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# HIGHER–ORDER QUASIMONOTONICITY AND INTEGRAL INEQUALITIES

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*Abstract.* The classical Hermite–Hadamard inequality is not merely a consequence of convexity, but also characterizes convexity. Such inequalities hold in the case of higher-order monotonicity in sense of Hopf and Popoviciu with the same characteristic feature. The aim of this note is to extend these results, when the underlying monotonicity is induced by so-called quasipolynomial Chebyshev systems.

## 1. Introduction

The motivation of our investigations is the well-known inequality of Hermite [9] and Hadamard [7]. (For historical comments, see the note of Mitrinović and Lacković [13].) A converse of their result is also known, and can be found in the books of Hardy, Littlewood and Pólya [8, p. 98], of Kuczma [12, Exercise 8. p. 205], of Niculescu and Persson [15, pp. 50–51] or of Roberts and Varberg [17, Problem Q. p. 15]. An excellent essay on the topic was presented by Niculescu and Persson [14].

The notion of classical convexity can be extended via the next concept. Let $I$ be a real interval. A continuous mapping $\boldsymbol{\omega}\colon I\to\mathbb{R}^n$ is called a *Chebyshev system* if $\det\big(\boldsymbol{\omega}(t_1)\dots\boldsymbol{\omega}(t_n)\big)>0$ remains true for any elements $t_1<\dots<t_n$ of the domain.

Having a Chebyshev system $\boldsymbol{\omega}$ over $I$, a function $f\colon I\to\mathbb{R}$ is termed *monotone with respect to* $\boldsymbol{\omega}$ (or briefly: $\boldsymbol{\omega}$-monotone), if, for all elements $t_0\leqslant\dots\leqslant t_n$ of $I$, the next inequality holds:

$$\det\begin{pmatrix}\boldsymbol{\omega}(t_0) & \dots & \boldsymbol{\omega}(t_n)\\ f(t_0) & \dots & f(t_n)\end{pmatrix}\geqslant 0.$$

If $f$ satisfies the aboves with equality, then it is called $\boldsymbol{\omega}$*-affine*. One of the most important example for a Chebyshev system is the *polynomial system* $\boldsymbol{\pi}$, defined by $\boldsymbol{\pi}(t)=(1,t,\dots,t^{n-1})$. In this case, $\boldsymbol{\omega}$-monotonicity is also called as $n$*th order monotonicity*. The particular settings of $n=1$ and $n=2$ correspond to the usual notions of monotonicity and convexity, respectively.

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The book of Karlin and Studden [11] gives an excellent overview of the theory of Chebyshev systems. The polynomial system seems to appear first in the dissertation of Hopf [10], and was studied intensively by Popoviciu [16].

Hermite–Hadamard-type inequalities (that is, estimations for the integral average involving the values of a function) can be achieved in the case of higher-order monotonicity [3] or, in more general, when the monotonicity notion is induced by an arbitrary Chebyshev system [4]. For details, we refer to the papers [1] and [2]. Therefore the question arises, whether the obtained inequalities *characterize* the underlying monotonicity notion or not. In the particular settings when the monotonicity is induced by a two dimensional Chebyshev system [5] or a polynomial one [6], the answer is *positive*. However, the general case still remains an open problem.

The aim of this paper is to make further steps towards this problem, extending the results of [6] for a wider class of Chebyshev systems then the polynomial. These Chebyshev systems are obtained by transforming the polynomial system with a strictly monotone increasing and continuous function. The characteristic inequalities, in this so-called higher-order quasimonotone case, involve certain weighted quasiarithmetic means and Riemann–Stieltjes integral means.

## 2. The motivating results

Recalling the corresponding part of [2], first we subsume the most important properties of some distinguished orthogonal polynomial system. For further details, see [19]. Consider the polynomials $G_m$, $L_{m-1}$ and $R_m$, named after Gauss, Lobatto and Radau, defined by the next formula:

$$G_m(t) := \begin{vmatrix} 1 & 1 & \cdots & \frac{1}{m} \\ t & \frac{1}{2} & \cdots & \frac{1}{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ t^m & \frac{1}{m+1} & \cdots & \frac{1}{2m} \end{vmatrix} \tag{1}$$

$$L_{m-1}(t) := \begin{vmatrix} 1 & \frac{1}{2\cdot 3} & \cdots & \frac{1}{m(m+1)} \\ t & \frac{1}{3\cdot 4} & \cdots & \frac{1}{(m+1)(m+2)} \\ \vdots & \vdots & \ddots & \vdots \\ t^{m-1} & \frac{1}{(m+1)(m+2)} & \cdots & \frac{1}{(2m-1)2m} \end{vmatrix} \tag{2}$$

$$R_m(t) := \begin{vmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{m+1} \\ t & \frac{1}{3} & \cdots & \frac{1}{m+2} \\ \vdots & \vdots & \ddots & \vdots \\ t^m & \frac{1}{m+2} & \cdots & \frac{1}{2m+1} \end{vmatrix}. \tag{3}$$

It is well-known, that each sequence $(G_m)$, $(L_{m-1})$, and $(R_m)$ is an orthogonal polynomial system on $[0,1]$ with respect to a suitable weight function. On this right, $G_m$, $L_{m-1}$, and $R_m$ has, in turn, $m$, $m-1$, and $m$ pairwise distinct zeros in $]0,1[$.

Denote the zeros of $G_m$ by $\nu_1,\dots,\nu_m$ and define the coefficient $\alpha_k$ via the representation

$$\alpha_k := \int_0^1 \frac{G_m(t)}{(t-\nu_k)G_m'(\nu_k)}dt. \tag{4}$$

Similarly, take the zeros $\mu_1,\dots,\mu_{m-1}$ of the polynomial $L_{m-1}$ and define the coefficients $\beta_k$ by

$$\begin{aligned}
\beta_0 &:= \frac{1}{L_{m-1}^2(0)}\int_0^1 (1-t)L_{m-1}^2(t)dt,\\
\beta_k &:= \frac{1}{(1-\mu_k)\mu_k}\int_0^1 \frac{t(1-t)L_{m-1}(t)}{(t-\mu_k)L_{m-1}'(\mu_k)}dt,\\
\beta_m &:= \frac{1}{L_{m-1}^2(1)}\int_0^1 tL_{m-1}^2(t)dt.
\end{aligned} \tag{5}$$

Finally, using now the zeros $\lambda_1,\dots,\lambda_m$ of $R_m$, introduce the coefficients $\gamma_k$ in the following way:

$$\begin{aligned}
\gamma_0 &:= \frac{1}{R_m^2(0)}\int_0^1 R_m^2(t)dt,\\
\gamma_k &:= \frac{1}{\lambda_k}\int_0^1 \frac{tR_m(t)}{(t-\lambda_k)R_m'(\lambda_k)}dt.
\end{aligned} \tag{6}$$

Fix an open interval $I$ and set $T = \{(x,y)\in I^2 \mid x<y\}$, the upper open triangle of $I^2$. Take the zeros of the orthogonal polynomials (1), (2) and (3), further the coefficients (4), (5) and (6). Define the mappings $\mathscr{G}_m, \mathscr{L}_{m-1}, \mathscr{R}_{m;l}, \mathscr{R}_{m;r} : \mathscr{C}(I,\mathbb{R}) \to \mathscr{C}(T,\mathbb{R})$ by

$$\begin{aligned}
\mathscr{G}_m(f)(x,y) &:= \sum_{k=1}^{m} \alpha_k f\big((1-\nu_k)x+\nu_k y\big),\\
\mathscr{L}_{m-1}(f)(x,y) &:= \beta_0 f(x) + \sum_{k=1}^{m-1} \beta_k f\big((1-\mu_k)x+\mu_k y\big) + \beta_m f(y),\\
\mathscr{R}_{m;l}(f)(x,y) &:= \gamma_0 f(x) + \sum_{k=1}^{m} \gamma_k f\big((1-\lambda_k)x+\lambda_k y\big),\\
\mathscr{R}_{m;r}(f)(x,y) &:= \sum_{k=1}^{m} \gamma_k f\big(\lambda_k x+(1-\lambda_k)y\big) + \gamma_0 f(y).
\end{aligned}$$

These mappings are bounded linear ones, and are stemming from the quadrature rules of Gauss, Lobatto and Radau. Using the integral average as a mapping $\mathscr{A} : \mathscr{C}(I,\mathbb{R}) \to \mathscr{C}(T,\mathbb{R})$ turns out to be quite convenient:

$$\mathscr{A}(f)(x,y) := \frac{1}{y-x}\int_x^y f(t)dt.$$

Now we can recall, in a slightly modified but still equivalent form, the main results of [6]. The results are split into two theorems, according to the parity of the order of the underlying polynomial system.

THEOREM A. *If $I$ is an open interval and $f\colon I\to\mathbb{R}$ is $2m$-monotone, then it fulfills the inequalities*

$$\mathscr{G}_m(f)\leqslant\mathscr{A}(f)\leqslant\mathscr{L}_{m-1}(f)$$

*on $T$. Conversely, if a continuous function $f\colon I\to\mathbb{R}$ satisfies any part of these inequalities on $T$, then it is $2m$-monotone. For continuous functions, equality occurs exactly when the function is $2m$-affine.*

THEOREM B. *If $I$ is an open interval and $f\colon I\to\mathbb{R}$ is $(2m+1)$-monotone, then it fulfills the inequalities*

$$\mathscr{R}_{m;l}(f)\leqslant\mathscr{A}(f)\leqslant\mathscr{R}_{m;r}(f)$$

*on $T$. Conversely, if a continuous function $f\colon I\to\mathbb{R}$ satisfies any part of these inequalities on $T$, then it is $(2m+1)$-monotone. For continuous functions, equality occurs exactly when the function is $(2m+1)$-affine.*

Observe that Theorem A reduces to the Hermite–Hadamard inequality in the particular setting $m=1$. Indeed, simple calculations yield $G_1(t)=t-1/2$ and $L_0(t)=1$. Hence we have $\alpha_1=1$ while $\beta_0=\beta_1=1/2$. Our aim is to extend Theorem A and Theorem B for Chebyshev systems which are obtained as strictly increasing and continuous transformations of the polynomial system.

## 3. The main results

To prove the main results, we shall need two lemmas. The first one presents a change of variables for the the Riemann–Stieltjes integral. Although it is valid in a more general form (see the book of Rudin [18, Theorem 6.19.]), we present the proof of that version which is convenient for us. The second lemma makes possible to modify Chebyshev systems in order to get adequate new ones.

LEMMA 1. *If $I$ and $J$ are real intervals, $\varphi\colon I\to J$ is a continuous, strictly increasing function, and $f\colon I\to\mathbb{R}$ is continuous, then*

$$\int_I f\,d\varphi=\int_{\varphi(I)} f\circ\varphi^{-1}.$$

*Proof.* Note first that the integrals above do exist. Indeed, the integrand of the right-hand side is continuous. On the left-hand side, $f$ is continuous and, in particular, $\varphi$ is a function of bounded variation. Hence the Riemann–Stieltjes integral exists.

Let $\tau = \{[t_{k-1}, t_k] \mid k = 1, \dots, n\}$ be a partition of $I$ and let $\xi = \{\xi_k \mid k = 1, \dots n\}$ be a selection of $\tau$. Then, $\rho = \{[s_{k-1}, s_k] \mid k = 1, \dots, n\}$ is a partition of $J$ and $\eta = \{\eta_k \mid k = 1, \dots n\}$ is a selection of $\rho$, where $s_k = \varphi(t_k)$ and $\eta_k = \varphi(\xi_k)$. Then,

$$\begin{aligned} S(f, \varphi, \tau, \xi) &= \sum_{k=1}^{n} f(\xi_k)(\varphi(t_k) - \varphi(t_{k-1})) \\ &= \sum_{k=1}^{n} f \circ \varphi^{-1}(\eta_k)(s_k - s_{k-1}) = S(f \circ \varphi^{-1}, \rho, \eta). \end{aligned}$$

Applying normal partition sequences, a limiting process completes the proof. $\square$

LEMMA 2. *Assume that $I$ and $J$ are real intervals, $\varphi\colon I \to J$ is a continuous, strictly increasing function, and $\boldsymbol{\omega}\colon J \to \mathbb{R}$ is a continuous mapping. Then $\boldsymbol{\omega}$ is Chebyshev system on $J$ if and only if the mapping $\boldsymbol{\omega}_\varphi = \boldsymbol{\omega} \circ \varphi$ is a Chebyshev system on $I$. Moreover, $f\colon I \to \mathbb{R}$ is $\boldsymbol{\omega}_\varphi$-monotone if and only if the function $g := f \circ \varphi^{-1}$ is $\boldsymbol{\omega}$-monotone on $J$.*

*Proof.* Clearly, $\boldsymbol{\omega}_\varphi$ is a continuous mapping. Let $t_1 < \ldots < t_n$ be fixed elements of $I$, and define $x_k = \varphi(t_k)$ for all $k = 1, \dots, n$. Then, the strict monotonicity of $\varphi$ ensures that $x_1 < \ldots < x_n$. Therefore,

$$\det\big(\boldsymbol{\omega}(x_1) \ldots \boldsymbol{\omega}(x_n)\big) = \det\big(\boldsymbol{\omega}_\varphi(t_1) \ldots \boldsymbol{\omega}_\varphi(t_n)\big).$$

This identity proves the first statement. For the second one, fix the elements $t_0 \leqslant \cdots \leqslant t_n$ of $I$. Define again $x_k = \varphi(t_k)$ for all $k = 0, \dots, n$. Then, $x_0 \leqslant \ldots \leqslant x_n$ and $g(x_k) = f(t_k)$ hold. Therefore,

$$\det\begin{pmatrix} \boldsymbol{\omega}(x_0) & \ldots & \boldsymbol{\omega}(x_n) \\ g(x_0) & \ldots & g(x_n) \end{pmatrix} = \det\begin{pmatrix} \boldsymbol{\omega}_\varphi(t_0) & \ldots & \boldsymbol{\omega}_\varphi(t_n) \\ f(t_0) & \ldots & f(t_n) \end{pmatrix}.$$

This identity completes the proof. $\square$

This lemma enables us to extend the notion of higher-order monotonicity. Let $I$ and $J$ be real intervals, and consider the polynomial system $\boldsymbol{\pi}\colon J \to \mathbb{R}^n$. If $\varphi\colon I \to J$ is a strictly increasing, continuous function, then the mapping $\boldsymbol{\pi}_\varphi$ is a Chebyshev system over $I$. A function on $I$ is termed to be *$\varphi$-quasimonotone/$\varphi$-quasiaffine of order $n$*, if it is monotone/affine with respect to $\boldsymbol{\pi}_\varphi$. On the other hand, the function $\varphi$ and a parameter $\lambda \in [0,1]$ generate a weighted arithmetic mean:

$$M_{\varphi,\lambda}(x,y) := \varphi^{-1}\big((1-\lambda)\varphi(x) + \lambda\varphi(y)\big) \qquad (x, y \in I).$$

Using the zeros of the polynomials (1), (2) and (3) as weights, and the coefficients (4), (5), (6), define the mappings $\mathscr{G}_{\varphi;m}, \mathscr{L}_{\varphi;m-1}, \mathscr{R}_{\varphi;m;l}, \mathscr{R}_{\varphi;m;r}, \mathscr{A}_\varphi \colon \mathscr{C}(I,\mathbb{R}) \to \mathscr{C}(T,\mathbb{R})$ as follows (here we keep the notation $T$ for the upper open triangle).

$$\begin{aligned}
\mathscr{G}_{\varphi;m}(f)(x,y) &:= \sum_{k=1}^{m} \alpha_k f\big(M_{\varphi,\nu_k}(x,y)\big),\\
\mathscr{L}_{\varphi;m-1}(f)(x,y) &:= \beta_0 f(x) + \sum_{k=1}^{m-1} \beta_k f\big(M_{\varphi,\mu_k}(x,y)\big) + \beta_m f(y),\\
\mathscr{R}_{\varphi;m;l}(f)(x,y) &:= \gamma_0 f(x) + \sum_{k=1}^{m} \gamma_k f\big(M_{\varphi,\lambda_k}(x,y)\big),\\
\mathscr{R}_{\varphi;m;r}(f)(x,y) &:= \sum_{k=1}^{m} \gamma_k f\big(M_{\varphi,\lambda_k}(y,x)\big) + \gamma_0 f(y).
\end{aligned}$$

and

$$\mathscr{A}_\varphi(f)(x,y) := \frac{1}{\varphi(y)-\varphi(x)} \int_x^y f(t)d\varphi(t).$$

Distinguishing the parity of the order of the quasimonotonicity, we can present our main results. We shall focus only on the proof of the first theorem.

THEOREM 1. *If $I$ and $J$ are open intervals, $\varphi\colon I \to J$ is strictly increasing and continuous, and $f\colon I \to \mathbb{R}$ is $\varphi$-quasimonotone of order $2m$, then it fulfills the inequalities*

$$\mathscr{G}_{\varphi;m}(f) \leqslant \mathscr{A}_\varphi(f) \leqslant \mathscr{L}_{\varphi;m-1}(f)$$

*on $T$. Conversely, if a continuous function $f\colon I \to \mathbb{R}$ satisfies any part of these inequalities on $T$, then it is $\varphi$-quasimonotone of order $2m$. For continuous functions, equality occurs exactly when the function $\varphi$-quasiaffine of order $2m$.*

*Proof.* For arbitrary $x,y \in I$, consider the elements $u,v \in J$ given by $\varphi(x) = u$ and $\varphi(y) = v$. Furthermore, define $g\colon J \to \mathbb{R}$ by the formula $g = f \circ \varphi^{-1}$. Then,

$$\begin{aligned}
\mathscr{G}_{\varphi;m}(f)(x,y) &= \sum_{k=1}^{m} \alpha_k f\Big(\varphi^{-1}\big((1-\nu_k)\varphi(x) + \nu_k\varphi(y)\big)\Big)\\
&= \sum_{k=1}^{m} \alpha_k g\big((1-\nu_k)u + \nu_k v\big) = \mathscr{G}_m(g)(u,v).
\end{aligned}$$

Similar calculations provide that $\mathscr{L}_{\varphi;m-1}(f)(x,y) = \mathscr{L}_{m-1}(g)(u,v)$ also holds. The change of variable low of Lemma 1 yields

$$\mathscr{A}_\varphi(f)(x,y) = \frac{1}{\varphi(y)-\varphi(x)} \int_x^y f d\varphi = \frac{1}{\varphi(y)-\varphi(x)} \int_{\varphi(x)}^{\varphi(y)} f\circ\varphi^{-1} = \mathscr{A}(g)(u,v).$$

Thus, we have

$$\mathscr{G}_{\varphi;m}(f) = \mathscr{G}_m(g)\circ\Phi, \qquad \mathscr{A}_\varphi(f) = \mathscr{A}(g)\circ\Phi, \qquad \mathscr{L}_{\varphi;m-1}(f) = \mathscr{L}_{m-1}(g)\circ\Phi$$

where $\Phi\colon I^2 \to J^2$ stands for the coordinate-wise substitution $\Phi(x,y) = \big(\varphi(x),\varphi(y)\big)$. These identities, taking into consideration the statements of Theorem A and Lemma 2, complete the proof. $\square$

THEOREM 2. *If $I$ and $J$ are open intervals, $\varphi\colon I \to J$ is strictly increasing and continuous, and $f\colon I \to \mathbb{R}$ is $\varphi$-quasimonotone of order $(2m+1)$, then it fulfills the inequalities*

$$\mathscr{R}_{\varphi;m;l}(f) \leqslant \mathscr{A}_{\varphi}(f) \leqslant \mathscr{R}_{\varphi;m;r}(f)$$

*on $T$. Conversely, if a continuous function $f\colon I \to \mathbb{R}$ satisfies any part of these inequalities on $T$, then it is $\varphi$-quasimonotone of order $(2m+1)$. For continuous functions, equality occurs if and only if the function $\varphi$-quasiaffine of order $(2m+1)$.*

The main results formally covers the motivating ones, as well. However, they are not proper generalization of Theorem A and Theorem B, as their proofs depend on these motivating theorems.

The prototype of our investigations was originally the exponential system defined by $\boldsymbol{\omega}(t) = (1, \mathrm{e}^t, \dots, \mathrm{e}^{(n-1)t})$. Note that many Chebyshev systems can be obtained in an analogous way. For example, the system $\boldsymbol{\omega}(t) = (1, \sinh t, \dots, \sinh(n-1)t)$ reflects the same feature. Therefore, the next conspicuous question arises: Does there exist other quasipolynomial system, besides the exponential one, sharing a similar property? To give the astonishing answer, we need the next concept. A sequence of functions $(\psi_k)$ is a *Chebyshev sequence*, if its any finite section $(\psi_1, \dots, \psi_n)$ generates a Chebyshev system.

THEOREM 3. *A function $\psi\colon \mathbb{R}_+ \to \mathbb{R}$ generates a quasipolynomial Chebyshev sequence $\big(\psi(kt)\big)_{k=0}^{\infty}$ if and only if there exists $\alpha > 0$ such that $\psi(t) = \mathrm{e}^{\alpha t}$.*

*Proof.* Assume that $\big(\psi(kt)\big)_{k=0}^{\infty}$ is $\varphi$-quasipolynomial where $\varphi\colon \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing, continuous function. Then, for all $t \geqslant 0$ and $k \in \mathbb{N} \cup \{0\}$, we have

$$\psi(kt) = \varphi^k(t).$$

The particular choice $k=1$ shows that $\psi = \varphi$ and hence $\psi$ is continuous and strictly increasing. The case $k=2$ yields $\psi(2t) = \varphi^2(t) \geqslant 0$. That is, the range of $\psi$ is contained by the nonnegative reals. On the other hand, $\psi$ cannot have a zero: In the opposite case, it would also take negative values by the strict increasing property. Hence we can write $\psi$ instead of $\varphi$ in the equation above, and then we can take the logarithm of both sides. Then the equation obtained shows that $h = \log \circ \psi$ is homogeneous for all $k \in \mathbb{N}$. Using standard arguments, one can easily conclude that $h$ is homogeneous for all positive rationals. Finally, applying the continuity of $h$,

$$h(ct) = ch(t)$$

follows whenever $c > 0$ and $t \geqslant 0$. Therefore, $h(t) = th(1)$ and hence $\psi(t) = \mathrm{e}^{\alpha t}$ follows, where $\alpha = h(1)$. Taking into account the addition rules of the exponential function, the converse statement is obvious. □

## 4. Examples and remarks

Let us present here two specific cases of the main results. Firstly, let $\alpha > 0$ be a positive parameter and consider the function $\varphi(t) := \mathrm{e}^{\alpha t}$. Then, $\varphi\colon \mathbb{R} \to \mathbb{R}$ is strictly monotone increasing and continuous. Therefore, the system $\boldsymbol{\pi}_\varphi$ given in Lemma 2 is a Chebyshev system, indeed. With this particular setting, Theorem 1 and Theorem 2 reduce to the corollaries below. The quasiarithmetic mean induced by $\varphi$ can be easily expressed in the explicit forms involved. The proofs are omitted.

COROLLARY 1. *If $\alpha > 0$, then $\boldsymbol{\omega}(t) = (1, \mathrm{e}^{\alpha t}, \dots, \mathrm{e}^{(2m-1)\alpha t})$ is a Chebyshev system on any subinterval $I$ of $\mathbb{R}$. Furthermore, the next statements are equivalent:*

*(i)* $f\colon I \to \mathbb{R}$ *quasimonotone with respect to* $\boldsymbol{\omega}$;

*(ii)* $f$ *is continuous and, for all* $x, y \in I$ *with* $x < y$,

$$\sum_{k=1}^{m} \alpha_k f\big(\log \sqrt[\alpha]{(1-\nu_k)\,\mathrm{e}^{\alpha x} + \nu_k\,\mathrm{e}^{\alpha y}}\big) \leqslant \frac{1}{\mathrm{e}^{\alpha y} - \mathrm{e}^{\alpha x}} \int_x^y f(t)\, d\,\mathrm{e}^{\alpha t};$$

*(iii)* $f$ *is continuous and, for all* $x, y \in I$ *with* $x < y$,

$$\frac{1}{\mathrm{e}^{\alpha y} - \mathrm{e}^{\alpha x}} \int_x^y f(t)\, d\,\mathrm{e}^{\alpha t} \leqslant$$

$$\beta_0 f(x) + \sum_{k=1}^{m-1} \beta_k f\big(\log \sqrt[\alpha]{\mu_k\,\mathrm{e}^{\alpha x} + (1-\mu_k)\,\mathrm{e}^{\alpha y}}\big) + \beta_m f(y);$$

*(iv)* $f$ *is continuous and, for all* $x, y \in I$ *with* $x < y$,

$$\sum_{k=1}^{m} \alpha_k f\big(\log \sqrt[\alpha]{(1-\nu_k)\,\mathrm{e}^{\alpha x} + \nu_k\,\mathrm{e}^{\alpha y}}\big)$$

$$\leqslant \beta_0 f(x) + \sum_{k=1}^{m-1} \beta_k f\big(\log \sqrt[\alpha]{\mu_k\,\mathrm{e}^{\alpha x} + (1-\mu_k)\,\mathrm{e}^{\alpha y}}\big) + \beta_m f(y).$$

*Finally, a continuous function is $\boldsymbol{\omega}$-affine if and only if it fulfills one of the cases with equality.*

COROLLARY 2. *If $\alpha > 0$, then $\boldsymbol{\omega}(t) = (1, \mathrm{e}^{\alpha t}, \dots, \mathrm{e}^{2m\alpha t})$ is a Chebyshev system on any subinterval $I$ of $\mathbb{R}$. Furthermore, the next statements are equivalent:*

*(i)* $f\colon I \to \mathbb{R}$ *quasimonotone with respect to* $\boldsymbol{\omega}$;

*(ii)* $f$ *is continuous and, for all* $x, y \in I$ *with* $x < y$,

$$\gamma_0 f(x) + \sum_{k=1}^{m} \gamma_k f\big(\log \sqrt[\alpha]{(1-\lambda_k)\,\mathrm{e}^{\alpha x} + \lambda_k\,\mathrm{e}^{\alpha y}}\big) \leqslant \frac{1}{\mathrm{e}^{\alpha y} - \mathrm{e}^{\alpha x}} \int_x^y f(t)\, d\,\mathrm{e}^{\alpha t};$$

*(iii)* *$f$ is continuous and, for all $x,y \in I$ with $x<y$,*

$$\frac{1}{e^{\alpha y}-e^{\alpha x}}\int_x^y f(t)d\,e^{\alpha t} \leqslant \sum_{k=1}^{m}\gamma_k f\left(\log\sqrt[\alpha]{\lambda_k e^{\alpha x}+(1-\lambda_k)e^{\alpha y}}\right)+\gamma_0 f(y);$$

*(iv)* *$f$ is continuous and, for all $x,y \in I$ with $x<y$,*

$$\gamma_0 f(x)+\sum_{k=1}^{m}\gamma_k f\left(\log\sqrt[\alpha]{(1-\lambda_k)e^{\alpha x}+\lambda_k e^{\alpha y}}\right)$$
$$\leqslant \sum_{k=1}^{m}\gamma_k f\left(\log\sqrt[\alpha]{\lambda_k e^{\alpha x}+(1-\lambda_k)e^{\alpha y}}\right)+\gamma_0 f(y).$$

*Finally, a continuous function is $\boldsymbol{\omega}$-affine if and only if it fulfills one of the cases with equality.*

The second example is the modified polynomial system. Let $\alpha>0$ be a positive parameter and define the function $\varphi\colon \mathbb{R}\to\mathbb{R}$ by $\varphi(t):=t^{\alpha}$. Then, as direct calculations show, Theorem 1 and Theorem 2 take the next forms:

COROLLARY 3. *If $\alpha>0$, then $\boldsymbol{\omega}(t)=(1,t^{\alpha},\dots,t^{(2m-1)\alpha})$ is a Chebyshev system on any subinterval $I$ of the nonnegative reals. Furthermore, the next statements are equivalent:*

*(i)* *$f\colon I\to\mathbb{R}$ quasimonotone with respect to $\boldsymbol{\omega}$;*

*(ii)* *$f$ is continuous and, for all $x,y \in I$ with $x<y$,*

$$\sum_{k=1}^{m}\alpha_k f\left(\sqrt[\alpha]{(1-\nu_k)x^{\alpha}+\nu_k y^{\alpha}}\right)\leqslant \frac{1}{y^{\alpha}-x^{\alpha}}\int_x^y f(t)dt^{\alpha};$$

*(iii)* *$f$ is continuous and, for all $x,y \in I$ with $x<y$,*

$$\frac{1}{y^{\alpha}-x^{\alpha}}\int_x^y f(t)dt^{\alpha}\leqslant$$

$$\beta_0 f(x)+\sum_{k=1}^{m-1}\beta_k f\left(\sqrt[\alpha]{\mu_k x^{\alpha}+(1-\mu_k)y^{\alpha}}\right)+\beta_m f(y);$$

*(iv)* *$f$ is continuous and, for all $x,y \in I$ with $x<y$,*

$$\sum_{k=1}^{m}\alpha_k f\left(\sqrt[\alpha]{(1-\nu_k)x^{\alpha}+\nu_k y^{\alpha}}\right)$$
$$\leqslant \beta_0 f(x)+\sum_{k=1}^{m-1}\beta_k f\left(\sqrt[\alpha]{\mu_k x^{\alpha}+(1-\mu_k)y^{\alpha}}\right)+\beta_m f(y).$$

*Finally, a continuous function is $\boldsymbol{\omega}$-affine if and only if it fulfills one of the cases with equality.*

COROLLARY 4. *If $\alpha > 0$, then $\boldsymbol{\omega}(t) = (1, t^{\alpha}, \ldots, t^{2m\alpha})$ is a Chebyshev system on any subinterval $I$ of the nonnegative reals. Furthermore, the next statements are equivalent:*

*(i)* $f\colon I \to \mathbb{R}$ *quasimonotone with respect to* $\boldsymbol{\omega}$*;*

*(ii)* $f$ *is continuous and, for all* $x, y \in I$ *with* $x < y$,

$$\gamma_0 f(x) + \sum_{k=1}^{m} \gamma_k f\big(\sqrt[\alpha]{(1-\lambda_k)x^{\alpha} + \lambda_k y^{\alpha}}\big) \leqslant \frac{1}{y^{\alpha} - x^{\alpha}} \int_x^y f(t)\,dt^{\alpha};$$

*(iii)* $f$ *is continuous and, for all* $x, y \in I$ *with* $x < y$,

$$\frac{1}{y^{\alpha} - x^{\alpha}} \int_x^y f(t)\,dt^{\alpha} \leqslant \sum_{k=1}^{m} \gamma_k f\big(\sqrt[\alpha]{\lambda_k x^{\alpha} + (1-\lambda_k) y^{\alpha}}\big) + \gamma_0 f(y);$$

*(iv)* $f$ *is continuous and, for all* $x, y \in I$ *with* $x < y$,

$$\begin{aligned}\gamma_0 f(x) + &\sum_{k=1}^{m} \gamma_k f\big(\sqrt[\alpha]{(1-\lambda_k)x^{\alpha} + \lambda_k y^{\alpha}}\big) \\ &\leqslant \sum_{k=1}^{m} \gamma_k f\big(\sqrt[\alpha]{\lambda_k x^{\alpha} + (1-\lambda_k) y^{\alpha}}\big) + \gamma_0 f(y).\end{aligned}$$

*Finally, a continuous function is $\boldsymbol{\omega}$-affine if and only if it fulfills one of the cases with equality.*

The main results may also have some impact in Numerical Analysis. Now we focus to applications in the field of Functional Equations. We present the proof only of the first Corollary.

COROLLARY 5. *Assume that $I$ is an open interval, and $\varphi\colon I \to \mathbb{R}$ a continuously differentiable function with $\varphi' > 0$. Let further $\nu_1, \ldots, \nu_m$ be the zeros of (1) and let $\alpha_1, \ldots, \alpha_m$ be given by (4). Then a continuous function $f\colon I \to \mathbb{R}$ and a function $F\colon I \to \mathbb{R}$ are the solutions of the functional equation*

$$\big(\varphi(y) - \varphi(x)\big) \sum_{k=1}^{m} \alpha_k f\big(M_{\varphi,\nu_k}(x,y)\big) = F(y) - F(x)$$

*for all elements $x < y$ of $I$ if and only if $f$ is $\varphi$-quasiaffine of order $2m$, and $F$ is the antiderivative of $f \cdot \varphi'$.*

*Proof.* Evidently, $\varphi$ is continuous and strictly monotone. Observe also, that the functional equation above can be written into the equivalent form

$$\frac{\varphi(y)-\varphi(x)}{y-x}\cdot\mathscr{G}_{\varphi;m}(f)(x,y)=\frac{F(y)-F(x)}{y-x}.$$

The limit of the first term in the left-hand side exists if $y\to x$ and is equal to $\varphi'(x)$. On the other hand, $f$ and the quasiarithmetic means $M_{\varphi,\nu_k}$ are continuous, therefore

$$\lim_{y\to x}\mathscr{G}_{\varphi;m}(f)(x,y)=\lim_{y\to x}\sum_{k=1}^{m}\alpha_k f\big(M_{\varphi,\nu_k}(x,y)\big)=\sum_{k=1}^{m}\alpha_k f\big(M_{\varphi,\nu_k}(x,x)\big)=f(x),$$

since the coefficients $\alpha_k$ are convex ones. That is, the limit of the right-hand side also exists, yielding $F'=f\varphi'$. Rearranging the original functional equation, and using the Newton–Leibniz Theorem and the well-known connection between the Riemann and the Riemann–Stieltjes integrals, $\mathscr{G}_{\varphi;m}(f)=\mathscr{A}_{\varphi}(f)$ follows. By Theorem 1, this means that $f$ is $\varphi$-quasiaffine. The converse statement is trivial. □

COROLLARY 6. *Assume that $I$ is an open interval, and $\varphi\colon I\to\mathbb{R}$ a continuously differentiable function with $\varphi'>0$. Let further $\mu_1,\dots,\mu_{m-1}$ be the zeros of (2) and let $\beta_1,\dots,\beta_m$ be given by (5). Then a continuous function $f\colon I\to\mathbb{R}$ and a function $F\colon I\to\mathbb{R}$ are the solutions of the functional equation*

$$\big(\varphi(y)-\varphi(x)\big)\Big(\beta_0 f(x)+\sum_{k=1}^{m-1}\beta_k f\big((1-\mu_k)x+\mu_k y\big)+\beta_m f(y)\Big)=F(y)-F(x)$$

*for all elements $x<y$ of $I$ if and only if $f$ is $\varphi$-quasiaffine of order $2m$, and $F$ is the antiderivative of $f\cdot\varphi'$.*

COROLLARY 7. *Assume that $I$ is an open interval, and $\varphi\colon I\to\mathbb{R}$ a continuously differentiable function with $\varphi'>0$. Let further $\lambda_1,\dots,\lambda_m$ be the zeros of (3) and let $\gamma_1,\dots,\gamma_m$ be given by (6). Then a continuous function $f\colon I\to\mathbb{R}$ is a solution of the functional equation*

$$\big(\varphi(y)-\varphi(x)\big)\Big(\gamma_0 f(x)+\sum_{k=1}^{m}\gamma_k f\big((1-\lambda_k)x+\lambda_k y\big)\Big)=F(y)-F(x)$$

*for all elements $x<y$ of $I$ if and only if $f$ is $\varphi$-quasiaffine of order $(2m+1)$, and $F$ is the antiderivative of $f\cdot\varphi'$.*

COROLLARY 8. *Assume that $I$ is an open interval, and $\varphi\colon I\to\mathbb{R}$ a continuously differentiable function with $\varphi'>0$. Let further $\lambda_1,\dots,\lambda_m$ be the zeros of (3) and let $\gamma_1,\dots,\gamma_m$ be given by (6). Then a continuous function $f\colon I\to\mathbb{R}$ and a function $F\colon I\to\mathbb{R}$ are the solutions of the functional equation*

$$\big(\varphi(y)-\varphi(x)\big)\Big(\sum_{k=1}^{m}\gamma_k f\big(\lambda_k x+(1-\lambda_k)y\big)+\gamma_0 f(y)\Big)=F(y)-F(x)$$

*for all elements $x<y$ of $I$ if and only if $f$ is $\varphi$-quasiaffine of order $(2m+1)$, and $F$ is the antiderivative of $f\cdot\varphi'$.*

To illustrate the aboves, consider the special case $m = 1$ of Corollary 1 and Corollary 5. Then, it turns out that a continuous functions $f$ and a function $F$ fulfills the functional equation

$$\left(e^{\alpha y} - e^{\alpha x}\right) f\left(\log \sqrt[\alpha]{\frac{e^{\alpha x} + e^{\alpha y}}{2}}\right) = F(y) - F(x)$$

for all elements $x < y$ of an open interval $I$ if and only if

$$f(t) = c_1 e^{\alpha t} + c_2, \qquad F(t) = \alpha e^{\alpha t}(c_1 e^{\alpha t} + c_2).$$

To summarize very briefly, higher-order quasimonotonicity can be characterized via integral inequalities of Hermite–Hadamard types. However, the question still remains open: What other kind of Chebyshev systems enjoy this property? Finding the proper answer might be the topic of further research.

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