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SOME INEQUALITIES FOR MEANS DEFINED ON THE LORENTZ CONE

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Abstract. In this paper, we define various means associated with Lorentz cones (also known as second-order cones), which are new concepts and natural extensions of traditional arithmetic mean, harmonic mean, and geometric mean, logarithmic mean. Based on these means defined on the Lorentz cone, some inequalities and trace inequalities are established.

1. Introduction

A *mean* is a binary map $m : (0,\infty)\times(0,\infty) \to (0,\infty)$ satisfying the following:

(a) $m(a,b) > 0$;

(b) $\min\{a,b\} \leqslant m(a,b) \leqslant \max\{a,b\}$;

(c) $m(a,b) = m(b,a)$;

(d) $m(a,b)$ is increasing in a,b;

(e) $m(\alpha a,\alpha b) = \alpha m(a,b)$, for all $\alpha > 0$;

(f) $m(a,b)$ is continuous in a,b.

Many types of means have been investigated in the literature, to name a few, the arithmetic mean, geometric mean, harmonic mean, logarithmic mean, identric mean, contra-harmonic mean, quadratic (or root-square) mean, first Seiffert mean, second Seiffert mean, and Neuman-Sandor mean, etc.. In addition, many inequalities describing the relationship among different means have been established. For instance, for any two positive real number a,b, it is well-known that

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$$\min\{a,b\} \leqslant H(a,b) \leqslant G(a,b) \leqslant L(a,b) \leqslant A(a,b) \leqslant \max\{a,b\}, \tag{1}$$

where

$$\begin{aligned} H(a,b) &= \frac{2ab}{a+b}, \\ G(a,b) &= \sqrt{ab}, \\ L(a,b) &= \begin{cases} \dfrac{a-b}{\ln a - \ln b} & \text{if } a \neq b, \\ a & \text{if } a = b, \end{cases} \\ A(a,b) &= \frac{a+b}{2}, \end{aligned}$$

represents the harmonic mean, geometric mean, logarithmic mean, and arithmetic mean, respectively. For more details regarding various means and their inequalities, please refer to [10, 17].

Recently, the matrix versions of means have been generalized from the classical means, see [4, 6, 7, 8]. In particular, the matrix version of Arithmetic Geometric Mean Inequality (AGM) is proved in [4, 5], and has attracted much attention. Indeed, let A and B be two $n \times n$ positive definite matrices, the following inequalities hold under the partial order induced by positive semidefinite matrices cone $\mathbb{S}^n_+$:

$$(A : B) \preceq A\#B \preceq \frac{1}{2}(A+B), \tag{2}$$

where

$$\begin{aligned} A : B &= 2\left(A^{-1} + B^{-1}\right)^{-1}, \\ A\#B &= A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2} A^{1/2}, \end{aligned}$$

denotes the matrix harmonic mean, the matrix geometric mean, respectively. For more details about matrix means and their inequalities, please see [4, 6, 7, 8, 19] and references therein.

Note that the nonnegative orthant, the cone of positive semidefinite matrices, and the second-order cone (denoted by $\mathcal{K}^n$ and will be introduced later) belong to the so-called symmetric cones [15]. In addition, Lim [22] generalized the geometric mean from the cone of positive semidefinite matrices into the symmetric cone, and some applications are established in [21, 23]. This motivates us to consider further extension of means, that is, the means associated with second-order cone (SOC means for short). In this paper, we generalize some well-known means to the setting of second-order cone and build up some inequalities under the partial order induced by second-order cone $\mathcal{K}^n$. Moreover, two trace inequalities are established as well.

2. Preliminary

In this section, we recall some background materials regarding Lorentz cones, also known as second-order cones. The second-order cone (SOC for short) in $\mathbb{R}^n$, is defined by

$$\mathscr{K}^n = \left\{ x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|x_2\| \leqslant x_1 \right\}.$$

For $n = 1$, $\mathscr{K}^n$ denotes the set of nonnegative real number $\mathbb{R}_+$. For any x, y in $\mathbb{R}^n$, we write $x \succeq_{\mathscr{K}^n} y$ if $x - y \in \mathscr{K}^n$ and write $x \succ_{\mathscr{K}^n} y$ if $x - y \in \mathrm{int}(\mathscr{K}^n)$. In other words, we have $x \succeq_{\mathscr{K}^n} 0$ if and only if $x \in \mathscr{K}^n$ and $x \succ_{\mathscr{K}^n} 0$ if and only if $x \in \mathrm{int}(\mathscr{K}^n)$. The relation $\succeq_{\mathscr{K}^n}$ is a partial ordering but not a linear ordering in $\mathscr{K}^n$, i.e., there exist $x, y \in \mathscr{K}^n$ such that neither $x \succeq_{\mathscr{K}^n} y$ nor $y \succeq_{\mathscr{K}^n} x$. To see this, for $n = 2$, let $x = (1,1)$ and $y = (1,0)$, we have $x - y = (0,1) \notin \mathscr{K}^n$, $y - x = (0,-1) \notin \mathscr{K}^n$.

For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define their *Jordan product* as

$$x \circ y = (x^T y\ ,\ y_1 x_2 + x_1 y_2).$$

We write x^2 to mean $x \circ x$ and write $x + y$ to mean the usual componentwise addition of vectors. Then, $\circ, +$, together with $e' = (1, 0, \ldots, 0)^T \in \mathbb{R}^n$ and for any $x, y, z \in \mathbb{R}^n$, the following basic properties [15, 16] hold: (1) $e' \circ x = x$, (2) $x \circ y = y \circ x$, (3) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, (4) $(x + y) \circ z = x \circ z + y \circ z$. Notice that the Jordan product is *not associative* in general. However, it is power associative, i.e., $x \circ (x \circ x) = (x \circ x) \circ x$ for all $x \in \mathbb{R}^n$. Thus, we may, without loss of ambiguity, write x^m for the product of m copies of x and $x^{m+n} = x^m \circ x^n$ for all positive integers m and n. Here, we set $x^0 = e'$. Besides, $\mathscr{K}^n$ is *not closed* under Jordan product.

For any $x \in \mathscr{K}^n$, it is known that there exists a unique vector in $\mathscr{K}^n$ denoted by $x^{1/2}$ such that $(x^{1/2})^2 = x^{1/2} \circ x^{1/2} = x$. Indeed,

$$x^{1/2} = \left(s, \frac{x_2}{2s} \right), \quad \text{where } s = \sqrt{\frac{1}{2}\left(x_1 + \sqrt{x_1^2 - \|x_2\|^2} \right)}.$$

In the above formula, the term x_2/s is defined to be the zero vector if $x_2 = 0$ and $s = 0$, i.e., $x = 0$. For any $x \in \mathbb{R}^n$, we always have $x^2 \in \mathscr{K}^n$, i.e., $x^2 \succeq_{\mathscr{K}^n} 0$. Hence, there exists a unique vector $(x^2)^{1/2} \in \mathscr{K}^n$ denoted by $|x|$. It is easy to verify that $|x| \succeq_{\mathscr{K}^n} 0$ and $x^2 = |x|^2$ for any $x \in \mathbb{R}^n$. It is also known that $|x| \succeq_{\mathscr{K}^n} x$. For any $x \in \mathbb{R}^n$, we define $[x]_+$ to be the nearest point (in Euclidean norm, since Jordan product does not induce a norm) projection of x onto $\mathscr{K}^n$, which is the same definition as in $\mathbb{R}^n_+$. In other words, $[x]_+$ is the optimal solution of the parametric SOCP: $[x]_+ = \arg\min\{\|x - y\| \,|\, y \in \mathscr{K}^n\}$. In addition, it can be verified that $[x]_+ = (x + |x|)/2$; see [15, 16].

Recently, there has found many optimization problems involved second-order cones in real world applications. For dealing with second-order cone programs (SOCP) and second-order cone complementarity problems (SOCCP), there needs *spectral decomposition* associated with SOC [14]. More specifically, for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the vector x can be decomposed as

$$x = \lambda_1 u_x^{(1)} + \lambda_2 u_x^{(2)}, \tag{3}$$

where λ_1, λ_2 and $u_x^{(1)}, u_x^{(2)}$ are the spectral values and the associated spectral vectors of x, respectively, given by

$$\lambda_i = x_1 + (-1)^i \|x_2\|, \tag{4}$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{x_2}{\|x_2\|}\right) & \text{if } x_2 \neq 0, \\ \frac{1}{2}(1, (-1)^i w) & \text{if } x_2 = 0. \end{cases} \tag{5}$$

for $i = 1,2$ with w being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$. If $x_2 \neq 0$, the decomposition is unique.

For any function $f : \mathbb{R} \to \mathbb{R}$, the following vector-valued function associated with $\mathscr{K}^n$ $(n \geqslant 1)$ was considered in [12, 13]:

$$f^{\mathrm{soc}}(x) = f(\lambda_1)u_x^{(1)} + f(\lambda_2)u_x^{(2)}, \quad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{6}$$

If f is defined only on a subset of $\mathbb{R}$, then f^{soc} is defined on the corresponding subset of $\mathbb{R}^n$. The definition (6) is unambiguous whether $x_2 \neq 0$ or $x_2 = 0$. The cases of $f^{\mathrm{soc}}(x) = x^{1/2}$, x^2, $\exp(x)$ are discussed in [15].

LEMMA 1. ([16, Proposition 3.3]) *For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with spectral decomposition (3)–(5), there have*

(a) $|x| = (x^2)^{1/2} = |\lambda_1| u_x^{(1)} + |\lambda_2| u_x^{(2)}$;

(b) $[x]_+ = [\lambda_1]_+ u_x^{(1)} + [\lambda_2]_+ u_x^{(2)} = \frac{1}{2}(x + |x|)$.

We point out that the relation $\succeq_{\mathscr{K}^n}$ is not a linear ordering. Hence, it is not possible to compare any two vectors (elements) via $\succeq_{\mathscr{K}^n}$. Nonetheless, we note that for any $a, b \in \mathbb{R}$

$$\begin{aligned} \max\{a,b\} &= b + [a-b]_+ = \frac{1}{2}(a + b + |a-b|), \\ \min\{a,b\} &= a - [a-b]_+ = \frac{1}{2}(a + b - |a-b|). \end{aligned}$$

This motivates us to define supremum and infimum of $\{x,y\}$, denoted by $x \vee y$ and $x \wedge y$ respectively, in the setting of second-order cone as follows. For any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} x \vee y &:= y + [x-y]_+ = \frac{1}{2}(x + y + |x-y|), \\ x \wedge y &:= \begin{cases} x - [x-y]_+ = \frac{1}{2}(x + y - |x-y|), & \text{if } x + y \succeq_{\mathscr{K}^n} |x-y|; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Next, we review the concepts of SOC-monotone and SOC-convex functions which are introduced in [12] and needed for subsequent analysis. For a real valued function $f : \mathbb{R} \to \mathbb{R}$, f is said to be *SOC-monotone* of order n if its corresponding vector-valued function f^{soc} defined as in (6) satisfies

$$x \succeq_{\mathscr{K}^n} y \implies f^{\mathrm{soc}}(x) \succeq_{\mathscr{K}^n} f^{\mathrm{soc}}(y).$$

The function f is said to be SOC-monotone if f is SOC-monotone of all order n. f is said to be *SOC-convex* of order n if its corresponding vector-valued function f^{soc} defined as in (6) satisfies

$$f^{\mathrm{soc}}((1-\lambda)x+\lambda y) \preceq_{\mathscr{K}^n} (1-\lambda)f^{\mathrm{soc}}(x)+\lambda f^{\mathrm{soc}}(y)$$

for all $x, y \in \mathbb{R}^n$ and $0 \leqslant \lambda \leqslant 1$. The function f is said to be SOC-convex if f is SOC-convex of all order n. In fact, it easily follows by induction that for each integer $m \geqslant 2$,

$$f^{\mathrm{soc}}\left(\sum_{i=1}^{m} \lambda_i x^{(i)}\right) \preceq_{\mathscr{K}^n} \sum_{i=1}^{m} \lambda_i f^{\mathrm{soc}}\left(x^{(i)}\right)$$

where each $x^{(i)} \in \mathbb{R}^n$ and $\sum_{i=1}^{m} \lambda_i = 1$ with $0 \leqslant \lambda_i \leqslant 1$.

The concepts of SOC-monotone and SOC-convex functions are analogous to matrix monotone and matrix convex functions [3, 18], and are special cases of operator monotone and operator convex functions [2, 8, 20]. Examples and characterizations of SOC-monotone and SOC-convex functions are given in [12, 13].

LEMMA 2. ([12, Proposition 3.3]) *Let* $f:(0,\infty)\to(0,\infty)$ *be* $f(t)=1/t$. *Then,*

(a) $-f$ *is SOC-monotone on* $(0,\infty)$;

(b) f *is SOC-convex on* $(0,\infty)$.

LEMMA 3. ([12, Proposition 3.7]) *Let* $f:[0,\infty)\to[0,\infty)$ *be* $f(t)=t^r$, $0\leqslant r\leqslant 1$. *Then,*

(a) f *is SOC-monotone on* $[0,\infty)$;

(b) $-f$ *is SOC-convex on* $[0,\infty)$.

3. Main results

Inspired by the definition of classical means, we define the means associated with Lorentz cones in a similar way. As introduced in Section 2, the Lorentz cone is also called second-order cone. For convenience, we use "SOC means" to denote our proposed means defined on the Lorentz cone.

In the setting of second-order cone, we call a binary operation $(x,y)\mapsto M(x,y)$ defined on $\mathrm{int}(\mathscr{K}^n)\times\mathrm{int}(\mathscr{K}^n)$ a SOC mean if the following are satisfied:

(i) $M(x,y) \succ_{\mathscr{K}^n} 0$;

(ii) $x\wedge y \preceq_{\mathscr{K}^n} M(x,y) \preceq_{\mathscr{K}^n} x\vee y$;

(iii) $M(x,y)=M(y,x)$;

(iv) $M(x,y)$ is monotone in x,y;

(v) $M(\alpha x,\alpha y)=\alpha M(x,y)$, $\alpha>0$;

(vi) $M(x,y)$ is continuous in x,y.

It is clear to see that the SOC arithmetic mean $A(x,y):\mathrm{int}(\mathscr{K}^n)\times\mathrm{int}(\mathscr{K}^n)\to\mathrm{int}(\mathscr{K}^n)$ given by

$$A(x,y)=\frac{x+y}{2} \tag{7}$$

satisfies all the above properties. Besides, it is not hard to verify that the SOC harmonic mean of x and y, $H(x,y):\mathrm{int}(\mathscr{K}^n)\times\mathrm{int}(\mathscr{K}^n)\to\mathrm{int}(\mathscr{K}^n)$, can be defined as

$$H(x,y)=\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \tag{8}$$

Note that some of the above properties are obvious, whereas some others are not.

THEOREM 1. *Let $A(x,y)$, $H(x,y)$ be defined as in (7) and (8), respectively. For any $x\succ_{\mathscr{K}^n}0$, $y\succ_{\mathscr{K}^n}0$, there holds*

$$x\wedge y\preceq_{\mathscr{K}^n}H(x,y)\preceq_{\mathscr{K}^n}A(x,y)\preceq_{\mathscr{K}^n}x\vee y.$$

Proof. (i) To verify the first inequality, if $\frac{1}{2}(x+y-|x-y|)\notin\mathscr{K}^n$, the inequality holds clearly. Suppose $\frac{1}{2}(x+y-|x-y|)\succeq_{\mathscr{K}^n}0$, we note that $\frac{1}{2}(x+y-|x-y|)\preceq_{\mathscr{K}^n}x$ and $\frac{1}{2}(x+y-|x-y|)\preceq_{\mathscr{K}^n}y$. Then, using the SOC-monotonicity of $f(t)=-t^{-1}$ (Lemma 3), we obtain

$$x^{-1}\preceq_{\mathscr{K}^n}\left(\frac{x+y-|x-y|}{2}\right)^{-1}\quad\text{and}\quad y^{-1}\preceq_{\mathscr{K}^n}\left(\frac{x+y-|x-y|}{2}\right)^{-1},$$

which imply

$$\frac{x^{-1}+y^{-1}}{2}\preceq_{\mathscr{K}^n}\left(\frac{x+y-|x-y|}{2}\right)^{-1}.$$

Next, applying the SOC-monotonicity again, we conclude that

$$\frac{x+y-|x-y|}{2}\preceq_{\mathscr{K}^n}\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1}.$$

(ii) To see the second inequality, we first observe that

$$\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1}\preceq_{\mathscr{K}^n}\frac{1}{2}(x^{-1})^{-1}+\frac{1}{2}(y^{-1})^{-1}=\frac{x+y}{2},$$

where the inequality comes from the SOC-convexity of $f(t)=t^{-1}$.

(iii) To check the last inequality, we observe that

$$\frac{x+y}{2}\preceq_{\mathscr{K}^n}\frac{x+y+|x-y|}{2}\quad\Longleftrightarrow\quad 0\preceq_{\mathscr{K}^n}\frac{|x-y|}{2},$$

where it is clear $|x-y| \succeq_{\mathscr{K}^n} 0$ always holds for any element x,y. Then, the desired result follows. $\square$

Now, we consider the SOC geometric mean, denoted by $G(x,y)$, which can be borrowed from the geometric mean of symmetric cone, see [22]. More specifically, let V be a Euclidean Jordan algebra, $\mathscr{K}$ be the set of all square elements of V (the associated symmetric cone), and $\Omega := \mathrm{int}\mathscr{K}$ (the interior symmetric cone). For $x \in V$, let $\mathscr{L}(x)$ denote the linear operator given by $\mathscr{L}(x)y := x \circ y$, and let $P(x) := 2\mathscr{L}(x)^2 - \mathscr{L}(x^2)$. The mapping P is called the *quadratic representation* of V. If x is invertible, then we have

$$P(x)\mathscr{K} = \mathscr{K} \quad \text{and} \quad P(x)\Omega = \Omega.$$

Suppose that $x,y \in \Omega$, the geometric mean, denoted by $x\#y$ of x and y is

$$x\#y := P(x^{\frac{1}{2}})(P(x^{-\frac{1}{2}})y)^{\frac{1}{2}}.$$

On the other hand, it turns out that the cone Ω admits a $G(\Omega)$-invariant Riemannian metric [15]. The unique geodesic curve joining x and y is

$$t \mapsto x\#_t y := P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})y\right)^t,$$

and the geometric mean $x\#y$ is the midpoint of the geodesic curve. In addition, Lim establishes the arithmetic-geometric-harmonic means inequalities [22, Theorem 2.8],

$$\left\{\frac{x^{-1}+y^{-1}}{2}\right\}^{-1} \preceq_{\mathscr{K}} x\#y \preceq_{\mathscr{K}} \frac{x+y}{2}, \tag{9}$$

where $\preceq_{\mathscr{K}}$ is the partial order induced by the closed convex cone $\mathscr{K}$. We note that inequality (9) includes the inequality (2) as a special case. For more details, please refer to [21, 22, 23]. As an example of Euclidean Jordan algebra, for any x and y in $\mathrm{int}(\mathscr{K}^n)$, we therefore adopt the geometric mean $G(x,y)$ as

$$G(x,y) := P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})y\right)^{\frac{1}{2}}. \tag{10}$$

Then, we immediately have the following parallel properties of SOC geometric mean.

PROPOSITION 1. Let $A(x,y)$, $H(x,y)$, $G(x,y)$ be defined as in (7), (8) and (10), respectively. Then, for any $x \succ_{\mathscr{K}^n} 0$ and $y \succ_{\mathscr{K}^n} 0$, we have

(a) $G(x,y) = G(y,x)$.

(b) $G(x,y)^{-1} = G(x^{-1},y^{-1})$.

(c) $H(x,y) \preceq_{\mathscr{K}^n} G(x,y) \preceq_{\mathscr{K}^n} A(x,y)$.

Next, we look into another type of SOC mean, the SOC logarithmic mean $L(x,y)$. First, for any two positive real numbers a,b, Carlson [11] has established the integral representation

$$L(a,b) = \left[\int_0^1 \frac{dt}{ta+(1-t)b}\right]^{-1},$$

whereas Neuman [24] has also given an alternative integral representation

$$L(a,b) = \int_0^1 a^{1-t}b^t\,dt.$$

Moreover, Bhatia [4, p. 229] proposes the matrix logarithmic mean of two positive definite matrices A and B as

$$L(A,B) = A^{1/2}\int_0^1 \left(A^{-1/2}BA^{-1/2}\right)^t dt\, A^{1/2}.$$

In other words,

$$L(A,B) = \int_0^1 A\#_t B\,dt,$$

where $A\#_t B =: A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2} = P(A^{1/2})(P(A^{-1/2})B)^t$ is also called the t-weighted geometric mean. For general operator setting, Bourin and Hiai [9] establish an operator mean, called geodesic mean, which is defined as

$$L(A,B) = \int_0^1 A\#_t B\,d\nu(t)$$

for positive definite matrices A, B, and some probability measure on $[0,1]$. From the above, we observe that $A\#_t B = A^{1-t}B^t$ for $AB = BA$, and the definition of logarithmic mean coincides with the one of real numbers. These two integral representations motivate us to define the SOC logarithmic mean on $\mathrm{int}(\mathcal{K}^n)\times\mathrm{int}(\mathcal{K}^n)$ as

$$L(x,y) = \int_0^1 x\#_t y\,dt. \tag{11}$$

To verify it is an SOC mean, we need the following technical lemmas. The first lemma is the symmetric cone version of Bernoulli inequality.

LEMMA 4. *Let V be a Euclidean Jordan algebra, $\mathcal{K}$ be the associated symmetric cone, and e be the Jordan identity. Then,*

$$(e+s)^t \preceq_{\mathcal{K}} e+ts,$$

where $0\leqslant t\leqslant 1$, $s\succeq_{\mathcal{K}} -e$, and the partial order is induced by the closed convex cone $\mathcal{K}$.

Proof. For any $s \in V$, we denote the spectral decomposition of s as $\sum_{i=1}^{r} \lambda_i c_i$. Since $s \succeq_{\mathscr{K}} -e$, we obtain that each eigenvalue $\lambda_i \geqslant -1$. Then, we have

$$\begin{aligned}(e+s)^t &= (1+\lambda_1)^t c_1 + (1+\lambda_2)^t c_2 + \cdots + (1+\lambda_r)^t c_r \\ &\preceq_{\mathscr{K}} (1+t\lambda_1)c_1 + (1+t\lambda_2)c_2 + \cdots + (1+t\lambda_r)c_r \\ &= e+ts,\end{aligned}$$

where the inequality holds by the real number version of Bernoulli inequality. □

LEMMA 5. *Suppose that $u(t) : \mathbb{R} \to \mathbb{R}^n$ is integrable on $[a,b]$.*

(a) *If $u(t) \succeq_{\mathscr{K}^n} 0$ for any $t \in [a,b]$, then $\int_a^b u(t)dt \succeq_{\mathscr{K}^n} 0$.*

(b) *If $u(t) \succ_{\mathscr{K}^n} 0$ for any $t \in [a,b]$, then $\int_a^b u(t)dt \succ_{\mathscr{K}^n} 0$.*

Proof. (a) Consider the partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$ with $t_k = a + k(b-a)/n$ and some $\bar{t}_k \in [t_{k-1}, t_k]$, we have

$$\int_a^b u(t)dt = \lim_{n\to\infty} \sum_{k=1}^{n} u(\bar{t}_k)\frac{b-a}{n} \succeq_{\mathscr{K}^n} 0$$

because $u(t) \succeq_{\mathscr{K}^n} 0$ and $\mathscr{K}^n$ is closed.

(b) For convenience, we write $u(t) = (u_1(t), u_2(t)) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and let

$$\begin{aligned}\bar{u}(t) &= (\|u_2(t)\|, u_2(t)), \\ \tilde{u}(t) &= (u_1(t) - \|u_2(t)\|, \mathbf{0}).\end{aligned}$$

Then, we have

$$u(t) = \bar{u}(t) + \tilde{u}(t) \quad \text{and} \quad \begin{cases} \bar{u}(t) \succeq_{\mathscr{K}^n} 0, \\ u_1(t) - \|u_2(t)\| > 0. \end{cases}$$

Note that $\int_a^b \tilde{u}(t)dt = (\int_a^b (u_1(t) - \|u_2(t)\|)dt, \mathbf{0}) \succ_{\mathscr{K}^n} 0$ since $u_1(t) - \|u_2(t)\| > 0$. This together with $\int_a^b \bar{u}(t)dt \succeq_{\mathscr{K}^n} 0$ (from (i)) yields that

$$\int_a^b u(t)dt = \int_a^b \bar{u}(t)dt + \int_a^b \tilde{u}(t)dt \succ_{\mathscr{K}^n} 0.$$

Thus, the proof is complete. □

In general, it is not hard to have an extension of Lemma 5 as below.

PROPOSITION 2. Suppose that $u(t) : \mathbb{R} \to \mathbb{R}^n$ and $v(t) : \mathbb{R} \to \mathbb{R}^n$ are integrable on $[a,b]$.

(a) If $u(t) \succeq_{\mathscr{K}^n} v(t)$ for any $t \in [a,b]$, then $\int_a^b u(t)dt \succeq_{\mathscr{K}^n} \int_a^b v(t)$.

(b) If $u(t) \succ_{\mathcal{K}^n} v(t)$ for any $t \in [a,b]$, then $\int_a^b u(t)dt \succ_{\mathcal{K}^n} \int_a^b v(t)$.

THEOREM 2. *Let $A(x,y)$, $G(x,y)$, and $L(x,y)$ be defined as in (7), (10), and (11), respectively. For any $x \succ_{\mathcal{K}^n} 0$, $y \succ_{\mathcal{K}^n} 0$, there holds*

$$G(x,y) \preceq_{\mathcal{K}^n} L(x,y) \preceq_{\mathcal{K}^n} A(x,y),$$

and hence $L(x,y)$ is an SOC mean.

Proof. (i) To verify the first inequality, we first note that

$$G(x,y) = P(x^{\frac{1}{2}})(P(x^{-\frac{1}{2}})y)^{\frac{1}{2}} = \int_0^1 P(x^{\frac{1}{2}})(P(x^{-\frac{1}{2}})y)^{\frac{1}{2}}dt.$$

Let $s = P(x^{-\frac{1}{2}})y = \lambda_1 u_s^{(1)} + \lambda_2 u_s^{(2)}$. Then, we have

$$\begin{aligned}
& L(x,y) - G(x,y) \\
= & \int_0^1 P(x^{\frac{1}{2}})(P(x^{-\frac{1}{2}})y)^t \, dt - P(x^{\frac{1}{2}})(P(x^{-\frac{1}{2}})y)^{\frac{1}{2}} \\
= & \int_0^1 P(x^{\frac{1}{2}}) \left(\lambda_1^t u_s^{(1)} + \lambda_2^t u_s^{(2)}\right) \, dt - P(x^{\frac{1}{2}}) \left(\sqrt{\lambda_1} u_s^{(1)} + \sqrt{\lambda_2} u_s^{(2)}\right) \\
= & \left[\int_0^1 \lambda_1^t dt\right] P(x^{\frac{1}{2}}) u_s^{(1)} + \left[\int_0^1 \lambda_2^t dt\right] P(x^{\frac{1}{2}}) u_s^{(2)} - P(x^{\frac{1}{2}}) \left(\sqrt{\lambda_1} u_s^{(1)} + \sqrt{\lambda_2} u_s^{(2)}\right) \\
= & \left[\frac{\lambda_1 - 1}{\ln \lambda_1 - \ln 1} - \sqrt{\lambda_1}\right] P(x^{\frac{1}{2}}) u_s^{(1)} + \left[\frac{\lambda_2 - 1}{\ln \lambda_2 - \ln 1} - \sqrt{\lambda_2}\right] P(x^{\frac{1}{2}}) u_s^{(2)} \\
= & [L(\lambda_1, 1) - G(\lambda_1, 1)] P(x^{\frac{1}{2}}) u_s^{(1)} + [L(\lambda_2, 1) - G(\lambda_2, 1)] P(x^{\frac{1}{2}}) u_s^{(2)} \\
\succeq_{\mathcal{K}^n} & \; 0.
\end{aligned}$$

where last inequality holds by (1) and $P(x^{\frac{1}{2}}) u_s^{(i)} \in \mathcal{K}^n$. Thus, we obtain the first inequality.

(ii) To see the second inequality, we let $s = P(x^{-\frac{1}{2}})y - e$. Then, we have $s \succeq_{\mathcal{K}^n} -e$, and applying Lemma 4 gives

$$\left(e + P(x^{-\frac{1}{2}})y - e\right)^t \preceq_{\mathcal{K}^n} e + t\left[P(x^{-\frac{1}{2}})y - e\right],$$

which is equivalent to

$$0 \preceq_{\mathcal{K}^n} (1-t)e + t\left[P(x^{-\frac{1}{2}})y\right] - \left(P(x^{-\frac{1}{2}})y\right)^t.$$

Since $P(x^{\frac{1}{2}})$ is invariant on $\mathcal{K}^n$, we have

$$\begin{aligned}
0 \preceq_{\mathcal{K}^n} & \; P(x^{\frac{1}{2}}) \left((1-t)e + t\left[P(x^{-\frac{1}{2}})y\right] - \left(P(x^{-\frac{1}{2}})y\right)^t\right) \\
= & \; (1-t)x + ty - x\#_t y
\end{aligned}$$

Hence, by Lemma 5, we obtain

$$L(x,y) = \int_0^1 x\#_t y\, dt \preceq_{\mathcal{K}^n} \int_0^1 [(1-t)x + ty]\, dt = A(x,y).$$

The proof is complete. $\square$

Finally, for SOC quadratic mean, it is natural to consider the following

$$Q(x,y) := \left(\frac{x^2+y^2}{2}\right)^{1/2}.$$

It is easy to verify $A(x,y) \preceq_{\mathcal{K}^n} Q(x,y)$. However, $Q(x,y)$ does not satisfy the property (ii) mentioned in the definition of SOC mean. Indeed, taking $x = \begin{bmatrix} 31 \\ 10 \\ -20 \end{bmatrix} \in \mathcal{K}^n$ and $y = \begin{bmatrix} 10 \\ 9 \\ 0 \end{bmatrix} \in \mathcal{K}^n$, it is obvious that $x \succ_{\mathcal{K}^n} y$. In addition, by simple calculation, we have

$$\left(\frac{x^2+y^2}{2}\right)^{1/2} = \begin{bmatrix} s \\ \frac{400}{2s} \\ \frac{-620}{2s} \end{bmatrix} \approx \begin{bmatrix} 24.30 \\ 8.23 \\ -12.76 \end{bmatrix},$$

where $s = \sqrt{\frac{1}{2}\left(821 + \sqrt{821^2 - (400^2 + 620^2)}\right)} \approx 24.30$. However,

$$x \vee y - \left(\frac{x^2+y^2}{2}\right)^{1/2} \approx \begin{bmatrix} 6.7 \\ 1.77 \\ -7.24 \end{bmatrix}$$

is not in $\mathcal{K}^n$. Hence, this definition of $Q(x,y)$ cannot officially serve as a SOC mean.

To sum up, we already have the following inequalities

$$x \wedge y \preceq_{\mathcal{K}^n} H(x,y) \preceq_{\mathcal{K}^n} G(x,y) \preceq_{\mathcal{K}^n} L(x,y) \preceq_{\mathcal{K}^n} A(x,y) \preceq_{\mathcal{K}^n} x \vee y,$$

but we do not have SOC quadratic mean. Nevertheless, we still can generalize all the means inequalities as in (1) to SOC setting when the dimension is 2. To see this, the Jordan product on second-order cone of order 2 satisfies the associative law and closedness such that the geometric mean

$$G(x,y) = x^{1/2} \circ y^{1/2}$$

and the logarithmic mean

$$L(x,y) = \int_0^1 x^{1-t} \circ y^t\, dt$$

are well-defined (note this is true only when $n = 2$) and coincide with the definition (10), (11). Then, the following inequalities

$$x \wedge y \preceq_{\mathcal{K}^2} H(x,y) \preceq_{\mathcal{K}^2} G(x,y) \preceq_{\mathcal{K}^2} L(x,y) \preceq_{\mathcal{K}^2} A(x,y) \preceq_{\mathcal{K}^2} Q(x,y) \preceq_{\mathcal{K}^2} x \vee y.$$

hold as well.

4. Two trace inequalities

In this section, we build up two trace inequalities based on the aforementioned SOC means. To this end, we recall a technical lemma, which is explored in [12].

LEMMA 6. ([12, Proposition 2.1]) *For any $x,y \in \mathbb{R}^n$, the following hold.*

(a) *If $x \preceq_{\mathcal{K}^n} y$, then* $\mathrm{tr}(x) \leqslant \mathrm{tr}(y)$.

(b) $\mathrm{tr}(x \circ y) \leqslant \lambda_1(x)\lambda_1(y) + \lambda_2(x)\lambda_2(y)$.

By applying Lemma 6(i), we immediately obtain one trace inequality for SOC mean.

THEOREM 3. *Let $A(x,y)$, $H(x,y)$, $G(x,y)$ and $L(x,y)$ be defined as in (7)–(8), (10)–(11), respectively. For any $x \succ_{\mathcal{K}^n} 0$, $y \succ_{\mathcal{K}^n} 0$, there holds*

$$\mathrm{tr}(x \wedge y) \leqslant \mathrm{tr}(H(x,y)) \leqslant \mathrm{tr}(G(x,y)) \leqslant \mathrm{tr}(L(x,y)) \leqslant \mathrm{tr}(A(x,y)) \leqslant \mathrm{tr}(x \vee y).$$

At the end of this section, we establish the SOC trace version of Young's inequality. In 1995, Ando [1] showed the singular value version of Young's inequality that

$$s_j(AB) \leqslant s_j\left(\frac{A^p}{p} + \frac{B^q}{q}\right) \qquad \text{for all } 1 \leqslant j \leqslant n,$$

where A and B are positive definite matrices, and $1/p + 1/q = 1$. Originally, we try to derive the eigenvalue version of Young's inequality in the setting of second-order cone:

$$\lambda_j(x \circ y) \leqslant \lambda_j\left(\frac{x^p}{p} + \frac{y^q}{q}\right), \quad j = 1,2.$$

But, it is very complicated to derive and prove the inequalities directly. Eventually, we give up. Instead, we establish that SOC trace version of Young's inequality as below.

THEOREM 4. *For any $x,y \in \mathbb{R}^n$, there holds* $\mathrm{tr}(x \circ y) \leqslant \mathrm{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right)$.

Proof. First, we note $x \circ y = \left(x_1y_1 + x_2^Ty_2, x_1y_2 + y_1x_2\right)$ and $\dfrac{|x|^p}{p} + \dfrac{|y|^q}{q} = (w_1, w_2)$ where

$$\begin{aligned} w_1 &= \frac{|\lambda_1(x)|^p + |\lambda_2(x)|^p}{2p} + \frac{|\lambda_1(y)|^q + |\lambda_2(y)|^q}{2q}, \\ w_2 &= \frac{|\lambda_2(x)|^p - |\lambda_1(x)|^p}{2p}\frac{x_2}{\|x_2\|} + \frac{|\lambda_2(y)|^q - |\lambda_1(y)|^q}{2q}\frac{y_2}{\|y_2\|}. \end{aligned}$$

Then, the desired result follows by

$$\begin{aligned}
\operatorname{tr}(x\circ y) &\leqslant \lambda_1(x)\lambda_1(y)+\lambda_2(x)\lambda_2(y)\\
&\leqslant |\lambda_1(x)\lambda_1(y)|+|\lambda_2(x)\lambda_2(y)|\\
&\leqslant \left(\frac{|\lambda_1(x)|^p}{p}+\frac{|\lambda_1(y)|^q}{q}\right)+\left(\frac{|\lambda_2(x)|^p}{p}+\frac{|\lambda_2(y)|^q}{q}\right)\\
&= \operatorname{tr}\left(\frac{|x|^p}{p}+\frac{|y|^q}{q}\right),
\end{aligned}$$

where the last inequality holds by Young's inequality on real number. □

REFERENCES

[1] T. ANDO, *Matrix Young inequalities*, Operator Theory: Advances and Applications **75** (1995), 33–38.

[2] J. S. AUJLA AND H. L. VASUDEVA, *Convex and monotone operator functions*, Annales Polonici Mathematici **62** (1995), 1–11.

[3] R. BHATIA, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[4] R. BHATIA, *Positive definite matrices*, Princeton university press, 2005.

[5] R. BHATIA, *Interpolating the arithmetic-geometric mean inequality and its operator version*, Linear Algebra and Its Applications **413** (2006), 355–363.

[6] R. BHATIA AND C. DAVIS, *More matrix forms of the arithmetic–geometric mean inequality*, SIAM Journal on Matrix Analysis and Applications **14** (1993), 132–136.

[7] R. BHATIA AND F. KITTANEH, *Notes on matrix arithmetic–geometric mean inequalities*, Linear Algebra and Its Applications **308** (2000), 203–211.

[8] P. BHATIA AND K.P. PARTHASARATHY, *Positive definite functions and operator inequalities*, Bulletin of the London Mathematical Society **32** (2000), 214–228.

[9] J.-C. BOURIN AND F. HIAI, *Jensen and Minkowski inequalities for operator means and anti-norms*, Linear Algebra and its Applications **456** (2014), 22–53

[10] P. S. BULLEN, *Handbook of Means and Their Inequalities*, Mathematics and Its Applications **560**, Kluwer Academic Publishers, 2003.

[11] B. C. CARLSON, *Some inequalities for hypergeometric functions*, Proceedings of the American Mathematical Society **17** (1966), 32–39.

[12] J.-S. CHEN, *The convex and monotone functions associated with second-order cone*, Optimization **55** (2006), 363–385.

[13] J.-S. CHEN, X. CHEN, S.-H. PAN, AND J. ZHANG, *Some characterizations for SOC-monotone and SOC-convex functions*, Journal of Global Optimization **45** (2009), 259–279.

[14] J.-S. CHEN, X. CHEN, AND P. TSENG, *Analysis of nonsmooth vector-valued functions associated with second-order cones*, Mathematical Programming **101** (2004), 95–117.

[15] J. FARAUT AND A. KORÁNYI, *Analysis on Symmetric Cones*, Oxford Mathematical Monographs, New York: Oxford University Press, 1994.

[16] M. FUKUSHIMA, Z.-Q. LUO, AND P. TSENG, *Smoothing functions for second-order-cone complementarity problems*, SIAM Journal on Optimization **12** (2002), 436–460.

[17] M. HAJJA, P. S. BULLEN, J. MATKOWSKI, E. NEUMAN, AND S. SIMIC, *Means and Their Inequalities*, International Journal of Mathematics and Mathematical Sciences **2013** 2013.

[18] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, 1985.

[19] F. KITTANEH AND M. KRNIC, *Refined Heinz operator inequalities*, Linear Multilinear Algebra **61** (2013), 1148–1157.

[20] A. KORÁNYI, *Monotone functions on formally real Jordan algebras*, Mathematische Annalen **269** (1984), 73–76.

[21] H. LEE AND Y. LIM, *Metric and spectral geometric means on symmetric cones*, Kyungpook Mathematical Journal **47**, 1 (2007), 133–150.

[22] Y. LIM, *Geometric means on symmetric cones*, Archiv der Mathematik **75**, 1 (2000), 39–45.
[23] Y. LIM, *Applications of geometric means on symmetric cones*, Mathematische Annalen **319**, 3 (2001), 457–468.
[24] E. NEUMAN, *The weighted logarithmic mean*, Journal of Mathematical Analysis and Applications, **188**, 3 (1994), 885–900.

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