

## LIMITING CASE HARDY INEQUALITIES ON THE SPHERE

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*Abstract.* We give sharp limiting case Hardy inequalities on the sphere  $\mathbb{S}^2$  and show that their optimal constants are unattainable by any  $f \in H^1(\mathbb{S}^2) \setminus \{0\}$ . The singularity of the problem is related to the geodesic distance from a point on the sphere.

### 1. Introduction

The classical Hardy inequality

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \quad (1)$$

is valid in dimensions  $n \geq 3$  for all functions  $u \in H^1(\mathbb{R}^n)$  ([1]). It obviously fails on  $\mathbb{R}^2$  as the right hand side of (1) no longer makes sense. In order to obtain a version of (1) in the critical case  $n = 2$  on bounded domains, a logarithmic weight can be introduced to tame the singularity. In [2, 4–8, 10], for instance, inequalities of the type

$$\int_B |\nabla u|^n dx \geq C_n(B) \int_B \frac{|u|^n}{|x|^n \left(\log \frac{1}{|x|}\right)^n} dx$$

were analysed for  $u \in W_0^{1,n}(B)$  where  $B$  is the unit ball in  $\mathbb{R}^n$ .

Let  $n \geq 3$  and  $\mathbb{S}^n$  be the unit sphere equipped with its Lebesgue surface measure  $\sigma_n$  in  $\mathbb{R}^{n+1}$ . Denote by  $d(\cdot, p) : \mathbb{S}^n \rightarrow [0, \pi]$  the geodesic distance from  $p \in \mathbb{S}^n$ , and by  $\nabla_{\mathbb{S}^n}$  the gradient on  $\mathbb{S}^n$ . Recently, Xiao [11] proved that if  $f \in C^\infty(\mathbb{S}^n)$  then

$$\bar{c}_n \int_{\mathbb{S}^n} f^2 d\sigma_n + \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}^n} f|^2 d\sigma_n \geq c_n^2 \int_{\mathbb{S}^n} \left( \frac{f^2}{d(x,p)^2} + \frac{f^2}{(\pi - d(x,p))^2} \right) d\sigma_n \quad (2)$$

with  $\bar{c}_n = \left(\frac{4}{3} + \frac{1}{\pi^2}\right) c_n^2 + c_n$ ,  $c_n = \frac{n-2}{2}$ . It was also shown in [11] that the constant  $c_n$  in (2) is sharp in the sense that

$$c_n^2 = \inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{D_n(f)}{\int_{\mathbb{S}^n} \frac{f^2}{d(x,p)^2} d\sigma_n} = \inf_{f \in C^\infty(\mathbb{S}^n) \setminus \{0\}} \frac{D_n(f)}{\int_{\mathbb{S}^n} \frac{f^2}{(\pi - d(x,p))^2} d\sigma_n}$$

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where

$$D_n(f) := \bar{c}_n \int_{\mathbb{S}^n} f^2 d\sigma_n + \int_{\mathbb{S}^n} |\nabla_{\mathbb{S}^n} f|^2 d\sigma_n, \quad f \in C^\infty(\mathbb{S}^n).$$

Recently, Xiao’s result was extended to the case  $p \neq 2$  in [9].

We prove  $L^2$  Hardy inequalities with optimal constants on the sphere  $\mathbb{S}^2$  in  $\mathbb{R}^3$ . This is a critical exponent case as the integral  $\int_{\mathbb{S}^2} \theta^{-1+\lambda} d\sigma_2$ , where  $\theta$  is the polar angle, diverges for  $\lambda \leq -1$ . We also argue the lack of maximizers for our inequalities. Our approach denies the possibility of an equality in Xiao’s inequality (2) as well.

### 2. Preliminaries

A point on the sphere  $\mathbb{S}^2$  will have the standard spherical coordinate parametrization  $(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$  where  $\theta \in [0, \pi]$  refers to the polar angle and  $\varphi \in [0, 2\pi[$  is the azimuthal angle. Then the surface measure induced by the Lebesgue measure on  $\mathbb{R}^3$  is  $d\sigma_2 = \sin \theta d\theta d\varphi$ , the gradient and the Laplace-Beltrami operator, respectively, are given by

$$\nabla_{\mathbb{S}^2} = \hat{\theta} \frac{\partial}{\partial \theta} + \hat{\varphi} \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi}, \quad \Delta_{\mathbb{S}^2} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}.$$

Here  $\hat{\theta}, \hat{\varphi}$  denote the orthogonal three-dimensional unit vectors in the direction where  $\theta, \varphi$  increase, respectively. The Sobolev space  $H^1(\mathbb{S}^2)$  is the completion of  $C^\infty(\mathbb{S}^2)$  in the norm

$$\|f\|_{H^1(\mathbb{S}^2)} := \left( \|f\|_{L^2(\mathbb{S}^2)}^2 + \|\nabla f\|_{L^2(\mathbb{S}^2)}^2 \right)^{\frac{1}{2}}.$$

In order to find the geodesic distance  $d(x, p)$  from a point  $x \in \mathbb{S}^2$  to a given a point  $p \in \mathbb{S}^2$ , we rotate the axes, if necessary, to put  $p$  on the zenith direction then place the great circle passing through  $p$  and  $x$  in the azimuth reference direction so that we have  $d(x, p) = \theta$ .

For simplicity, we henceforth denote  $d\sigma_2, \nabla_{\mathbb{S}^2}$  and  $\Delta_{\mathbb{S}^2}$  by  $d\sigma, \nabla$  and  $\Delta$ , respectively.

### 3. Main results

Let  $\phi : ]0, \pi[ \rightarrow [1, \infty[$  be defined by  $\phi(t) := \log(\pi e/t)$ ,  $\psi : ]0, \pi[ \rightarrow [1 + \log \pi, \infty[$  be such that  $\psi(t) := \phi(\sin t)$ , and  $\rho_\phi(t) := t\phi(t)$ . Let  $A > 0$ . Denote by  $S, T_A$ , and  $\mathcal{Q}(\cdot; \phi)$  the positive nonlinear functionals on  $H^1(\mathbb{S}^2)$  given by

$$\begin{aligned} S(f) &:= \int_{\mathbb{S}^2} |\hat{\theta} \cdot \nabla f|^2 d\sigma + \frac{1}{2\pi^2} \int_{\mathbb{S}^2} f^2 d\sigma, \\ T_A(f) &:= \int_{\mathbb{S}^2} |\nabla f|^2 d\sigma_2 + \frac{A}{4} \int_{\mathbb{S}^2} f^2 d\sigma_2, \\ \mathcal{Q}(f; \phi) &:= \frac{1}{4} \int_{\mathbb{S}^2} \left( \frac{f^2}{\rho_\phi^2(d(x, p))} + \frac{f^2}{\rho_\phi^2(\pi - d(x, p))} \right) d\sigma_2. \end{aligned}$$

**THEOREM 1.** Assume that  $f \in H^1(\mathbb{S}^2)$ . Then there exists constants  $A, B > 0$ , independent of  $f$ , such that

$$Q(f; \phi) \leq T_A(f), \tag{3}$$

$$Q(f; \psi) \leq T_B(f). \tag{4}$$

Both inequalities (3) and (4) are optimal, but an equality is impossible in either one:

**THEOREM 2.**

$$\sup_{f \in H^1(\mathbb{S}^2) \setminus \{0\}} \frac{Q(f; \phi)}{T_A(f)} = 1, \tag{5}$$

$$\sup_{f \in H^1(\mathbb{S}^2) \setminus \{0\}} \frac{Q(f; \psi)}{T_B(f)} = 1. \tag{6}$$

**THEOREM 3.** There does not exist  $f \in H^1(\mathbb{S}^2) \setminus \{0\}$  such that  $Q(f; \phi) = T_A(f)$ , or  $Q(f; \psi) = T_B(f)$ .

A variant of the above-mentioned results follows via a different approach:

**THEOREM 4.** Let  $f \in H^1(\mathbb{S}^2)$ . Then

$$\frac{1}{4} \int_{\mathbb{S}^2} \frac{f^2}{\rho_\phi^2(d(x,p))} d\sigma \leq S(f) + \frac{1}{2\pi} \int_{\mathbb{S}^2} \frac{f^2}{\pi - d(x,p)} d\sigma, \tag{7}$$

$$\frac{1}{4} \int_{\mathbb{S}^2} \frac{f^2}{\rho_\phi^2(\pi - d(x,p))} d\sigma \leq S(f) + \frac{1}{2\pi} \int_{\mathbb{S}^2} \frac{f^2}{d(x,p)} d\sigma. \tag{8}$$

Moreover

$$\sup_{f \in H^1(\mathbb{S}^2) \setminus \{0\}} \frac{\frac{1}{4} \int_{\mathbb{S}^2} \frac{f^2}{\rho_\phi^2(d(x,p))} d\sigma}{S(f) + \frac{1}{2\pi} \int_{\mathbb{S}^2} \frac{f^2}{\pi - d(x,p)} d\sigma} = \sup_{f \in H^1(\mathbb{S}^2) \setminus \{0\}} \frac{\frac{1}{4} \int_{\mathbb{S}^2} \frac{f^2}{\rho_\phi^2(\pi - d(x,p))} d\sigma}{S(f) + \frac{1}{2\pi} \int_{\mathbb{S}^2} \frac{f^2}{d(x,p)} d\sigma} = 1, \tag{9}$$

and the suprema in (9) are not attained in  $H^1(\mathbb{S}^2) \setminus \{0\}$ .

### 4. Proof of Theorem 1

*Proof.* Let  $f \in C^\infty(\mathbb{S}^2)$ . Notice that  $\psi > 1$  and write  $f(\theta, \varphi) = \sqrt{\psi(\theta)}g(\theta, \varphi)$ . We have

$$\begin{aligned} |\nabla f|^2 &= |\psi^{\frac{1}{2}} \nabla g + g \nabla \psi^{\frac{1}{2}}|^2 \\ &= \psi |\nabla g|^2 + \langle \psi^{\frac{1}{2}} \nabla g, g \psi^{-\frac{1}{2}} \nabla \psi \rangle + \left| \frac{1}{2} \psi^{-\frac{1}{2}} \nabla \psi \right|^2 g^2 \\ &= \psi |\nabla g|^2 + \frac{1}{2} \langle \nabla \psi, \nabla g^2 \rangle + \frac{1}{4} \frac{1}{\psi} |\nabla \psi|^2 g^2. \end{aligned} \tag{10}$$

Integrating both sides of (10) over  $\mathbb{S}^2$  we get

$$\begin{aligned} \int_{\mathbb{S}^2} |\nabla f|^2 d\sigma &= \int_{\mathbb{S}^2} \left( \psi |\nabla g|^2 + \frac{1}{2} \langle \nabla \psi, \nabla g^2 \rangle + \frac{1}{4} \frac{1}{\psi} |\nabla \psi|^2 g^2 \right) d\sigma \\ &\geq \frac{1}{4} \int_{\mathbb{S}^2} \frac{1}{\psi} |\nabla \psi|^2 g^2 d\sigma + \frac{1}{2} \int_{\mathbb{S}^2} \langle \nabla \psi, \nabla g^2 \rangle d\sigma \end{aligned} \tag{11}$$

$$= \frac{1}{4} \int_{\mathbb{S}^2} \frac{1}{\psi} |\psi'|^2 g^2 d\sigma - \frac{1}{2} \int_{\mathbb{S}^2} g^2 \Delta \psi d\sigma \tag{12}$$

by partial integration over the closed manifold  $\mathbb{S}^2$ . Calculating, we find

$$\Delta \psi = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \psi \right) = 1. \tag{13}$$

Returning  $g$  to  $f/\sqrt{\psi}$  and substituting for  $\Delta \psi$  from (13) into (12), we obtain

$$\int_{\mathbb{S}^2} |\nabla f|^2 d\sigma \geq \frac{1}{4} \int_{\mathbb{S}^2} \frac{f^2 \cos^2 \theta}{\psi^2 \sin^2 \theta} d\sigma - \frac{1}{2} \int_{\mathbb{S}^2} \frac{f^2}{\psi} d\sigma. \tag{14}$$

Adding the finite integral  $\frac{1}{4} \int_{\mathbb{S}^2} \left( \frac{1}{\theta^2 \phi^2(\theta)} + \frac{1}{(\pi - \theta)^2 \phi^2(\pi - \theta)} \right) f^2 d\sigma$  to both sides of (14) transforms it into the inequality

$$\begin{aligned} \frac{1}{4} \int_{\mathbb{S}^2} \left( \frac{1}{\theta^2 \phi^2(\theta)} + \frac{1}{(\pi - \theta)^2 \phi^2(\pi - \theta)} \right) f^2 d\sigma \\ \leq \int_{\mathbb{S}^2} |\nabla f|^2 d\sigma + \frac{1}{4} \int_{\mathbb{S}^2} F(\theta) f^2 d\sigma, \end{aligned} \tag{15}$$

where

$$F(t) := \frac{1}{t^2 \phi^2(t)} + \frac{1}{(\pi - t)^2 \phi^2(\pi - t)} - \frac{\cos^2 t}{\sin^2 t} \frac{1}{\phi^2(\sin t)} + \frac{2}{\phi(\sin t)}.$$

Obviously,  $F$  is continuous on  $]0, \pi[$  and, as expected from the facts that  $\phi(t) \rightarrow +\infty$  when  $t \rightarrow 0^+$ ,  $\sin t = t + o(t)$  as  $t \rightarrow 0$ , it turns out

$$\lim_{t \rightarrow 0^+} F(t) = \lim_{t \rightarrow \pi^-} F(t) = \frac{1}{\pi^2}.$$

Hence,  $F$  can be extended to a uniformly continuous, consequently a bounded, function on  $[0, \pi]$ . Noting this in (15) implies (3). Direct computation also shows

$$A = \sup_{[0, \pi]} |F| = F\left(\frac{\pi}{2}\right) = \frac{2}{1 + \log \pi} + \frac{8}{(1 + \log 2)^2} \frac{1}{\pi^2}.$$

To prove (4), we add to both sides of (14) the well-defined integral

$$\frac{1}{4} \int_{\mathbb{S}^2} \left( \frac{1}{\theta^2} + \frac{1}{(\pi - \theta)^2} \right) \frac{f^2}{\psi^2(\theta)} d\sigma.$$

We then obtain the following analogue of (15):

$$\begin{aligned} \frac{1}{4} \int_{\mathbb{S}^2} \left( \frac{1}{\theta^2} + \frac{1}{(\pi - \theta)^2} \right) \frac{f^2}{\psi^2(\theta)} d\sigma \\ \leq \int_{\mathbb{S}^2} |\nabla f|^2 d\sigma + \frac{1}{4} \int_{\mathbb{S}^2} G(\theta) f^2 d\sigma, \end{aligned} \tag{16}$$

where

$$\begin{aligned} G(t) &:= \frac{M(t)}{\psi^2(t)} + \frac{2}{\psi(t)}, \\ M(t) &:= \frac{1}{t^2} + \frac{1}{(\pi - t)^2} - \frac{\cos^2 t}{\sin^2 t}. \end{aligned} \tag{17}$$

Once the boundedness of  $G$  is ensured, we see that (16) yields the inequality (4). Evidently,  $G$  has the same features as  $F$ . Since

$$\lim_{\theta \rightarrow 0} M(\theta) = \lim_{\theta \rightarrow \pi} M(\theta) = \frac{2}{3} + \frac{1}{\pi^2}, \quad \lim_{\theta \rightarrow 0^+} \psi(t) = \lim_{\theta \rightarrow \pi^-} \psi(t) = +\infty \tag{18}$$

then  $M \in C[0, \pi]$ , and  $\lim_{t \rightarrow 0^+} G(t) = \lim_{t \rightarrow \pi^-} G(t) = 0$ , which makes  $G$  bounded on  $[0, \pi]$ . Moreover

$$B = \sup_{[0, \pi]} |G| = G\left(\frac{\pi}{2}\right) = \frac{2}{1 + \log \pi} + \frac{8}{(1 + \log \pi)^2 \pi^2}. \quad \square$$

### 5. Proof of Theorem 2

*Proof.* First, we would like to define the weak Laplace-Beltrami gradient of a function  $f \in L^1(\mathbb{S}^2)$ . Suppose  $f \in C^\infty(\mathbb{S}^2)$  and  $v(\theta, \varphi) = v_\theta(\theta, \varphi)\hat{\theta} + v_\varphi(\theta, \varphi)\hat{\varphi}$  with  $v_\theta, v_\varphi \in C^\infty(\mathbb{S}^2)$ . Then

$$\begin{aligned} \int_{\mathbb{S}^2} \frac{\partial f}{\partial \theta} v_\theta d\sigma &= \int_{\mathbb{S}^2} \nabla f \cdot \hat{\theta} v_\theta d\sigma = - \int_{\mathbb{S}^2} f \nabla \cdot (v_\theta \hat{\theta}) d\sigma, \\ \int_{\mathbb{S}^2} \frac{1}{\sin \theta} \frac{\partial f}{\partial \varphi} v_\varphi d\sigma &= \int_{\mathbb{S}^2} \nabla f \cdot \hat{\varphi} v_\varphi d\sigma = - \int_{\mathbb{S}^2} f \nabla \cdot (v_\varphi \hat{\varphi}) d\sigma. \end{aligned}$$

Adding these identities we get

$$\int_{\mathbb{S}^2} \nabla f \cdot V d\sigma = - \int_{\mathbb{S}^2} f \nabla \cdot V d\sigma \tag{19}$$

for any vector field  $V \in C^\infty(\mathbb{S}^2 \rightarrow T(\mathbb{S}^2))$  where  $T(\mathbb{S}^2)$  is the tangent bundle of the smooth manifold  $\mathbb{S}^2$ . Motivated by (19),  $f$  is weakly differentiable if there is a vector field  $\vartheta_f \in L^1(\mathbb{S}^2 \rightarrow T(\mathbb{S}^2))$  such that

$$\int_{\mathbb{S}^2} \vartheta_f \cdot V d\sigma = - \int_{\mathbb{S}^2} f \nabla \cdot V d\sigma, \quad \forall V \in C^\infty(\mathbb{S}^2 \rightarrow T(\mathbb{S}^2)). \tag{20}$$

This, unique up to a set of zero measure, vector field  $\vartheta_f$  is the weak surface gradient of  $f$ . According to ([3], Proposition 3.2., page 15)

$$H^1(\mathbb{S}^2) = W^{1,2}(\mathbb{S}^2) := \{f \in L^2(\mathbb{S}^2) : |\vartheta_f| \in L^2(\mathbb{S}^2)\}.$$

We start with (5). By Theorem 1, it suffices to prove the existence of a sequence  $\{f_n\}_{n \geq 1}$  in  $H^1(\mathbb{S}^2)$  such that

$$\lim_{n \rightarrow \infty} \frac{Q(f_n; \phi)}{T_A(f_n)} = 1. \tag{21}$$

Consider the functions

$$f_n(\theta, \varphi) := \phi(\theta)^{\frac{1}{2} - \frac{1}{n}}. \tag{22}$$

The functions  $f_n$  are independent of  $\varphi$ , hence

$$\frac{Q(f_n; \phi)}{T_A(f_n)} = \frac{\int_0^\pi \frac{f_n^2 \sin \theta}{\theta^2 \phi^2(\theta)} d\theta + \int_0^\pi \frac{f_n^2 \sin \theta}{(\pi - \theta)^2 \phi^2(\pi - \theta)} d\theta}{4 \int_0^\pi \left(\frac{\partial f_n}{\partial \theta}\right)^2 \sin \theta d\theta + A \int_0^\pi f_n^2 \sin \theta d\theta} \tag{23}$$

where the derivative  $\partial f_n / \partial \theta$  is understood in the weak sense discussed above. Since  $\phi \in L^1_{\text{loc}}(\mathbb{R})$  and  $\phi \geq 1$  on  $[0, \pi]$ , then

$$\int_0^\pi f_n^2 \sin \theta d\theta = \int_0^\pi \phi(\theta)^{1 - \frac{2}{n}} \sin \theta d\theta \leq \int_0^\pi \phi(\theta) d\theta \approx 1. \tag{24}$$

Thus  $f_n \in L^2(\mathbb{S}^2)$  for all  $n \geq 1$ . Notice also that  $f_n$  is smooth on  $[0, \pi] \setminus \{0\}$  and its weak derivative

$$\frac{\partial f_n}{\partial \theta} = \frac{\frac{1}{n} - \frac{1}{2}}{\theta \phi^{\frac{1}{2} + \frac{1}{n}}}. \tag{25}$$

Therefore

$$\int_0^\pi \left(\frac{\partial f_n}{\partial \theta}\right)^2 \sin \theta d\theta = \frac{a_n}{4} \int_0^\pi \frac{1}{\theta \phi^{1 + \frac{2}{n}}} \frac{\sin \theta}{\theta} d\theta, \quad a_n := \left(1 - \frac{2}{n}\right)^2.$$

And since  $\int_0^\pi \frac{d\theta}{\theta \phi^{1 + \frac{2}{n}}} = \frac{n}{2}$ ,  $\sin \theta \leq \theta$ , then  $\partial f_n / \partial \theta \in L^2(\mathbb{S}^2)$  for all  $n \geq 1$ . Substituting for  $f_n$  from (22) and for  $\partial f_n / \partial \theta$  from (25) into (23) implies

$$\frac{Q(f_n; \phi)}{T_A(f_n)} = \frac{\alpha_n + \beta_n}{a_n \alpha_n + \gamma_n} = \frac{1}{a_n} \left(1 + \frac{\beta_n - \gamma_n / a_n}{\alpha_n + \gamma_n / a_n}\right) \tag{26}$$

where

$$\begin{aligned} \alpha_n &:= \int_0^\pi \frac{1}{\theta \phi^{1 + \frac{2}{n}}} \frac{\sin \theta}{\theta} d\theta, \\ \beta_n &:= \int_0^\pi \frac{\phi^{1 - \frac{2}{n}}(\theta) \sin \theta}{(\pi - \theta)^2 \phi^2(\pi - \theta)} d\theta, \\ \gamma_n &:= A \int_0^\pi \phi^{1 - \frac{2}{n}} \sin \theta d\theta. \end{aligned}$$

Observe that  $\lim_{n \rightarrow +\infty} a_n = 1$ . We shall show that, while  $\lim_{n \rightarrow +\infty} \alpha_n = +\infty$ , the sequences  $\{\beta_n\}_{n \geq 1}$  and  $\{\gamma_n\}_{n \geq 1}$  are both convergent. Using this in (26) proves (21).

Exploiting the continuity and positivity of  $\sin \theta / \left(\theta^2 \phi^{1+\frac{2}{n}}\right)$  on  $[\pi/2, \pi]$ , then applying the inequality  $\sin \theta / \theta \geq 2/\pi$  when  $0 \leq \theta \leq \pi/2$ , we obtain

$$\begin{aligned} \alpha_n &= \int_0^{\pi/2} \frac{1}{\theta \phi^{1+\frac{2}{n}}} \frac{\sin \theta}{\theta} d\theta + \int_{\pi/2}^{\pi} \frac{\sin \theta}{\theta^2 \phi^{1+\frac{2}{n}}} d\theta \\ &\geq \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{\theta \phi^{1+\frac{2}{n}}} d\theta = \frac{n}{\pi(1 + \log(2))^{\frac{2}{n}}}. \end{aligned} \tag{27}$$

This proves the divergence of  $\{\alpha_n\}$ . Next, by the dominated convergence theorem and (24) we readily find

$$\lim_{n \rightarrow +\infty} \gamma_n = A \lim_{n \rightarrow +\infty} \int_0^{\pi} \phi^{1-\frac{2}{n}}(\theta) \sin \theta d\theta = \int_0^{\pi} \phi(\theta) \sin \theta d\theta \lesssim 1.$$

Finally, since  $\theta \mapsto \sin \theta / \left((\pi - \theta)^2 \phi^2(\pi - \theta)\right) \in C([0, \pi/2])$ , then using the local integrability of  $\phi$  and the dominated convergence theorem again implies

$$\lim_{n \rightarrow \infty} \int_0^{\pi/2} \frac{\phi^{1-\frac{1}{n}}(\theta) \sin \theta}{(\pi - \theta)^2 \phi^2(\pi - \theta)} d\theta = \int_0^{\pi/2} \frac{\phi(\theta) \sin \theta}{(\pi - \theta)^2 \phi^2(\pi - \theta)} d\theta \lesssim 1. \tag{28}$$

Furthermore, since  $\phi \in C([\pi/2, \pi])$ , and  $\frac{\sin \theta}{\pi - \theta} = \frac{\sin(\pi - \theta)}{\pi - \theta} \leq 1$ , on  $[\pi/2, \pi]$ , then

$$\int_{\pi/2}^{\pi} \frac{\phi^{1-\frac{1}{n}}(\theta) \sin \theta}{(\pi - \theta)^2 \phi^2(\pi - \theta)} d\theta \lesssim \int_{\pi/2}^{\pi} \frac{d\theta}{(\pi - \theta) \phi^2(\pi - \theta)} \approx 1. \tag{29}$$

The convergence of  $\{\beta_n\}$  follows from (28) together with (29).

The proof of (6) shares the main idea of (5). The functions  $g_n(\theta, \varphi) := \psi(\theta)^{\frac{1}{2}-\frac{1}{n}} \in L^2(\mathbb{S}^2)$ ,  $n \geq 1$ , and satisfy  $\lim_{n \rightarrow \infty} \frac{Q(g_n; \Psi)}{T_B(g_n)} = 1$ . Indeed, we have

$$\begin{aligned} \frac{Q(g_n; \Psi)}{T_B(g_n)} &= \frac{\int_0^{\pi} \frac{g_n^2 \sin \theta}{\theta^2 \psi^2(\theta)} d\theta + \int_0^{\pi} \frac{g_n^2 \sin \theta}{(\pi - \theta)^2 \psi^2(\pi - \theta)} d\theta}{4 \int_0^{\pi} \left(\frac{\partial g_n}{\partial \theta}\right)^2 \sin \theta d\theta + B \int_0^{\pi} g_n^2 \sin \theta d\theta} \\ &= \frac{\tilde{\alpha}_n}{a_n \tilde{\alpha}_n + \tilde{\beta}_n} = \frac{1}{a_n} \left(1 - \frac{\tilde{\beta}_n/a_n}{\tilde{\alpha}_n + \tilde{\beta}_n/a_n}\right) \end{aligned}$$

where

$$\begin{aligned} \tilde{\alpha}_n &:= \int_0^{\pi} \frac{\sin \theta d\theta}{\theta^2 \psi^{1+\frac{2}{n}}} + \int_0^{\pi} \frac{\sin \theta d\theta}{(\pi - \theta)^2 \psi^{1+\frac{2}{n}}} = 2 \int_0^{\pi} \frac{\sin \theta d\theta}{\theta^2 \psi^{1+\frac{2}{n}}}, \\ \tilde{\beta}_n &:= B \int_0^{\pi} \psi^{1-\frac{2}{n}} \sin \theta d\theta - a_n \int_0^{\pi} M(\theta) \frac{\sin \theta}{\psi^{1+\frac{2}{n}}} d\theta. \end{aligned}$$

Similarly to (27), we have

$$\begin{aligned} \tilde{\alpha}_n &= 2 \int_0^1 \frac{\sin \theta}{\theta^2} \frac{1}{\psi^{1+\frac{2}{n}}} d\theta + 2 \int_1^\pi \frac{\sin \theta}{\theta^2} \frac{1}{\psi^{1+\frac{2}{n}}} d\theta \\ &\geq 2 \int_0^1 \frac{\sin \theta}{\theta^2} \frac{1}{\psi^{1+\frac{2}{n}}} d\theta = 2 \int_0^1 \frac{\sin^2 \theta}{\theta^2 \cos \theta} \frac{1}{\psi^{1+\frac{2}{n}}} \frac{\cos \theta}{\sin \theta} d\theta \\ &\geq \frac{8}{\pi^2} \int_0^1 \frac{1}{\psi^{1+\frac{2}{n}}} \frac{\cos \theta}{\sin \theta} d\theta = \frac{4n}{\pi^2} \frac{1}{(1 + \log \pi)^{\frac{2}{n}}}. \end{aligned}$$

Hence  $\lim_{n \rightarrow \infty} \tilde{\alpha}_n = \infty$ . Recall from (17) and (18) that  $M \in C([0, \pi])$ . Also, since  $\psi \in L^1_{\text{loc}}(\mathbb{R})$ ,  $\psi > 1$  uniformly, then  $\lim_{n \rightarrow \infty} \tilde{\beta}_n$  exists by the dominated convergence theorem.  $\square$

### 6. Proof of Theorem 3

*Proof.* The transition to the inequalities (3) and (4) from their respective stronger versions, (15) and (16), comes from the bounds

$$\int_{\mathbb{S}^2} F(\theta) f^2 d\sigma \leq A \int_{\mathbb{S}^2} f^2 d\sigma, \quad \int_{\mathbb{S}^2} G(\theta) f^2 d\sigma \leq B \int_{\mathbb{S}^2} f^2 d\sigma$$

where the bounded functions  $F$  and  $G$  are both positive and independent of  $f$ . Interestingly, as seen in Section 5, the size of  $0 < A, B < \infty$  played no role in optimising (3) and (4).

Up to the inequality (15) or (16) an equality relation persists except for the only inequality (11). So a sufficient and necessary condition for an equality in (15) or (16) (and a necessary condition for an equality in (3) and (4)) is an equality in (11). But an equality in (11) occurs if and only if

$$\int_{\mathbb{S}^2} \psi |\nabla g|^2 d\sigma = 0. \tag{30}$$

Recalling that  $g = f/\sqrt{\psi}$ , we compute

$$\begin{aligned} \psi |\nabla g|^2 &= \psi \left| \frac{\nabla f}{\sqrt{\psi}} - \frac{1}{2} \frac{f}{\psi^{\frac{3}{2}}} \frac{\partial \psi}{\partial \theta} \hat{\theta} \right|^2 \\ &= |\nabla f|^2 - \frac{f}{\psi} \frac{\partial \psi}{\partial \theta} \nabla f \cdot \hat{\theta} + \frac{1}{4} \frac{f^2}{\psi^2} \left( \frac{\partial \psi}{\partial \theta} \right)^2 \\ &= |\nabla f|^2 - \left( \frac{\partial f}{\partial \theta} \right)^2 + \left( \frac{\partial f}{\partial \theta} \right)^2 - \frac{f}{\psi} \frac{\partial \psi}{\partial \theta} \frac{\partial f}{\partial \theta} + \frac{1}{4} \frac{f^2}{\psi^2} \left( \frac{\partial \psi}{\partial \theta} \right)^2 \\ &= |\nabla f|^2 - \left( \frac{\partial f}{\partial \theta} \right)^2 + \left( \frac{\partial f}{\partial \theta} - \frac{1}{2} \frac{f}{\psi} \frac{\partial \psi}{\partial \theta} \right)^2. \end{aligned} \tag{31}$$



Since  $|\nabla f|^2 - \left(\frac{\partial f}{\partial \theta}\right)^2 = \frac{1}{\sin^2 \theta} \left(\frac{\partial f}{\partial \varphi}\right)^2 \geq 0$ , then, by (31), the equality (30) is equivalent to

$$\int_{\mathbb{S}^2} |\nabla f|^2 - \left(\frac{\partial f}{\partial \theta}\right)^2 d\sigma = \int_{\mathbb{S}^2} \left(\frac{\partial f}{\partial \theta} - \frac{1}{2} \frac{f}{\psi} \frac{\partial \psi}{\partial \theta}\right)^2 d\sigma = 0. \tag{32}$$

The equalities (32) are, in their turn, equivalent to

$$\frac{1}{\sin \theta} \left| \frac{\partial f}{\partial \varphi} \right| = \left| \frac{\partial f}{\partial \theta} - \frac{1}{2} \frac{f}{\psi} \frac{\partial \psi}{\partial \theta} \right| = 0. \tag{33}$$

Suppose that  $f$  is not the zero function. Then (33) are possible if and only if

$$f = f(\theta), \quad \frac{df}{f} = \frac{1}{2} \frac{d\psi}{\psi}.$$

That is  $f = c\sqrt{\psi}$ ,  $c$  is a constant. But such  $f \notin H^1(\mathbb{S}^2)$  because

$$\begin{aligned} \int_{\mathbb{S}^2} |\nabla f|^2 d\sigma &= 2\pi \int_0^\pi \left(\frac{\partial f}{\partial \theta}\right)^2 \sin \theta d\theta \gtrsim \int_0^1 \frac{\cos^2 \theta}{\sin \theta} \frac{1}{\psi} d\theta \\ &\gtrsim \int_0^1 \frac{d\theta}{\sin \theta \phi(\sin \theta)} \approx \int_0^1 \frac{d\theta}{\theta \phi(\theta)} = +\infty. \quad \square \end{aligned}$$

### 7. Proof of Theorem 4

*Proof.* Write

$$\frac{1}{\theta} \frac{1}{\phi^2(\theta)} = \nabla \left( \frac{1}{\phi(\theta)} \right) \cdot \hat{\theta}.$$

Assume that  $f$  is smooth. Then integrating by parts w.r.t. the surface measure  $\sigma$  we get

$$\begin{aligned} \int_{\mathbb{S}^2} \frac{f^2}{\theta^2 \phi^2(\theta)} d\sigma &= \int_{\mathbb{S}^2} \nabla \left( \frac{1}{\phi(\theta)} \right) \cdot \frac{f^2}{\theta} \hat{\theta} d\sigma \\ &= - \int_{\mathbb{S}^2} \frac{1}{\phi(\theta)} \nabla \cdot \left( \frac{f^2}{\theta} \hat{\theta} \right) d\sigma \\ &= -2 \int_{\mathbb{S}^2} \frac{f \nabla f \cdot \hat{\theta}}{\theta \phi(\theta)} d\sigma + \int_{\mathbb{S}^2} \frac{f^2}{\theta^2 \phi(\theta)} d\sigma - \int_{\mathbb{S}^2} \frac{f^2}{\theta \phi(\theta)} \frac{\cos \theta}{\sin \theta} d\sigma. \end{aligned} \tag{34}$$

Observe here that each of the last two integrals on the right hand side of (34) can diverge. They suffer nonintegrable singularities at  $\theta = 0$ . But, when put together, their sum

$$I := \int_{\mathbb{S}^2} \frac{f^2}{\theta^2 \phi(\theta)} d\sigma - \int_{\mathbb{S}^2} \frac{f^2}{\theta \phi(\theta)} \frac{\cos \theta}{\sin \theta} d\sigma = \int_{\mathbb{S}^2} \frac{1}{\theta \phi(\theta)} \left( \frac{1}{\theta} - \frac{\cos \theta}{\sin \theta} \right) f^2 d\sigma \tag{35}$$

is convergent. In fact

$$\lim_{\theta \rightarrow 0^+} \frac{1}{\theta \phi(\theta)} \left( \frac{1}{\theta} - \frac{\cos \theta}{\sin \theta} \right) = 0.$$

Also,  $\theta \mapsto 1/(\theta^2 \phi(\theta))$  is continuous on a neighborhood of  $\theta = \pi$ . Furthermore, if we fix  $\delta > 0$  and let  $D := \{x(\theta, \varphi) \in \mathbb{S}^2 : 0 \leq \theta < \delta\}$ , then the integral

$$\int_{\mathbb{S}^2 \setminus D} \frac{f^2}{\theta \phi(\theta)} \frac{\cos \theta}{\sin \theta} d\sigma$$

does exist. Unfortunately, we can not control the integral  $I$  by  $\int_{\mathbb{S}^2} f^2 d\sigma$ , up to a constant factor. The reason is

$$\lim_{\theta \rightarrow \pi^-} \frac{1}{\theta \phi(\theta)} \frac{\cos \theta}{\sin \theta} = -\infty.$$

But since

$$\lim_{\theta \rightarrow \pi^-} \left( \frac{1}{\theta \phi(\theta)} \frac{\cos \theta}{\sin \theta} + \frac{1}{\pi} \frac{1}{(\pi - \theta)} \right) = 0$$

then, we may introduce the convergent integral  $J := \frac{1}{\pi} \int_{\mathbb{S}^2} \frac{f^2}{\pi - \theta} d\sigma$  to the integral  $I$  to get

$$I = I - J + J = \int_{\mathbb{S}^2} K(\theta) f^2 d\sigma + J \tag{36}$$

where

$$K(\theta) := \frac{1}{\theta \phi(\theta)} \left( \frac{1}{\theta} - \frac{\cos \theta}{\sin \theta} \right) - \frac{1}{\pi} \frac{1}{(\pi - \theta)}.$$

By the continuity of  $K$  on  $]0, \pi[$  and since

$$\lim_{\theta \rightarrow 0^+} K(\theta) = -\lim_{\theta \rightarrow \pi} K(\theta) = -\frac{1}{\pi^2}$$

then  $K$  is bounded on  $[0, \pi]$ . Actually,  $K$  is monotonically increasing. Thus

$$\sup_{[0, \pi]} |K| = \frac{1}{\pi^2}. \tag{37}$$

Using (37) in (36) we deduce that

$$I \leq \frac{1}{\pi^2} \int_{\mathbb{S}^2} f^2 d\sigma + J. \tag{38}$$

Returning with (38) to the inequality (34) in the light of (35) we obtain

$$\int_{\mathbb{S}^2} \frac{f^2}{\theta^2 \phi^2(\theta)} d\sigma \leq -2 \int_{\mathbb{S}^2} \frac{f \nabla f \cdot \hat{\theta}}{\theta \phi(\theta)} d\sigma + \frac{1}{\pi^2} \int_{\mathbb{S}^2} f^2 d\sigma + \frac{1}{\pi} \int_{\mathbb{S}^2} \frac{f^2}{\pi - \theta} d\sigma. \tag{39}$$

Applying Cauchy’s inequality with an  $\varepsilon$  we find

$$-2 \int_{\mathbb{S}^2} \frac{f \nabla f \cdot \hat{\theta}}{\theta \phi(\theta)} d\sigma \leq 2\varepsilon \int_{\mathbb{S}^2} \frac{f^2}{\theta^2 \phi^2(\theta)} d\sigma + \frac{1}{2\varepsilon} \int_{\mathbb{S}^2} |\hat{\theta} \cdot \nabla f|^2 d\sigma. \tag{40}$$

Therefore, it follows from (39) and (40) that

$$2\varepsilon(1 - 2\varepsilon) \int_{\mathbb{S}^2} \frac{f^2}{\theta^2 \phi^2(\theta)} d\sigma \leq \int_{\mathbb{S}^2} |\hat{\theta} \cdot \nabla f|^2 d\sigma + \frac{2\varepsilon}{\pi^2} \int_{\mathbb{S}^2} f^2 d\sigma + \frac{2\varepsilon}{\pi} \int_{\mathbb{S}^2} \frac{f^2}{\pi - \theta} d\sigma, \quad 0 < \varepsilon < \frac{1}{2}. \tag{41}$$

The choice  $\varepsilon = 1/4$  maximizes the factor  $2\varepsilon(1 - 2\varepsilon)$  and, consequently, the left hand side of (41). This proves (7). The inequality (8) can be obtained analogously.

In the fashion of the proof of Theorem 2, the sequence  $f_n = \phi^{\frac{1}{2} - \frac{1}{n}}$  clearly satisfies

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{4} \int_0^\pi \frac{f_n^2}{\rho_\phi^2(\theta)} \sin \theta d\theta}{U(f_n) + \frac{1}{2\pi} \int_0^\pi \frac{f_n^2}{\pi - \theta} \sin \theta d\theta} = \lim_{n \rightarrow \infty} \frac{\frac{1}{4} \int_0^\pi \frac{f_n^2}{\rho_\phi^2(\pi - \theta)} \sin \theta d\theta}{U(f_n) + \frac{1}{2\pi} \int_0^\pi \frac{f_n^2}{\theta} \sin \theta d\theta} = 1$$

where

$$U(f) = \int_0^\pi \left( \frac{\partial f}{\partial \theta} \right)^2 \sin \theta d\theta + \frac{1}{2\pi^2} \int_0^\pi f^2 \sin \theta d\theta.$$

One only needs to inspect the convergence of  $\int_0^\pi \left( \phi^{1 - \frac{2}{n}} \sin \theta / \theta \right) d\theta$ ,  $\int_0^\pi \left( \phi^{1 - \frac{2}{n}} \sin \theta / (\pi - \theta) \right) d\theta$  as  $n \rightarrow \infty$ . This is obvious from the bound  $\sin \theta \leq \min\{\theta, \pi - \theta\}$  on  $[0, \pi]$  and the fact  $\phi \in L^1([0, \pi])$ .

Finally, careful review of the proof of (7) above reveals that a necessary condition for a function  $f \in H^1(\mathbb{S}^2) \setminus \{0\}$  to achieve an equality in (7) is that it yields an equality in (40). This is equivalent to

$$\nabla f \cdot \hat{\theta} = -\frac{1}{2} \frac{f}{\theta \phi(\theta)}. \tag{42}$$

Suppose (42) was true. Then by (34) and (35) we must have

$$\int_{\mathbb{S}^2} \frac{h(\theta) f^2}{\theta \phi(\theta)} d\sigma = 0 \tag{43}$$

where

$$h(\theta) := \frac{1}{\theta} - \frac{\cos \theta}{\sin \theta}.$$

On the other hand

$$\lim_{\theta \rightarrow 0^+} h(\theta) = 0, \quad h'(\theta) = \frac{\theta^2 - \sin^2 \theta}{\theta^2 \sin^2 \theta} > 0, \quad 0 < \theta < \pi.$$

This shows  $h$  is strictly positive on  $]0, \pi[$  and since  $\theta \phi(\theta) \geq 0$  then (43) is a contradiction.  $\square$

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