

AN EXTENSION OF HARTFIEL'S DETERMINANT INEQUALITY

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Abstract. Let A and B be $n \times n$ positive definite matrices, Hartfiel obtained a lower bound for $\det(A + B)$. In this paper, we first extend his result to $\det(A + B + C)$, where A, B and C are $n \times n$ positive definite matrices, and then show a generalization of this to the case of matrices whose numerical ranges are contained in a sector.

1. Introduction

Let A, B be $n \times n$ positive semidefinite matrices, it is well known that

$$\det(A + B) \geq \det A + \det B. \quad (1)$$

In [5], Haynsworth proved the following refinement of (1),

$$\det(A + B) \geq \left(1 + \sum_{k=1}^{n-1} \frac{\det B_k}{\det A_k}\right) \det A + \left(1 + \sum_{k=1}^{n-1} \frac{\det A_k}{\det B_k}\right) \det B. \quad (2)$$

where $A_k, B_k, k = 1, \dots, n-1$, denote the k -th leading principal submatrices of positive definite matrices A and B , respectively.

Later, Hartfiel [4] obtained an improvement of (2) as follows:

$$\begin{aligned} \det(A + B) \geq & \left(1 + \sum_{k=1}^{n-1} \frac{\det B_k}{\det A_k}\right) \det A + \left(1 + \sum_{k=1}^{n-1} \frac{\det A_k}{\det B_k}\right) \det B \\ & + (2^n - 2n) \sqrt{\det A \det B}. \end{aligned} \quad (3)$$

where A_k, B_k are under the same condition as in (2). And the author also gave an interesting corollary:

$$\det(A + B) \geq \det A + \det B + (2^n - 2) \sqrt{\det A \det B}. \quad (4)$$

In this paper, we first extend Hartfiel's result by giving a lower bound for $\det(A + B + C)$, where A, B and C are $n \times n$ positive definite matrices. And then, we show some generalizations of the matrix form of the new determinant inequalities to a larger class of matrices, namely, matrices whose numerical ranges are contained in a sector.

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2. Auxiliary results

Let \mathbb{M}_n be the set of $n \times n$ complex matrices. If A is positive semidefinite, we put $A \geq 0$, for two Hermitian matrices $A, B \in \mathbb{M}_n$, $A \geq B$ means $A - B$ is positive semidefinite. If A is positive definite, we put $A > 0$.

For $A \in \mathbb{M}_n$, recall the Cartesian decomposition (see, e.g. [6, p. 7])

$$A = \Re A + i\Im A.$$

where

$$\Re A = \frac{1}{2}(A + A^*), \quad \Im A = \frac{1}{2i}(A - A^*).$$

We say $A \in \mathbb{M}_n$ is accretive-dissipative matrix if $\Re A$ and $\Im A$ are positive definite. For more details about this class of matrices, please refer to [3, 8, 9, 12].

The numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax \mid x \in \mathbb{C}^n, x^*x = 1\}.$$

Also, for $\alpha \in [0, \frac{\pi}{2})$, let S_α be the sector in the complex plane given by

$$S_\alpha = \{z \in \mathbb{C} \mid \Re z > 0, |\Im z| \leq (\Re z) \tan(\alpha)\}$$

Clearly, if $W(A) \subset S_0$, then A is positive definite. As $0 \notin S_\alpha$, if $W(A) \subset S_\alpha$, we can get that A is necessarily nonsingular.

Relevant studies on matrices with numerical ranges in a sector can be found in [1, 2, 11, 7, 13].

Next, we present some lemmas which are useful for our proofs.

LEMMA 1. [6, Theorem 7.8.19] *Let $A \in \mathbb{M}_n$. If $\Re(A) > 0$, then*

$$\det(\Re(A)) \leq |\det(A)|.$$

LEMMA 2. [11, Lemma 2.6] *Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\alpha$. Then*

$$\sec^n(\alpha) \det(\Re(A)) \geq |\det(A)|.$$

LEMMA 3. [10, Theorem 1.1] *Let A, B, C be $n \times n$ positive definite matrices. Then*

$$\begin{aligned} \det(A + B + C) + \det(C) - (\det(A + C) + \det(B + C)) \\ \geq \det(A + B) - (\det(A) + \det(B)). \end{aligned} \tag{5}$$

LEMMA 4. [11, Proposition 2.1] *Let $A, B, C \in \mathbb{M}_n$, and partition A as $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with A_{11} square. If $W(A), W(B), W(C) \subset S_\alpha$, then $W(A + B + C) \subset S_\alpha$ and $W(A_{11}) \subset S_\alpha$.*

3. Main results

In this section, we begin with extending inequality (3) and inequality (4) as follows:

THEOREM 1. *Let A, B, C be $n \times n$ positive definite matrices, $A_k, B_k, C_k, k = 1, \dots, n - 1$, denote the k -th leading principal submatrices of A, B, C , respectively. Then*

$$\begin{aligned} \det(A + B + C) &\geq (1 + \sum_{k=1}^{n-1} \frac{\det B_k + \det C_k}{\det A_k}) \det A \\ &\quad + (1 + \sum_{k=1}^{n-1} \frac{\det A_k + \det C_k}{\det B_k}) \det B \\ &\quad + (1 + \sum_{k=1}^{n-1} \frac{\det A_k + \det B_k}{\det C_k}) \det C \\ &\quad + (2^n - 2n)(\sqrt{\det AB} + \sqrt{\det AC} + \sqrt{\det BC}). \end{aligned} \tag{6}$$

Proof. According to Lemma 3, we have

$$\begin{aligned} \det(A + B + C) &\geq \det(A + B) + \det(A + C) + \det(B + C) \\ &\quad - \det A - \det B - \det C. \end{aligned} \tag{7}$$

Then we can obtain, by (3), that

$$\begin{aligned} \det(A + B + C) &\geq (1 + \sum_{k=1}^{n-1} \frac{\det B_k}{\det A_k}) \det A + (1 + \sum_{k=1}^{n-1} \frac{\det A_k}{\det B_k}) \det B \\ &\quad + (2^n - 2n)\sqrt{\det AB} + (1 + \sum_{k=1}^{n-1} \frac{\det C_k}{\det A_k}) \det A \\ &\quad + (1 + \sum_{k=1}^{n-1} \frac{\det A_k}{\det C_k}) \det C + (2^n - 2n)\sqrt{\det AC} \\ &\quad + (1 + \sum_{k=1}^{n-1} \frac{\det C_k}{\det B_k}) \det B + (1 + \sum_{k=1}^{n-1} \frac{\det B_k}{\det C_k}) \det C \\ &\quad + (2^n - 2n)\sqrt{\det BC} - \det A - \det B - \det C \\ &\geq (1 + \sum_{k=1}^{n-1} \frac{\det B_k + \det C_k}{\det A_k}) \det A + (1 + \sum_{k=1}^{n-1} \frac{\det A_k + \det C_k}{\det B_k}) \det B \\ &\quad + (1 + \sum_{k=1}^{n-1} \frac{\det A_k + \det B_k}{\det C_k}) \det C \\ &\quad + (2^n - 2n)(\sqrt{\det AB} + \sqrt{\det AC} + \sqrt{\det BC}). \quad \square \end{aligned}$$

By inequality (4) and Lemma 3, the following interesting theorem can be obtained.

THEOREM 2. *Let A, B, C be $n \times n$ positive definite matrices. Then*

$$\det(A + B + C) \geq \det A + \det B + \det C + (2^n - 2)(\sqrt{\det AB} + \sqrt{\det AC} + \sqrt{\det BC}). \tag{8}$$

Now, we extend Theorem 1 and Theorem 2 to the case of matrices whose numerical ranges are contained in a sector.

THEOREM 3. *Let $A, B, C \in \mathbb{M}_n$, $W(A), W(B), W(C) \subset S_\alpha, \alpha \in [0, \frac{\pi}{2})$ and let $A_k, B_k, C_k, k = 1, \dots, n - 1$, denote the k -th leading principal submatrices of A, B, C , respectively. Then*

$$\begin{aligned} \sec^n(\alpha) |\det(A + B + C)| &\geq (1 + \sum_{k=1}^{n-1} \cos^k(\alpha) \frac{|\det B_k| + |\det C_k|}{|\det A_k|}) |\det A| \\ &\quad + (1 + \sum_{k=1}^{n-1} \cos^k(\alpha) \frac{|\det A_k| + |\det C_k|}{|\det B_k|}) |\det B| \\ &\quad + (1 + \sum_{k=1}^{n-1} \cos^k(\alpha) \frac{|\det A_k| + |\det B_k|}{|\det C_k|}) |\det C| \\ &\quad + (2^n - 2n)(\sqrt{|\det AB|} + \sqrt{|\det AC|} + \sqrt{|\det BC|}). \end{aligned}$$

Proof. According to Lemma 4, we have $W(A_k), W(B_k), W(C_k) \subset S_\alpha$. Compute

$$\begin{aligned} |\det(A + B + C)| &\geq \det(\Re(A + B + C)) \\ &\geq (1 + \sum_{k=1}^{n-1} \frac{\det \Re(B_k) + \det \Re(C_k)}{\det \Re(A_k)}) \det \Re(A) \\ &\quad + (1 + \sum_{k=1}^{n-1} \frac{\det \Re(A_k) + \det \Re(C_k)}{\det \Re(B_k)}) \det \Re(B) \\ &\quad + (1 + \sum_{k=1}^{n-1} \frac{\det \Re(A_k) + \det \Re(B_k)}{\det \Re(C_k)}) \det \Re(C) \\ &\quad + (2^n - 2n)(\sqrt{\det \Re(A) \det \Re(B)} + \sqrt{\det \Re(A) \det \Re(C)} \\ &\quad + \sqrt{\det \Re(B) \det \Re(C)}) \\ &\geq (1 + \sum_{k=1}^{n-1} \cos^k(\alpha) \frac{|\det B_k| + |\det C_k|}{|\det A_k|}) \cos_\alpha^n |\det A| \\ &\quad + (1 + \sum_{k=1}^{n-1} \cos^k(\alpha) \frac{|\det A_k| + |\det C_k|}{|\det B_k|}) \cos_\alpha^n |\det B| \\ &\quad + (1 + \sum_{k=1}^{n-1} \cos^k(\alpha) \frac{|\det A_k| + |\det B_k|}{|\det C_k|}) \cos_\alpha^n |\det C| \\ &\quad + (2^n - 2n) \cos^n(\alpha) (\sqrt{|\det AB|} + \sqrt{|\det AC|} + \sqrt{|\det BC|}). \end{aligned}$$

where the first inequality above is by Lemma 1; the second is due to the inequality (6) and the last inequality holds by Lemma 1 and Lemma 2.

Multiplying both sides of the inequality by $\sec^n(\alpha)$ yields the desired inequality, which completes the proof. \square

When $\alpha = 0$, Theorem 3 reduces to Theorem 1. Note that if A is accretive-dissipative, then $W(e^{-i\pi/4}A) \subset S_{\pi/4}$. Thus, we have the following corollary.

COROLLARY 1. *Suppose $A, B, C \in \mathbb{M}_n$ are accretive-dissipative, let $A_k, B_k, C_k, k = 1 \cdots n - 1$, denote the k -th leading principal submatrices of A, B, C , respectively. Then*

$$\begin{aligned} 2^{\frac{n}{2}} |\det(A + B + C)| &\geq \left(1 + \sum_{k=1}^{n-1} \frac{1}{2^{k/2}} \frac{|\det B_k| + |\det C_k|}{|\det A_k|}\right) |\det A| \\ &\quad + \left(1 + \sum_{k=1}^{n-1} \frac{1}{2^{k/2}} \frac{|\det A_k| + |\det C_k|}{|\det B_k|}\right) |\det B| \\ &\quad + \left(1 + \sum_{k=1}^{n-1} \frac{1}{2^{k/2}} \frac{|\det A_k| + |\det B_k|}{|\det C_k|}\right) |\det C| \\ &\quad + (2^n - 2n)(\sqrt{|\det AB|} + \sqrt{|\det AC|} + \sqrt{|\det BC|}). \end{aligned}$$

For the generalization of Theorem 2, the following results can be obtained by Lemma 1 and Lemma 2.

THEOREM 4. *Let $A, B, C \in \mathbb{M}_n, W(A), W(B), W(C) \subset S_\alpha, \alpha \in [0, \frac{\pi}{2})$. Then*

$$\begin{aligned} \sec^n(\alpha) |\det(A + B + C)| &\geq |\det A| + |\det B| + |\det C| \\ &\quad + (2^n - 2)(\sqrt{|\det AB|} + \sqrt{|\det AC|} + \sqrt{|\det BC|}). \end{aligned}$$

COROLLARY 2. *Let $A, B, C \in \mathbb{M}_n$ be accretive-dissipative. Then*

$$\begin{aligned} 2^{\frac{n}{2}} |\det(A + B + C)| &\geq |\det A| + |\det B| + |\det C| \\ &\quad + (2^n - 2)(\sqrt{|\det AB|} + \sqrt{|\det AC|} + \sqrt{|\det BC|}). \end{aligned}$$

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REFERENCES

- [1] S. DRURY, M. LIN, *Singular value inequalities for matrices with numerical ranges in a sector*, Oper. Matrices. **8**, 4 (2014), 1143–1148.
- [2] X. FU, Y. LIU, *Rofel'd inequality for partitioned matrices with numerical ranges in a sector*, Linear Multilinear Algebra **64**, 1 (2016), 105–109.
- [3] A. GEORGE, K. H. D. IKRAMOV, *On the properties of Accretive-Dissipative Matrices*, Math. Notes. **77**, 5–6 (2005), 767–776.

- [4] D. J. HARTFIEL, *An extension of Haynsworth's determinant inequality*, Proc. Amer. Math. Soc. **41**, 2 (1973), 463–465.
- [5] E. V. HAYNSWORTH, *Applications of an inequality for the Schur complement*, Proc. Amer. Math. Soc. **21**, 3 (1970), 512–516.
- [6] R. A. HORN, C. R. JOHNSON, *Matrix Analysis*, 2nd ed., Cambridge University Press, Cambridge, 2013.
- [7] L. HOU, D. ZHANG, *Concave Functions of partitioned matrices with numerical ranges in a sector*, Math. Inequal. Appl. **20**, 2 (2017), 583–589.
- [8] K. H. D. IKRAMOV, *Determinantal Inequalities for Accretive-Dissipative Matrices*, J. Math. Sci. **121**, 4 (2004), 2458–2464.
- [9] M. LIN, *Fischer type determinantal inequalities for accretive-dissipative matrices*, Linear Algebra Appl. **438**, 6 (2013), 2808–2812.
- [10] M. LIN, *A determinantal inequality for positive definite matrices*, Electron J. Linear Algebra **27**, 1 (2014), 821–826.
- [11] M. LIN, *Extension of a result of Hanynsworth and Hartfiel*, Arch. Math. **104**, 1 (2015), 93–100.
- [12] M. LIN, D. ZHOU, *Norm inequalities for accretive-dissipative operator matrices*, J. Math. Anal. Appl. **407**, 2 (2013), 436–442.
- [13] J. LIU, *Generalizations of the Brunn-Minkowski inequality*, Linear Algebra and its Applications **508**, 1 (2016), 206–213.

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