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BASIS PROPERTIES OF p–EXPONENTIAL FUNCTION OF LINDQVIST AND PEETRE TYPE

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Abstract. We show that a p-exponential function defined by the p-trigonometric functions of Lindqvist and Peetre form a basis in the Lebesgue space $L^r\big((-1,1)^n\big)$ for any $r \in (1,\infty)$, provided $n \leqslant 3$ and $p > p_n \geqslant 1$.

1. History, introduction and preliminaries

A history of generalized trigonometric functions, which is now quite long, goes back at least to the year 1879. In that year E. Lundberg in his thesis studied functions which are related to today's generalized trigonometric functions $\sin_{p,q}$ (see [17], where a nice historic review can be found). His work was then forgotten and rediscovered much later by J. Peetre. In 1938 V. I. Levin (see [13]) found the exact value of the norm of the Hardy operator $(Hf)(x) = \int_0^x f(t)\,\mathrm{d}t$ on $L^p\big((0,1)\big)$ and described the extremal functions explicitly. Notice that his extremal functions correspond to $\sin_p$-functions in the first quadrant. In 1940 E. Schmidt obtained independently (see [21]) the exact value for norm of the Hardy operator H from $L^p\big((0,1)\big)$ into $L^q\big((0,1)\big)$ and described the extremal functions which, in the first quadrant, correspond to $\sin_{p,q}$-functions.

Then, in 1979, Á. Elbert in [11] began to study properties of non-linear equations which might be rewritten to the form

$$(y')^p + y^p = 1 \quad \text{on } (0,\infty).$$

(Let us note that the solutions of these equations on $(0,\infty)$ are just the generalized trigonometric functions $\sin_p$ and $\cos_p$.) He fully described solutions of the above equation and studied their properties in detail. This paper, which was the first mathematical work devoted to such a detailed study of generalized trigonometric functions, can thus be considered as a gateway to the topic.

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Just a few years after Elbert's paper the generalized trigonometric functions were studied independently in the same context by J. Peetre ([19]). In the context of eigenfunctions of the p-Laplacian these functions were later studied and independently introduced by M. Ôtani ([18]) and P. Lindqvist ([15]). Let us mention that these functions appeared also in connection with the approximation theory (see [6], [7] and [20]), the theory of nonlinear operators ([8], [14]), later they were intensively studied by P. Lindqvist and J. Peetre ([16]) and others (see e.g. [5] and [12] for more information).

The main purpose of this paper is to continue the study of generalized trigonometric functions, related to a differential equation

$$(y')^{p'} + y^p = 1 \quad \text{on } (0,\infty),$$

and show that they can be used to introduce Fourier type analysis by observing that a suitable p-exponential function can be defined as a generalization of the standard exponential function via use of generalized trigonometric functions. This note is a continuation of [1], where similar questions were studied in the context of generalized p-exponential functions of Lindqvist and Peetre.

We open the discussion by recalling some well-known facts. For a complex function $f \in L^1\big((-1,1)\big)$ its k-th *Fourier coefficient*, $k \in \mathbb{Z}$, is defined by

$$\widehat{f}(k) = \frac{1}{2}\int_{-1}^{1} f(t)\, e^{-ik\pi t}\, \mathrm{d}t$$

(i denotes the imaginary unit). The formal trigonometric series

$$\sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ik\pi t} \tag{1}$$

is called the *Fourier series of the function* f. By a result of M. Riesz (see e.g. [10, Section 10 of Chapter 12]), the sequence $\{e^{ik\pi x}\}_{k\in\mathbb{Z}}$ is a basis of any $L^r\big((-1,1)\big)$, $r \in (1,\infty)$, that is, for all $f \in L^r\big((-1,1)\big)$, one has

$$\lim_{N\to\infty} \Big\| f - \sum_{|k|\leqslant N} \widehat{f}(k)\, e^{ik\pi x} \Big\|_r = 0,$$

which means that the series (1) converges to the function f in the Lebesgue space $L^r\big((-1,1)\big)$ for any $r \in (1,\infty)$.

In a recent paper by Boulton and Melkonian [4] it was shown that a similar result holds when we consider the p-exponential function (with $p > p_0$ for some $p_0 > 1$)

$$\exp_p(iy) = \cos_p(y) + i\sin_p(y), \quad y \in \mathbb{R},$$

instead of $\exp(iy) = e^{iy}$, $y \in \mathbb{R}$. Here $\cos_p = \cos_{p,p}$, $\sin_p = \sin_{p,p}$ are the p-trigonometric functions (see Subsection 1.1 below). The aim of this note is to derive a multidimensional analogue of this result with a p-exponential function made from the p-trigonometric functions of Lindqvist and Peetre $\mathrm{C}_{1/p} = (\cos_{p,p'})^{p-1}$ and $\mathrm{S}_{1/p} = \sin_{p,p'}$

(where $p' = p/(p-1)$). It will be seen that these functions are more suitable than the functions $\cos_p$, $\sin_p$, considered by Boulton and Melkonian, for obtaining such a result.

We underline the fact that Boulton and Lord [3], after making certain numerical computations, observed that the numerical solutions for some non-linear problems (like p-Poisson boundary value problem with non-smooth right hand side) are sometimes obtained faster by using bases created from generalized trigonometric functions than by bases generated by classical trigonometric functions. This, perhaps, suggests that the Gibbs phenomenon for Fourier type series based on the generalized trigonometric functions could be mitigated, and so, to investigate these types of Fourier series in dimension 2 might be of some interest for the image processing.

In what follows we recall basic definitions and introduce notation and properties of generalized trigonometric functions. Since the approach is an analogue of that used in [1] we do not repeat all the proofs here.

1.1. Generalized sine and cosine functions

Let $1 < p,q < \infty$. Define the function

$$F_{p,q}(x) = \int_0^x (1-t^q)^{-1/p}\,\mathrm{d}t, \quad x \in [0,1].$$

Since this is strictly increasing it has an inverse, which we denote by $\sin_{p,q}$,

$$\sin_{p,q} = (F_{p,q})^{-1},$$

to emphasize the connection with the usual sine function (note that $F_{2,2} = \sin^{-1}$). The function $\sin_{p,q}$ is defined on the interval $[0, \pi_{p,q}/2]$, where

$$\pi_{p,q} = 2\int_0^1 (1-t^q)^{-1/p}\,\mathrm{d}t.$$

The constants $\pi_{p,q}$ can be evaluated by means of the Beta or Gamma functions

$$\pi_{p,q} = \frac{2B(1/p',1/q)}{q} = \frac{2\Gamma(1/p')\Gamma(1/q)}{q\Gamma(1/p'+1/q)}. \tag{2}$$

Observing that $\sin_{p,q}0 = 0$ and $\sin_{p,q}(\pi_{p,q}/2) = 1$, we can extend $\sin_{p,q}$ to $[0,\pi_{p,q}]$ by defining

$$\sin_{p,q}x = \sin_{p,q}(\pi_{p,q}-x) \quad \text{for } x \in [\pi_{p,q}/2, \pi_{p,q}]; \tag{3}$$

a further extension to $[-\pi_{p,q}, \pi_{p,q}]$ is made by oddness and finally $\sin_{p,q}$ is extended to the whole real line by $2\pi_{p,q}$-periodicity. It is easy to see that this extension is continuously differentiable on $\mathbb{R}$ and infinitely differentiable everywhere except possibly at the points $\{k\pi_{p,q}/2; k \in \mathbb{Z}\}$.

Define $\cos_{p,q} : \mathbb{R} \to \mathbb{R}$ by

$$\cos_{p,q}x = \frac{\mathrm{d}}{\mathrm{d}x}\sin_{p,q}x, \quad x \in \mathbb{R}.$$

Clearly, $\cos_{p,q}$ is even, $2\pi_{p,q}$-periodic and odd about $\pi_{p,q}/2$. If $x \in [0, \pi_{p,q}/2]$ and we put $y = \sin_{p,q} x$, then

$$\cos_{p,q} x = (1 - y^q)^{1/p} = (1 - (\sin_{p,q} x)^q)^{1/p}.$$

Hence, $\cos_{p,q}$ is strictly decreasing on $[0, \pi_{p,q}/2]$, $\cos_{p,q} 0 = 1$, $\cos_{p,q}(\pi_{p,q}/2) = 0$ and

$$|\sin_{p,q} x|^q + |\cos_{p,q} x|^p = 1, \quad x \in \mathbb{R}. \tag{4}$$

For the first derivative of the cosine function we have the following formula (see [9, Proposition 3.1])

$$\frac{\mathrm{d}}{\mathrm{d}x} (\cos_{p,q} x)^{p-1} = -\frac{q(p-1)}{p} (\sin_{p,q} x)^{q-1}, \quad x \in \left[0, \tfrac{1}{2}\pi_{p,q}\right). \tag{5}$$

1.2. Trigonometric p-functions of Lindqvist and Peetre and their properties

Lindqvist and Peetre introduced in [16] and [15] a generalized sine $\mathrm{S}_{1/p}$ and generalized cosine $\mathrm{C}_{1/p}$ functions which are related to the above functions as shown below:

$$\mathrm{S}_{1/p}(x) = \sin_{p,p'}(x), \quad \mathrm{C}_{1/p}(x) = \left(\cos_{p,p'}(x)\right)^{p-1}, \quad x \in \mathbb{R},$$

where, for the sake of simplicity, we use the notation

$$\left(\cos_{p,p'}(x)\right)^{p-1} = \left|\cos_{p,p'}(x)\right|^{p-2} \cos_{p,p'}(x).$$

Next we present some basic properties of these functions (cf. (5) and (4)). For all $x \in \mathbb{R}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{C}_{1/p'}(x) = -\left(\mathrm{S}_{1/p'}(x)\right)^{p-1}, \quad \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{S}_{1/p'}(x) = \left(\mathrm{C}_{1/p'}(x)\right)^{p-1},$$
$$\left(\mathrm{S}_{1/p'}(x)\right)^{p} + \left(\mathrm{C}_{1/p'}(x)\right)^{p} = 1.$$

Another useful relation between $\mathrm{S}_{1/p}$ and $\mathrm{C}_{1/p}$ is given by

$$\mathrm{C}_{1/p}\left(\pi_{p,p'} t\right) = \mathrm{S}_{1/p}\left(\pi_{p,p'}(\tfrac{1}{2} - t)\right), \quad t \in [0, \tfrac{1}{2}]. \tag{6}$$

Observing that $\pi_{p,p'}$, $1 < p < \infty$, is equal to the area of the set $\mathbf{S}_{p'}$ enclosed by the p'-circle, that is,

$$\mathbf{S}_{p'} = \{(x,y) \in \mathbb{R}^2; |x|^{p'} + |y|^{p'} \leqslant 1\},$$

we obtain the estimate

$$2 \leqslant \pi_{p,p'} \leqslant 4 \tag{7}$$

(cf. [9, Lemma 2.4]), moreover, the function

$$p \mapsto \pi_{p,p'} \quad \text{is decreasing on } (1, \infty). \tag{8}$$

We finally present another essential property to be referred to in the sequel:

$$\frac{2}{\pi_{p,p'}} \leqslant \frac{\mathrm{S}_{1/p}(x)}{x} \leqslant 1, \quad x \in (0, \tfrac{1}{2}\pi_{p,p'}), \tag{9}$$

(see e.g. [9, Proposition 3.3]).

1.3. Generalized p-exponential function of Lindqvist and Peetre type

We introduce the p-exponential function $\mathrm{E}_{1/p}$ as

$$\mathrm{E}_{1/p}(iy) = \mathrm{C}_{1/p}(y) + i\mathrm{S}_{1/p}(y), \quad y \in \mathbb{R}.$$

As is customary, for any complex number in binomial form $z = a + bi$, we denote its conjugate by $\overline{z} = a - bi$.

1.4. Basis in $L^r\big((a,b)^n\big)$ generated by a function φ

We introduce the following notation. Let $n \in \mathbb{N}$, $a,b \in \mathbb{R}$, $a < b$ and $1 < r < \infty$. By $L^r\big((a,b)^n\big)$ we denote the Banach space of all complex Lebesgue measurable functions f on the rectangle $(a,b)^n$ with the finite norm $\|f\|_r = \big(\int_{(a,b)^n} |f(x)|^r \,\mathrm{d}x\big)^{1/r}$.

Let $a \in \mathbb{R}$, $\mathbf{m} = (m_1,\dots,m_n) \in \mathbb{Z}^n$, $\mathbf{k} = (k_1 \dots, k_n) \in \mathbb{Z}^n$ and $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, we write

$$a\mathbf{m} = (am_1,\dots,am_n), \quad \mathbf{m}x = (m_1x_1,\dots,m_nx_n), \quad \mathbf{mk} = (m_1k_1,\dots,m_nk_n)$$

and

$$|\mathbf{k}| \leqslant \mathbf{m} \quad \text{if } |k_i| \leqslant m_i \text{ for each } i \in \{1,\dots,n\}.$$

We also use the notation

$$\mathbf{1} = (1,\dots,1).$$

For a given function $\varphi : (a,b) \to \mathbb{C}$, $\mathbf{k} \in \mathbb{Z}^n$, and $x \in \mathbb{R}^n$, we write

$$\varphi_{\mathbf{k}}(x) = \varphi(k_1x_1)\cdots\varphi(k_nx_n). \tag{10}$$

DEFINITION 1.1. We say that the system $\{\varphi_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^n}$ is a basis in $L^r\big((a,b)^n\big)$, $r \in (1,\infty)$, $a,b \in \mathbb{R}$, $a < b$, if, given any $f \in L^r\big((a,b)^n\big)$, there is a unique sequence $\{a_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^n}$ of scalars such that

$$f = \sum_{\mathbf{k}\in\mathbb{Z}^n} a_{\mathbf{k}}\,\varphi_{\mathbf{k}} \quad \text{in } L^r(a,b)^n. \tag{11}$$

The above convergence is considered in the *Pringsheim sense*, that is,

$$\lim_{\min\{m_1,\dots,m_n\}\to\infty} \Big\| f - \sum_{|\mathbf{k}|\leqslant\mathbf{m}} a_{\mathbf{k}}\varphi_{\mathbf{k}} \Big\|_{L^r(a,b)^n} = 0.$$

2. Basis generated by $\mathrm{E}_{1/p}$

NOTATION 2.1. Let $e(t) = \frac{1}{\sqrt{2}}\exp(i\pi t)$, $t \in \mathbb{R}$. We denote by

$$e_k(t) = e(kt) = \frac{1}{\sqrt{2}}\exp(i\pi kt), \quad t \in \mathbb{R},\ k \in \mathbb{Z},$$

the family of complex functions which form an orthonormal basis in the complex Lebesgue space $L^2\big((-1,1)\big)$. Analogously, an orthonormal basis in the complex Lebesgue space $L^2\big((-1,1)^n\big)$ consists of the functions

$$e_{\mathbf{m}}(x) = e_{m_1}(x_1)\cdots e_{m_n}(x_n) = 2^{-n/2}\exp(i\pi m_1 x_1)\cdots\exp(i\pi m_n x_n), \quad x\in\mathbb{R}^n,\ \mathbf{m}\in\mathbb{Z}^n.$$

REMARK 2.2. The fact that the sequence $\{e_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{Z}^n}$ is orthonormal in $L^2\big((-1,1)^n\big)$ is easy to verify, that is,

$$\int_{(-1,1)^n} e_{\mathbf{m}}(x)\overline{e_{\mathbf{m}}(x)}\,\mathrm{d}x = 1 \quad \text{and} \quad \int_{(-1,1)^n} e_{\mathbf{m}}(x)\overline{e_{\mathbf{k}}(x)}\,\mathrm{d}x = 0 \text{ if } \mathbf{m}\neq\mathbf{k}.$$

PROPOSITION 2.3. *Let* $f\in L^r\big((-1,1)^n\big)$, *where* $r\in(1,\infty)$. *Denote*

$$\widehat{f}(\mathbf{k}) = \int_{(-1,1)^n} f(x)\overline{e_{\mathbf{k}}(x)}\,\mathrm{d}x, \quad \mathbf{k}=(k_1,\dots,k_n)\in\mathbb{Z}^n. \tag{12}$$

Then

$$f = \sum_{\mathbf{k}\in\mathbb{Z}^n}\widehat{f}(\mathbf{k})\,e_{\mathbf{k}} \tag{13}$$

in the sense of (11).

Proof. See Weisz [22]. □

Throughout this section assume that $1<p<\infty$ and put

$$\varphi(x) = \mathrm{E}_{1/p}(i\pi_{p,p'}x), \quad x\in\mathbb{R}. \tag{14}$$

Since each $\varphi_{\mathbf{n}}$, $\mathbf{n}\in\mathbb{Z}^n$, is continuous, it has a Fourier expansion (13) with coefficients (12), that is,

$$\varphi_{\mathbf{n}}(x) = \sum_{\mathbf{k}\in\mathbb{Z}^n}\widehat{\varphi_{\mathbf{n}}}(\mathbf{k})\,e_{\mathbf{k}}(x), \quad \text{where} \quad \widehat{\varphi_{\mathbf{n}}}(\mathbf{k}) = \int_{(-1,1)^n}\varphi_{\mathbf{n}}(x)\overline{e_{\mathbf{k}}(x)}\,\mathrm{d}x.$$

Due to the symmetry (3) of $\varphi = \mathrm{S}_{1/p}(\pi_{p,p'}\cdot) = \sin_{p,p'}(\pi_{p,p'}\cdot)$ about $t=1/2$, for every $\mathbf{k}=(k_1,\dots,k_n)$ with some even k_i, $i\in\{1,\dots,n\}$, we have $\widehat{\varphi_{\mathbf{1}}}(\mathbf{k})=0$, and (see also Remark 2.2)

$$\begin{aligned}
\widehat{\varphi_{\mathbf{n}}}(\mathbf{k}) &= \int_{(-1,1)^n}\varphi_{\mathbf{n}}(x)\overline{e_{\mathbf{k}}(x)}\,\mathrm{d}x \\
&= \sum_{\mathbf{m}\in\mathbb{Z}^n}\widehat{\varphi_{\mathbf{1}}}(\mathbf{m})\int_{(-1,1)^n} e_{\mathbf{mn}}(x)\overline{e_{\mathbf{k}}(x)}\,\mathrm{d}x \\
&= \begin{cases} \widehat{\varphi_{\mathbf{1}}}(\mathbf{m}) & \text{if } k_i = m_i n_i \text{ and } m_i \text{ is odd for all } i=1,\dots,n; \\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Let us put, for $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$,

$$\tau_{\mathbf{m}} = \prod_{j=1}^{n} \tau_{m_j} = \widehat{\varphi_{\mathbf{1}}}(\mathbf{m}), \tag{15}$$

where

$$\tau_{m_j} = \int_{-1}^{1} \varphi(x_j)\overline{e(m_j x_j)}\,\mathrm{d}x_j = \frac{1}{\sqrt{2}}\int_{-1}^{1} \varphi(x_j)\exp(-i\pi m_j x_j)\,\mathrm{d}x_j, \quad j = 1, \dots, n. \tag{16}$$

Given any function f on $[-1,1)^n$, extend it to a function $\widetilde{f}$ on $\mathbb{R}^n$ by setting $\widetilde{f}(x) = \widetilde{f}(2\mathbf{k}+x)$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$, such that $x_j \in [2k_j - 1, 2k_j + 1)$, $j = 1, \dots, n$. Define the mapping $\mathscr{E}_{\mathbf{m}}$ on $L^r\big((-1,1)^n\big)$, $\mathbf{m} \in \mathbb{Z}^n$, $r \in (1,\infty)$, by

$$\mathscr{E}_{\mathbf{m}} f(x) = \widetilde{f}(\mathbf{m}x) \tag{17}$$

and note that $\mathscr{E}_{\mathbf{m}}(e_{\mathbf{n}}) = e_{\mathbf{mn}}$. Just as in [2], we can show that $\mathscr{E}_{\mathbf{m}}$ is a linear isometry, $\|\mathscr{E}_{\mathbf{m}}\| = 1$, and that the map T,

$$Tf(x) = \sum_{\mathbf{m}\in\mathbb{Z}^n} \tau_{\mathbf{m}}\,\mathscr{E}_{\mathbf{m}} f(x), \tag{18}$$

(convergence is considered in the sense of (11)) is a bounded linear map of $L^r\big((-1,1)^n\big)$ to itself with the property that, for all $\mathbf{n} \in \mathbb{Z}^n$,

$$Te_{\mathbf{n}} = \varphi_{\mathbf{n}}.$$

It is sufficient to show that T is a homeomorphism, since then it follows that the $\varphi_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^n$, inherit from the $e_{\mathbf{n}}$ the property of forming a basis in $L^r\big((-1,1)^n\big)$ for every $r \in (1,\infty)$. In the following lemma we state a criterion for this operator T to be a homeomorphism on $L^r\big((-1,1)^n\big)$.

LEMMA 2.4. *The following properties hold:*

$$\tau_{2k} = 0, \quad \tau_{2k+1} = \frac{4}{\sqrt{2}}\int_0^1 \mathrm{S}_{1/p}(\pi_{p,p'}t)\sin\big((2k+1)\pi t\big)\,\mathrm{d}t, \quad \tau_{-k} = 0, \quad k \in \mathbb{N}_0.$$

Proof. For $m \in \mathbb{N}$ we obtain, using the oddness of the functions $\mathrm{S}_{1/p}$ and $\sin$, the eveness of $\mathrm{C}_{1/p}$ and $\cos$ and the relationship between sine and cosine functions (for $\mathrm{S}_{1/p}$ and $\mathrm{C}_{1/p}$ see (6)), that

$$\begin{aligned}
\tau_m &= \frac{1}{\sqrt{2}}\int_{-1}^{1}\varphi(t)\overline{\exp\big(im\pi t\big)}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2}}\int_{-1}^{1}\big(\mathrm{C}_{1/p}(\pi_{p,p'}t) + i\,\mathrm{S}_{1/p}(\pi_{p,p'}t)\big)\big(\cos(m\pi t) - i\sin(m\pi t)\big)\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2}}\int_{-1}^{1}\mathrm{C}_{1/p}(\pi_{p,p'}t)\cos(m\pi t)\,\mathrm{d}t + \frac{1}{\sqrt{2}}\int_{-1}^{1}\mathrm{S}_{1/p}(\pi_{p,p'}t)\sin(m\pi t)\,\mathrm{d}t \\
&= \frac{4}{\sqrt{2}}\int_0^1 \mathrm{S}_{1/p}(\pi_{p,p'}t)\sin(m\pi t)\,\mathrm{d}t.
\end{aligned}$$

Since the function $\mathrm{S}_{1/p}$ is symmetric about $t = 1/2$, we have

$$\tau_{2k} = 0 \quad \text{for all } k \in \mathbb{N}_0 .$$

Finally, for $k \in \mathbb{N}$,

$$\tau_{-k} = \frac{1}{\sqrt{2}} \int_{-1}^{1} \mathrm{C}_{1/p}(\pi_{p,p'}t) \cos(k\pi t)\,\mathrm{d}t - \frac{1}{\sqrt{2}} \int_{-1}^{1} \mathrm{S}_{1/p}(\pi_{p,p'}t) \sin(k\pi t)\,\mathrm{d}t = 0. \quad \square$$

Observe that, due to Lemma 2.4,

$$Tf(x) = \sum_{\mathbf{k}\in\mathbb{N}^n} \tau_{\mathbf{2k-1}} \mathscr{E}_{\mathbf{2k-1}} f(x).$$

LEMMA 2.5. *Let*

$$\sum_{\mathbf{k}\in\mathbb{N}^n,\ \mathbf{k}\neq\mathbf{1}} |\tau_{\mathbf{2k-1}}| < |\tau_{(1,\dots,1)}|. \tag{19}$$

Then T is a homeomorphism on $L^r\big((-1,1)^n\big)$.

Proof. A proof is the same as that of [1, Lemma 2.4]. $\square$

Aiming at finding an upper estimate for the left hand side of (19), we start by providing an upper bound for $|\tau_{2k-1}|$, $k \in \mathbb{N}$ (cf. (15)).

PROPOSITION 2.6. *The following estimate holds:*

$$|\tau_{2k-1}| \leqslant \frac{8\,\pi_{p,p'}}{\sqrt{2}\,\pi^2} \frac{1}{(2k-1)^2}, \quad k \in \mathbb{N}.$$

Proof. Using integration by parts, the properties

$$\mathrm{S}_{1/p}(0) = 0, \quad \mathrm{S}'_{1/p}\left(\tfrac{1}{2}\right) = 0 \quad \text{and } \mathrm{S}'_{1/p} \text{ is decreasing on } \left(0, \tfrac{1}{2}\right),$$

and the substitution $s = \mathrm{S}'_{1/p}(t)$ we obtain, due to Lemma 2.4 (cf. the proof of [1, Proposition 2.5]),

$$\tau_{2k-1} = \frac{8}{\sqrt{2}(2k-1)^2\pi^2} \int_0^{\mathrm{S}'_{1/p}(0)} \sin\left((2k-1)\,\pi\,(\mathrm{S}'_{1/p})^{-1}(s)\right) \mathrm{d}s.$$

Since $\mathrm{S}'_{1/p}(x) = \pi_{p,p'} \cos_{p,p'}(\pi_{p,p'}x)$, one has $\mathrm{S}'_{1/p}(0) = \pi_{p,p'}$. Consequently,

$$|\tau_{2k-1}| \leqslant \frac{8\,\mathrm{S}'_{1/p}(0)}{\sqrt{2}(2k-1)^2\pi^2} = \frac{8\,\pi_{p,p'}}{\sqrt{2}(2k-1)^2\pi^2}. \quad \square$$

PROPOSITION 2.7. *The following estimate holds*

$$\tau_1 \geqslant \frac{16}{\sqrt{2}\,\pi^2}. \tag{20}$$

Proof. Inequality (9) implies that

$$\mathrm{S}_{1/p}(t) \geqslant 2t, \quad t \in (0, \tfrac{1}{2}).$$

Thus, Lemma 2.4 in conjunction with the symmetry of functions $\mathrm{S}_{1/p}(t)$ and $\sin(\pi t)$ about $t = 1/2$ yields:

$$\tau_1 = \frac{8}{\sqrt{2}} \int_0^{1/2} \mathrm{S}_{1/p}(t) \sin(\pi t)\, \mathrm{d}t \geqslant \frac{16}{\sqrt{2}} \int_0^{1/2} t \sin(\pi t)\, \mathrm{d}t = \frac{16}{\sqrt{2}\pi^2},$$

which verifies the assertion. □

PROPOSITION 2.8. *Condition* (19) *is satisfied, if*

$$\left(\frac{\pi^2}{8}\right)^n - 1 < \left(\frac{2}{\pi_{p,p'}}\right)^n. \tag{21}$$

Proof. From Proposition 2.6, (15) and estimates (19), (20) and (21) we obtain

$$\sum_{\mathbf{k}\in\mathbb{N}^n,\ \mathbf{k}\neq\mathbf{1}} |\tau_{2\mathbf{k}-\mathbf{1}}| < \left(\frac{16}{\sqrt{2}\pi^2}\right)^n \leqslant |\tau_{(1,\dots,1)}|,$$

which completes the proof (cf. the proof of [1, Proposition 2.7]). □

THEOREM 2.9. *Let $p \in (1,\infty)$ and $n \in \mathbb{N}$ be so that the condition* (21) *is satisfied. Then the sequence $\left\{(\mathrm{E}_{1/p}(i\pi_{p,p'}\cdot))_{\mathbf{k}}\right\}_{\mathbf{k}\in\mathbb{Z}^n}$ is a basis in $L^r\big((-1,1)^n\big)$ for any $r \in (1,\infty)$.*

Proof. By Proposition 2.8, condition (21) implies that (19) holds. Thus, by Lemma 2.5, the operator T is a homeomorphism and the assertion follows. □

REMARK 2.10. By (7) we have, for any $p \in (1,\infty)$, that $2/\pi_{p,p'} \leqslant 1$. Since $\left(\pi^2/8\right)^n < 2$ if and only if $n \leqslant 3$, it is apparent that the above described method of proving that the sequence $\left\{(\mathrm{E}_{1/p}(i\pi_{p,p'}\cdot))_{\mathbf{k}}\right\}_{\mathbf{k}\in\mathbb{Z}^n}$ is a basis in $L^r\big((-1,1)^n\big)$ is confined to the cases $n = 1,2,3$.

COROLLARY 2.11. ($n = 1$) *Let $p \in (1,\infty)$. The sequence*

$$\left\{(\mathrm{E}_{1/p}(i\pi_{p,p'}kx))\right\}_{k=-\infty}^{\infty}$$

is a basis in $L^r\big((-1,1)\big)$ for any $r \in (1,\infty)$.

COROLLARY 2.12. ($n = 2$) *There exists $p_2 > 1$ such that, for every $p \in (p_2,\infty)$, the sequence*

$$\left\{\mathrm{E}_{1/p}(i\pi_{p,p'}k_1x_1)\,\mathrm{E}_{1/p}(i\pi_{p,p'}k_2x_2)\right\}_{k_1,k_2=-\infty}^{\infty}$$

is a basis in $L^r\big((-1,1)^2\big)$ for any $r \in (1,\infty)$.

COROLLARY 2.13. ($n = 3$) *There exists* $p_3 > 1$ *such that, for every* $p \in (p_3, \infty)$, *the sequence*

$$\left\{ \mathrm{E}_{1/p}(\pi_{p,p'} k_1 x_1)\, \mathrm{E}_{1/p}(\pi_{p,p'} k_2 x_2)\, \mathrm{E}_{1/p}(\pi_{p,p'} k_3 x_3) \right\}_{k_1,k_2,k_3=-\infty}^{\infty}$$

is a basis in $L^r\big((-1,1)^3\big)$ *for any* $r \in (1,\infty)$.

Proof. [Proof of Corollaries 2.11–2.13] The proofs can be handled analogously as that of Theorem 2.10 of [1], applying Theorem (2.9). We omit the details. □

REMARK 2.14. By numerical computation we obtain that these rough estimates: $p_2 \leqslant 3$, $p_3 \leqslant 21$.

REFERENCES

[1] O. BAKŞI, P. GURKA, J. LANG, AND O. MÉNDEZ, *Basis properties of Lindqvist-Peetre functions in* $L^r(0,1)^n$, Rev. Mat. Complut. **30** (1): 1–12, 2017.

[2] P. BINDING, L. BOULTON, J. ČEPIČKA, P. DRÁBEK, AND P. GIRG, *Basis properties of eigenfunctions of the* p*-Laplacian*, Proc. Amer. Math. Soc. **134** (12): 3487–3494 (electronic), 2006.

[3] L. BOULTON AND G. LORD, *Approximation properties of the* q*-sine bases*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **467** (2133): 2690–2711, 2011.

[4] L. BOULTON AND H. MELKONIAN, *Generalised cosine functions, basis and regularity properties*, Unpublished manuscript, 2016.

[5] P. J. BUSHELL AND D. E. EDMUNDS, *Remarks on generalized trigonometric functions*, Rocky Mountain J. Math. **42** (1): 25–57, 2012.

[6] A. P. BUSLAEV AND V. M. TIKHOMIROV, *The spectra of nonlinear differential equations and widths of Sobolev classes*, Mat. Sb. **181** (12): 1587–1606, 1990.

[7] A. P. BUSLAEV AND V. M. TIKHOMIROV, *Spectral theory of nonlinear equations and* n*-widths of Sobolev spaces*, In Methods of approximation theory in complex analysis and mathematical physics (Leningrad, 1991), vol. 1550 of Lecture Notes in Math., pages 19–30, Springer, Berlin, 1993.

[8] P. DRÁBEK AND R. MANÁSEVICH, *On the closed solution to some nonhomogeneous eigenvalue problems with* p*-Laplacian*, Differential Integral Equations **12** (6): 773–788, 1999.

[9] D. E. EDMUNDS, P. GURKA, AND J. LANG, *Properties of generalized trigonometric functions*, J. Approx. Theory **164** (1): 47–56, 2012.

[10] R. E. EDWARDS, *Fourier series, vol. 2*, vol. 85 of Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, second edition, 1982, A modern introduction.

[11] Á. ELBERT, *A half-linear second order differential equation*, In Qualitative theory of differential equations, vol. I, II (Szeged, 1979), vol. 30 of Colloq. Math. Soc. János Bolyai, pages 153–180. North-Holland, Amsterdam-New York, 1981.

[12] J. LANG AND D. E. EDMUNDS, *Eigenvalues, embeddings and generalised trigonometric functions*, volume 2016 of Lecture Notes in Mathematics, Springer, Heidelberg, 2011.

[13] V. LEVIN, *Notes on inequalities, II. On a class of integral inequalities*, Rec. Math. Moscou, n. Ser. **4**: 309–324, 1938.

[14] P. LINDQVIST, *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$, Proc. Amer. Math. Soc. **109** (1): 157–164, 1990.

[15] P. LINDQVIST, *Some remarkable sine and cosine functions*, Ricerche Mat. **44** (2): 269–290 (1996), 1995.

[16] P. LINDQVIST AND J. PEETRE, p*-arclength of the* q*-circle*, Math. Student **72** (1–4): 139–145 (2005), 2003.

[17] P. LINDQVIST AND J. PEETRE, *Comments on Erik Lundberg's 1879 thesis, especially on the work of Göran Dillner and his inuence on Lundberg*, Mem. dell'Istituto Lombardo, Accad. Sci. e Lett., Classe Sci. Mat. Nat. XXXI, Fasc. 1. Milano, 2004.

[18] M. ÔTANI, *A remark on certain nonlinear elliptic equations*, Proc. Fac. Sci. Tokai Univ. **19**: 23–28, 1984.
[19] J. PEETRE, *The differential equation* $y'^p - y^p = \pm 1 \ (p > 0)$, Ricerche Mat. **43** (1): 91–128, 1994.
[20] A. PINKUS, *n-widths of Sobolev spaces in L^p*, Constr. Approx. **1** (1): 15–62, 1985.
[21] E. SCHMIDT, *Über die Ungleichung, welche die Integrale über eine Potenz einer Funktion und über eine andere Potenz ihrer Ableitung verbindet*, Math. Ann. **117**: 301–326, 1940.
[22] F. WEISZ, *Summability of multi-dimensional trigonometric Fourier series*, Surv. Approx. Theory **7**: 1–179, 2012.

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