therefore, given a real number $s > 0$ and a point $(x, r) \in \text{rc}(\text{Epi}(f))$, we can write $(x, r + 2s) = \tau_{2s}(x, r) \in \text{rc}(\text{Epi}(f))$, which yields that the midpoint $\tau_s(x, r) = (x, r + s)$ of (x, r) and $(x, r + 2s)$ is in $\text{ri}(\text{Epi}(f))$ since $\text{Epi}(f)$ is strictly convex. This proves Point 2.

Point 2 \implies Point 3. Assume that Point 2 is satisfied.

Then, by Remark 3.1, we get $\tau_s(\text{rc}(\text{Epi}(f))) \subseteq \text{ri}(S) \times \mathbf{R}$ for any real number $s > 0$, and hence $\text{rc}(\text{Epi}(f)) \subseteq \tau_{-s}(\text{ri}(S) \times \mathbf{R}) = \text{ri}(S) \times \mathbf{R}$. This proves Point 3.

Point 3 \implies Point 4. Assume we have $\text{rc}(\text{Epi}(f)) \subseteq \text{ri}(S) \times \mathbf{R}$.

Then, using $\text{Epi}(f) \subseteq \text{rc}(\text{Epi}(f))$, we get $\text{Epi}(f) \subseteq \text{ri}(S) \times \mathbf{R}$, which gives $S \subseteq \text{ri}(S)$ by applying the projection of $V \times \mathbf{R}$ onto V . This proves Point 4. \square

Combining Lemma 3.2 and Lemma 3.3 with all the properties established in Section 2, we are now able to prove the Main Theorem.

Proof. [Proof of the Main Theorem] We may assume that C is not empty since this equivalence is obviously true otherwise.

* (\implies) Assume the left-hand side of \iff holds, let $(x, r), (y, s) \in \text{rc}(\text{Epi}(f))$ with $(x, r) \neq (y, s)$, $t \in (0, 1)$, and define the point $(a, \alpha) := (1 - t)(x, r) + t(y, s) = ((1 - t)x + ty, (1 - t)r + ts) \in V \times \mathbf{R}$.

By Lemma 3.2, we already have $(x, r), (y, s) \in \text{Epi}(f)$. Then, since C and f are both convex, $\text{Epi}(f)$ is convex by Proposition 1.1, which implies $(a, \alpha) \in \text{Epi}(f)$.

There are now two cases to be considered.

- Case $x = y$ and $r < s$.

Here we have $a = x = y$, which yields

$$f(a) = f(x) \leq r = (1 - t)r + tr < (1 - t)r + ts = \alpha.$$

- Case $x \neq y$.

By strict convexity of f , one can write

$$f(a) < (1 - t)f(x) + tf(y) \leq (1 - t)r + ts = \alpha.$$

In both cases, we get $(a, \alpha) \in \text{ri}(\text{Epi}(f))$ by Point 2 in Proposition 2.4 since f is upper semi-continuous at a .

This proves that $\text{Epi}(f)$ is strictly convex.

* (\impliedby) Assume the right-hand side of \iff holds.

First of all, C is convex by Proposition 1.1. Moreover, C is open in $\text{Aff}(C)$ by the implication Point 1 \implies Point 4 in Lemma 3.3 with $S := C$.

On the other hand, given $x, y \in C$ with $x \neq y$ and $t \in (0, 1)$, the points $(x, f(x))$ and $(y, f(y))$ are in $\text{Epi}(f) \subseteq \text{rc}(\text{Epi}(f))$, which yields

$$(a, \alpha) := ((1 - t)x + ty, (1 - t)f(x) + tf(y)) \in \text{ri}(\text{Epi}(f))$$

since $\text{Epi}(f)$ is strictly convex.

Therefore, we get $f((1-t)x+ty) = f(a) < \alpha = (1-t)f(x) + tf(y)$ by Point 1 in Proposition 2.8 with $X := C$ and $S := \text{Epi}(f)$ by using $\text{Aff}(\text{Epi}(f)) = \text{Aff}(C) \times \mathbf{R}$ (see the first equality in Lemma 3.1) and by Point 1 in Proposition 2.3 with $X := C$. This proves that the function f is strictly convex.

Now, since $\text{Epi}(f)$ is strictly convex and non-empty, we have $\text{ri}(\text{Epi}(f)) \neq \emptyset$ by Proposition 2.9. On the other hand, since we have $\text{Aff}(\text{Epi}(f)) = \text{Aff}(C) \times \mathbf{R}$ by the first equality in Lemma 3.1, we can apply Proposition 2.1 with $X := \text{Aff}(C)$ and $S := C$, and then obtain that f is bounded above on a subset of C which is non-empty and open in $\text{Aff}(C)$. But this implies that the function f is continuous by Proposition 21 in [2, Chapitre II, page 20].

Finally, given $z_0 \in \text{rb}(C)$, we have $(\{z_0\} \times \mathbf{R}) \cap \text{rc}(\text{Epi}(f)) = \emptyset$ by using the implication Point 1 \implies Point 3 in Lemma 3.3 with $S := C$ and $z_0 \notin \text{ri}(C)$. Hence, we get $f(z) \longrightarrow +\infty$ as $z \longrightarrow z_0$ by Proposition 2.5. \square

Proof of Proposition 1.2. * (\implies) Assume that $\text{Epi}(f)$ is strictly convex.

Since $\text{Epi}(f)$ is not empty, the same holds for $\widehat{\text{Epi}(f)}$ by Proposition 2.9.

* (\impliedby) Assume that $\widehat{\text{Epi}(f)}$ is not empty.

By Proposition 2.1 with $X := V$ and $S := V$, we get that f is bounded from above on a non-empty open subset of V , and hence it is continuous by Proposition 21 in [2, Chapitre II, page 20]. Therefore, the Main Theorem with $C := V$ implies that $\text{Epi}(f)$ is strictly convex. \square

Proof of Proposition 1.4. Point 1 \implies Point 2. Assume that Point 1 is satisfied. Given an affine subspace G of V , the inclusion

$$\text{Epi}(f|_{C \cap G}) = \text{Epi}(f) \cap (G \times \mathbf{R}) \subseteq \text{Epi}(f)$$

yields

$$\begin{aligned} \text{rc}(\text{Epi}(f|_{C \cap G})) &= \overline{\text{Epi}(f|_{C \cap G})} \cap (\text{Aff}(C \cap G) \times \mathbf{R}) \\ &\subseteq \overline{\text{Epi}(f|_{C \cap G})} \cap (\text{Aff}(C) \times \mathbf{R}) \\ &\subseteq \overline{\text{Epi}(f)} \cap (\text{Aff}(C) \times \mathbf{R}) = \text{rc}(\text{Epi}(f)). \end{aligned}$$

Therefore, for any two distinct points (x, r) and (y, s) in $\text{rc}(\text{Epi}(f|_{C \cap G}))$, we have $(x, r), (y, s) \in \text{rc}(\text{Epi}(f))$, which implies that each $(z, t) \in](x, r), (y, s)[$ is in $\text{ri}(\text{Epi}(f))$ since $\text{Epi}(f)$ is strictly convex.

So, there exist a real number $\varepsilon > 0$ and a neighborhood Ω of z in V such that the set $(\Omega \cap \text{Aff}(C)) \times (t - \varepsilon, t + \varepsilon)$ is included in $\text{Epi}(f)$, from which we deduce

$$\begin{aligned} (\Omega \cap \text{Aff}(C \cap G)) \times (t - \varepsilon, t + \varepsilon) &\subseteq (\Omega \cap \text{Aff}(C) \cap G) \times (t - \varepsilon, t + \varepsilon) \\ &= [(\Omega \cap \text{Aff}(C)) \times (t - \varepsilon, t + \varepsilon)] \cap (G \times \mathbf{R}) \\ &\subseteq \text{Epi}(f) \cap (G \times \mathbf{R}) = \text{Epi}(f|_{C \cap G}) \end{aligned}$$

since one has $\text{Aff}(C \cap G) \subseteq \text{Aff}(C) \cap G$ and $\text{Epi}(f) \cap (G \times \mathbf{R}) = \text{Epi}(f|_{C \cap G})$, proving that (z, t) is in $\text{rc}(\text{Epi}(f|_{C \cap G}))$.

This shows that $\text{Epi}(f|_{C \cap G})$ is strictly convex.

Point 2 \implies Point 1. This is clear.

Point 2 \implies Point 3. This is straightforward. \square

Before proving Proposition 1.5, we need two key lemmas.

LEMMA 3.4. *Given a convex subset C of a topological real vector space V and a straight line L in V , we have the implication*

$$\left(\begin{array}{l} C \text{ is open in } \text{Aff}(C) \text{ ,} \\ C \cap L \neq \emptyset \text{ ,} \\ \text{the subspace topology on } L \text{ is Hausdorff} \end{array} \right) \implies \text{rb}(C \cap L) = \text{rb}(C) \cap L .$$

Proof. Assume the left-hand side of \implies holds.

Using Point 3.b in Proposition 2.7 with $S := C$ and $W := L$, we already have the inclusion $\text{rb}(C \cap L) \subseteq \text{rb}(C) \cap L$.

Now, fix a point $b \in C \cap L \subseteq C = \text{ri}(C)$ and let a be an arbitrary point in $\text{rb}(C) \cap L$.

Since we have $a \in \text{rc}(C)$, Proposition 16 in [2, Chapitre II, page 15] implies $]a, b[\subseteq \text{ri}(C) = C$, which yields $]a, b[\subseteq C \cap L$ since a and b both lie in the convex set L .

The points a and b being distinct, we obtain $L = \text{Aff}(]a, b[) \subseteq \text{Aff}(C \cap L) \subseteq \text{Aff}(L) = L$, and hence $\text{Aff}(C \cap L) = L$.

Therefore, since we have $L = \text{Aff}(C \cap L) \subseteq \text{Aff}(C)$ and since C is open in $\text{Aff}(C)$, we then deduce that $C \cap L$ is open in L , which is equivalent to saying that $C \cap L$ is open in $\text{Aff}(C \cap L)$ by using again $\text{Aff}(C \cap L) = L$.

Conclusion: we get $\text{rb}(C \cap L) = \text{rc}(C \cap L) \setminus (C \cap L)$.

On the other hand, the inclusion $]a, b[\subseteq C \cap L$ established above yields $\text{rc}(]a, b[) \subseteq \text{rc}(C \cap L)$ by Point 1 in Proposition 2.7, which writes $]a, b[\subseteq \text{rc}(C \cap L)$ since we have $\text{Aff}(]a, b[) = L$ together with $(0, 1) =]0, 1[$ and since the map $\gamma : \mathbf{R} \rightarrow L$ defined by $\gamma(t) := a + t(b - a)$ is a homeomorphism as a consequence of Theorem 2 in [2, Chapitre I, page 14] and the fact that L is a finite dimensional affine space whose topology is Hausdorff.

So, we obtain in particular $a \in \text{rc}(C \cap L)$, and hence

$$a \in \text{rb}(C \cap L) = \text{rc}(C \cap L) \setminus (C \cap L)$$

since one has $a \in V \setminus C \subseteq V \setminus (C \cap L)$. \square

LEMMA 3.5. *Given a convex subset C of \mathbf{R}^n , a convex function $f : C \rightarrow \mathbf{R}$, and a point $a \in \text{rb}(C) \subseteq \bar{C}$, we have the equivalence*

$$\lim_{x \rightarrow a} f(x) = +\infty \iff$$

for any straight line L in \mathbf{R}^n passing through a , $\lim_{x \rightarrow a} f(x) = +\infty$ with $x \in C \cap L$.

Proof. * (\implies) This implication is obvious.

* (\impliedby) Assume that we do not have $f(x) \longrightarrow +\infty$ as $x \longrightarrow a$.

Since we can write $\text{rc}(C) = \text{rc}(\text{ri}(C))$ by Corollary 1 in [2, Chapitre II, bottom of page 15] and since we have $a \in \text{rc}(C)$, this means that there exist a real number $M > 0$ and a sequence $(x_k)_{k \geq 0}$ in $\text{ri}(C)$ that converges to a with $f(x_k) \leq M$ for every $k \in \mathbf{N}$.

Therefore, if we consider the set $X := \{x_k \mid k \in \mathbf{N}\}$, one obtains $f(x) \leq M$ for any $x \in \text{Conv}(X)$ according to Point 4 in Remark 2.6.

Now, noticing that a lies in \bar{X} , we get $a \in \text{rc}(\text{Conv}(X)) = \overline{\text{Conv}(X)} \cap \text{Aff}(X)$ since we have $\bar{X} \subseteq \overline{\text{Conv}(X)}$ and since $\text{Aff}(X)$ is closed in \mathbf{R}^n .

On the other hand, since $\text{Conv}(X)$ is not empty, the same holds for $\text{ri}(\text{Conv}(X))$, which insures the existence of a point $b \in \text{ri}(\text{Conv}(X))$.

Proposition 16 in [2, Chapitre II, page 15] then implies $]a, b[\subseteq \text{ri}(\text{Conv}(X))$.

Moreover, since $\text{ri}(C)$ is convex by Corollary 1 in [2, Chapitre II, bottom of page 15], we have $\text{ri}(\text{Conv}(X)) \subseteq \text{Conv}(X) \subseteq \text{Conv}(\text{ri}(C)) = \text{ri}(C)$, and hence a does not belong to $\text{ri}(\text{Conv}(X))$ since we have $a \notin \text{ri}(C)$, which yields $b \neq a$.

Finally, if L denotes the straight line in \mathbf{R}^n passing through a and b , we do not have $f(x) \longrightarrow +\infty$ as $x \longrightarrow a$ with $x \in C \cap L$ since the inequality $f(x) \leq M$ holds for any $x \in]a, b[\subseteq \text{ri}(\text{Conv}(X)) \cap L \subseteq \text{ri}(C) \cap L \subseteq C \cap L$. \square

Proof of Proposition 1.5. Since the equivalence is obvious when C is empty or reduced to a single point, we may assume that C has at least two distinct points.

Point 1 \implies Point 2. This implication has already been proved in Proposition 1.4.

Point 2 \implies Point 1. Assume that Point 2 is satisfied.

First of all, the intersection of C with any straight line L in \mathbf{R}^n is convex by applying the converse part of the Main Theorem to the function $f_{|_{C \cap L}}$. Hence, C is convex.

Let us then show that C is open in $\text{Aff}(C)$.

Fix a point $x_0 \in C$, and consider the subset $G := \{x - x_0 \mid x \in C\}$ of V .

Then the vector subspace $W := \{v - x_0 \mid v \in \text{Aff}(C)\}$ of V is generated by G since C is a generating set of the affine space $\text{Aff}(C)$.

Hence, there exists a subset B of G which is a basis of W .

Denoting by $d \in \{1, \dots, n\}$ the dimension of the affine subspace $\text{Aff}(C)$ of \mathbf{R}^n , we have $\dim(W) = d$, and hence there are points $x_1, \dots, x_d \in C$ such that we can write $B = \{x_i - x_0 \mid 1 \leq i \leq d\}$.

Now, for each index $i \in \{1, \dots, d\}$, if L_i denotes the straight line in \mathbf{R}^n passing through the points x_0 and x_i , then the strict convexity of $\text{Epi}(f_{|_{C \cap L_i}})$ implies that $C \cap L_i$ is open in $\text{Aff}(C \cap L_i) = L_i = \text{Aff}(\{x_0, x_i\})$ by the implication Point 1 \implies Point 4 in Lemma 3.3 with $S := C \cap L_i$, which insures the existence of a real number $r_i > 0$ such that the vector $v_i := r_i(x_i - x_0)$ satisfies

$$\{x_0 - v_i, x_0 + v_i\} \subseteq C \cap L_i \subseteq C.$$

Therefore, the convex hull U of $\bigcup_{i=1}^d \{x_0 - v_i, x_0 + v_i\}$ lies in the convex set C .

Since the map $f : \mathbf{R}^d \rightarrow \mathbf{R}^n$ defined by $f(\lambda_1, \dots, \lambda_d) := \sum_{i=1}^d \lambda_i v_i$ is linear and sends the canonical ordered basis (e_1, \dots, e_d) of \mathbf{R}^d to the family (v_1, \dots, v_d) of \mathbf{R}^n , we can write $U = x_0 + f(\Omega)$, where $\Omega := \{(\lambda_1, \dots, \lambda_d) \in \mathbf{R}^d \mid |\lambda_1| + \dots + |\lambda_d| \leq 1\}$ is the convex hull of $\bigcup_{i=1}^d \{-e_i, e_i\}$.

But (v_1, \dots, v_d) is a basis of W since B is, which implies that f satisfies $\text{Im}(f) = W$ and is a linear isomorphism onto its image. Therefore, $x_0 + f$ is a homeomorphism from \mathbf{R}^d (endowed with its usual topology) onto $x_0 + W = \text{Aff}(C)$.

As a consequence, we then get that U is a neighbourhood of x_0 in $\text{Aff}(C)$ since Ω is a neighborhood of the origin in \mathbf{R}^d , and hence C is itself a neighborhood of x_0 in $\text{Aff}(C)$.

On the other hand, the function f is strictly convex since its restriction to any straight line in \mathbf{R}^n is strictly convex and since strict convexity is an affine notion.

Moreover, the convexity of f on the open convex subset C of the finite dimensional affine space $\text{Aff}(C)$ equipped with the topology induced from that of \mathbf{R}^n implies that the function f is continuous according to the corollary given in [2, Chapitre II, page 20].

Finally, given any point $a \in \text{rb}(C)$ and any straight line L in \mathbf{R}^n passing through a , we have either $C \cap L = \emptyset$, which obviously yields $\lim_{x \rightarrow a} f(x) = +\infty$ with $x \in C \cap L$, or $C \cap L \neq \emptyset$, which first implies $a \in \text{rb}(C \cap L)$ by Lemma 3.4, and then $\lim_{x \rightarrow a} f(x) = +\infty$ with $x \in C \cap L$ by the Main Theorem since $\text{Epi}(f|_{C \cap L})$ is strictly convex.

In both cases, we obtain $\lim_{x \rightarrow a} f(x) = +\infty$ by Lemma 3.5. \square

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