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MONOTONICITY PROPERTIES AND BOUNDS INVOLVING THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND

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Abstract. In the article, we establish several monotonicity properties of the functions involving the complete elliptic integral of the first kind. As applications, we present sharp bounds for the complete elliptic integral of the first kind and the arithmetic-geometric mean.

1. Introduction

The well-known complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ [15] of the first and second kinds are defined as

$$\mathscr{K}(r)=\int_0^{\pi/2}\frac{dt}{\sqrt{1-r^2\sin^2(t)}},\quad \mathscr{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}dt\quad (0<r<1),$$

$$\mathscr{K}\left(0^+\right)=\mathscr{E}\left(0^+\right)=\frac{\pi}{2},\quad \mathscr{E}\left(1^-\right)=1,\quad \mathscr{K}\left(1^-\right)=\infty.$$

Both $\mathscr{K}(r)$ and $\mathscr{E}(r)$ are the particular cases of the Gaussian hypergeometric function [57–59, 63, 64, 75, 83]

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}\quad (-1<x<1), \tag{1.1}$$

where $(a,0)=1$ for $a\neq 0$, $(a,n)=\Gamma(a+n)/\Gamma(a)$ and $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ $(x>0)$ is the classical gamma function [74, 78, 79, 82, 84, 85]. In facts, $\mathscr{K}(r)$ and $\mathscr{E}(r)$ can be expressed by

$$\mathscr{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)^2}{(n!)^2}r^{2n}, \tag{1.2}$$

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$$\mathscr{E}(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2},n\right)\left(\frac{1}{2},n\right)}{(n!)^2}r^{2n}.$$

There are close connections between the complete elliptic integrals and bivariate means. For example, the Toader mean $T(a,b)$ [22, 24, 25, 27] and the arithmetic-geometric mean $AG(a,b)$ [13, 14, 17, 44, 51–56] of two positive real numbers a and b with $a>b$ can be expressed as

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2 t+b^2\sin^2 t}dt=\frac{2a}{\pi}\mathscr{E}\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right), \tag{1.3}$$

$$AG(a,b)=\frac{\pi a}{2\mathscr{K}\left(\sqrt{1-(\frac{b}{a})^2}\right)}. \tag{1.4}$$

The identity (1.4) is called Gaussian identity [11] and the arithmetic-geometric mean $AG(a,b)$ is defined as the common limit of the sequences $\{a_n\}$ and $\{b_n\}$ as follows:

$$a_0=a,\quad b_0=b,\quad a_{n+1}=\frac{a_n+b_n}{2},\quad b_{n+1}=\sqrt{a_nb_n}.$$

Recently, the complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ have been the subject of intensive research [19, 20, 23, 26, 48, 60–62, 65, 66, 76, 77]. In particular, many remarkable inequalities and applications for $\mathscr{K}(r)$, $\mathscr{E}(r)$ and other related special functions can be found in the literature [1–3, 6–9, 12, 18, 21, 28–38, 42, 50, 67–73].

Carlson and Gustafson [16] proved that the double inequality

$$\log\frac{4}{r'}<\mathscr{K}(r)<\frac{4}{3+r^2}\log\frac{4}{r'} \tag{1.5}$$

holds for all $r\in(0,1)$. Here and in what follows we denote $r'=\sqrt{1-r^2}$.

The lower bound given in (1.5) was improved by Kühnau [41] as follows

$$\mathscr{K}(r)>\frac{9}{8+r^2}\log\frac{4}{r'}$$

for all $r\in(0,1)$.

In [4, 10, 17, 49, 56], the authors proved that the two-sided inequalities

$$\frac{\log r'}{r'-1}<\mathscr{K}(r)<\frac{\pi\log r'}{2(r'-1)},$$

$$\left[1+\left(\frac{\pi}{4\log 2}-1\right)r'^2\right]\log\frac{4}{r'}<\mathscr{K}(r)<\left(1+\frac{1}{4}r'^2\right)\log\frac{4}{r'},$$

$$\frac{\pi}{2}\left[\frac{\tanh^{-1}(r)}{r}\right]^{1/2}<\mathscr{K}(r)<\frac{\pi}{2}\frac{\tanh^{-1}(r)}{r} \tag{1.6}$$

are valid for all $r > 0$, where $\tanh^{-1}(r) = \log[(1+r)/(1-r)]/2$ is the inverse hyperbolic tangent function.

Alzer and Qiu [5], and Yang et al. [81] improved the lower bound given in (1.6) independently as follows:

$$\mathscr{K}(r) > \frac{\pi}{2}\left[\frac{\tanh^{-1}(r)}{r}\right]^{3/4}$$

for all $r \in (0,1)$.

The main purpose of the article is to establish the monotonicity properties of the functions involving the complete elliptic integral $\mathscr{K}(r)$ and provide the sharp bounds for $\mathscr{K}(r)$ and $AG(1,r)$ in terms of elementary functions.

2. Lemmas

In order to prove our main results we need several formulas and lemmas, which we present in this section.

The hypergeometric function $F(a,b,c;x)$ and the complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ have the following formulas (See [11, (1.16), 1.19(4), 1.20(10), 1.48, (3.6)]):

$$\frac{d^n}{dx^n}F(a,b,c;x) = \frac{(a,n)(b,n)}{(c,n)}F(a+n,b+n;c+n;x), \tag{2.1}$$

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{2.2}$$

if $c > a+b$.

$$F(a,b;a+b+1;x) = (1-x)F(a+1,b+1;a+b+1;x), \tag{2.3}$$

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}F(a,b;a+b;x) + \log(1-x) + \psi(a) + \psi(b) + 2\gamma = O((1-x)\log(1-x)) \tag{2.4}$$

as $x \to 1$, where

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is the psi function and

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.57721566\cdots$$

is the Euler-Mascheroni constant.

$$\frac{d\mathscr{K}(r)}{dr} = \frac{\mathscr{E}(r) - r'^2\mathscr{K}(r)}{rr'^2}. \tag{2.5}$$

LEMMA 2.1. (See [80, Lemma 2.1]) *Let* $r > 0$, $\{a_k\}_{k=0}^{\infty}$ *be a nonnegative real sequence with* $a_m > 0$ *and* $\sum_{k=m+1}^{\infty} a_k > 0$, *and*

$$S(t) = \sum_{k=0}^{m} a_k t^k - \sum_{k=m+1}^{\infty} a_k t^k$$

be a convergent power series on the interval $(0,r)$. *Then* $S(t) > 0$ *for all* $t \in (0,r)$ *if* $S(r^-) \geqslant 0$.

LEMMA 2.2. (See [47, (2.13)]) *The inequality*

$$\frac{\Gamma^2(n+1/2)}{\Gamma^2(n+1)} < \frac{1}{n+1/4}$$

is valid for all $n \in \mathbb{N}$.

LEMMA 2.3. *Let* $n \in \mathbb{N}$ *and*

$$\lambda_n = \psi(n) - \frac{4n^2 - 17n + 1}{2(4n+1)} + \gamma. \tag{2.6}$$

Then $\lambda_n < 0$ *for* $n \geqslant 11$.

Proof. Elaborated computations lead to

$$\lambda_{11} = -\frac{107}{280} < 0, \tag{2.7}$$

$$\lambda_{n+1} - \lambda_n = -\frac{16n^3 - 8n^2 - 65n - 10}{2n(4n+1)(4n+5)} < 0 \tag{2.8}$$

for $n \geqslant 3$.

Therefore, Lemma 2.3 follows easily from (2.7) and (2.8). □

LEMMA 2.4. *Let* $n,k \in \mathbb{N}$ *with* $k \leqslant n$ *and*

$$v_k = \frac{1}{(k+1)(n-k+1)(k+1/4)(n-k+1/4)}, \tag{2.9}$$

$$\omega_n = \left(n + \frac{1}{4}\right) \sum_{k=0}^{n} v_k - \frac{6(2n+1)}{(n+1)(n+2)}. \tag{2.10}$$

Then $\omega_n < 0$ *for* $n \geqslant 8$.

Proof. Let $n \geqslant 8$, $1 \leqslant k \leqslant n-1$ and

$$\xi_k = \frac{1}{k(k+1)(n-k)(n-k+1)}. \tag{2.11}$$

Then we clearly see that

$$\xi_k = \frac{1}{n(n+1)}\left(\frac{1}{k}+\frac{1}{n-k}\right) - \frac{1}{(n+1)(n+2)}\left(\frac{1}{k+1}+\frac{1}{n-k+1}\right),$$

$$\sum_{k=1}^{n-1}\xi_k = \frac{2(\psi(n)+\gamma)}{n(n+1)} - \frac{2(\psi(n)+1/n+\gamma-1)}{(n+1)(n+2)} = \frac{2(2\psi(n)+n+2\gamma-1)}{n(n+1)(n+2)}. \tag{2.12}$$

Note that

$$\left(k+\frac{1}{4}\right)\left(n-k+\frac{1}{4}\right) > k(n-k) \tag{2.13}$$

for all $k,n \in \mathbb{N}$.

It follows (2.9), (2.11) and (2.13) that

$$v_0 + v_n = \frac{32}{(n+1)(4n+1)},$$

$$\sum_{k=0}^{n} v_k = \sum_{k=1}^{n-1} v_k + \frac{32}{(n+1)(4n+1)} < \sum_{k=1}^{n-1}\xi_k + \frac{32}{(n+1)(4n+1)}. \tag{2.14}$$

From (2.10), (2.12) and (2.14) we have

$$\begin{aligned}\omega_n &< \frac{2(2\psi(n)+n+2\gamma-1)(n+1/4)}{n(n+1)(n+2)} + \frac{32(n+1/4)}{(n+1)(4n+1)} - \frac{6(2n+1)}{(n+1)(n+2)} \\ &= \frac{4n+1}{n(n+1)(n+2)}\lambda_n,\end{aligned} \tag{2.15}$$

where λ_n is given by (2.6).

Elaborated computations lead to

$$\omega_8 = -\frac{1855051}{114400650} < 0, \quad \omega_9 = -\frac{3251242}{111035925} < 0, \quad \omega_{10} = -\frac{1777462049611}{46588453411500} < 0. \tag{2.16}$$

Therefore, Lemma 2.4 follows easily from Lemma 2.3, (2.15) and (2.16). □

LEMMA 2.5. *Let $k,n \in \mathbb{N}$ with $k \leqslant n$, and*

$$W_n = \frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}, \tag{2.17}$$

$$u_n = \pi\sum_{k=0}^{n}\frac{W_k^2 W_{n-k}^2}{(k+1)(n-k+1)} - \frac{6(2n+1)W_n^2}{(n+1)(n+2)}. \tag{2.18}$$

Then $u_n < 0$ for all $n \geqslant 8$.

Proof. From (2.17) we clearly see that the sequence $\{(k+1/4)W_k^2\}_{k=0}^{\infty}$ is strictly increasing, which leads to the conclusion that

$$\left(k+\frac{1}{4}\right)W_k^2 \leqslant \left(n+\frac{1}{4}\right)W_n^2,$$

$$W_k^2W_{n-k}^2 \leqslant \frac{(n+1/4)^2W_n^4}{(k+1/4)(n-k+1/4)} \tag{2.19}$$

for $0 \leqslant k \leqslant n$.

Let ω_n be defined by (2.10), then it follows from (2.9), (2.17), (2.18) and (2.19) together with Lemma 2.2 that

$$\begin{aligned}\frac{u_n}{W_n^2} &\leqslant \pi\sum_{k=0}^{n}\frac{(n+1/4)^2W_n^2}{(k+1)(n-k+1)(k+1/4)(n-k+1/4)} - \frac{6(2n+1)}{(n+1)(n+2)} \\ &< \sum_{k=0}^{n}\frac{n+1/4}{(k+1)(n-k+1)(k+1/4)(n-k+1/4)} - \frac{6(2n+1)}{(n+1)(n+2)} = \omega_n.\end{aligned} \tag{2.20}$$

Therefore, Lemma 2.5 follows from Lemma 2.4 and (2.20). □

3. Main results

THEOREM 3.1. *The function $r \mapsto r'^{p}e^{\mathscr{K}(r)}$ is strictly increasing on $(0,1)$ if and only if $p \leqslant \pi/4$ and strictly decreasing on $(0,1)$ if and only if $p \geqslant 1$.*

Proof. Let $x = r^2 \in (0,1)$ and

$$G_1(x) = (1-x)^{p/2}e^{\mathscr{K}(\sqrt{x})} = (1-x)^{p/2}e^{\frac{\pi}{2}F(1/2,1/2;1;x)} = r'^{p}e^{\mathscr{K}(r)}. \tag{3.1}$$

Then (2.1) and (2.3) lead to

$$\begin{aligned}G_1'(x) &= -\frac{p}{2}(1-x)^{p/2-1}e^{\mathscr{K}(\sqrt{x})} + \frac{\pi}{8}(1-x)^{p/2}F(3/2,3/2;2;x)e^{\mathscr{K}(\sqrt{x})} \\ &= -\frac{p}{2}(1-x)^{p/2-1}e^{\mathscr{K}(\sqrt{x})} + \frac{\pi}{8}(1-x)^{p/2-1}F(1/2,1/2;2;x)e^{\mathscr{K}(\sqrt{x})} \\ &= -\frac{1}{2}(1-x)^{p/2-1}e^{\mathscr{K}(\sqrt{x})}\left(p-\frac{\pi}{4}F(1/2,1/2;2;x)\right).\end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we clearly see that the function $r \rightarrow r'^{p}e^{\mathscr{K}(r)}$ is strictly increasing on $(0,1)$ if and only

$$p \leqslant \frac{\pi}{4}\inf_{x\in(0,1)} F\left(\frac{1}{2},\frac{1}{2};2;x\right)$$

and strictly decreasing on $(0,1)$ if and only if

$$p \geqslant \frac{\pi}{4}\sup_{x\in(0,1)} F\left(\frac{1}{2},\frac{1}{2};2;x\right).$$

It follows from (1.1) and (2.2) that

$$\inf_{x\in(0,1)} F\left(\frac{1}{2},\frac{1}{2};2;x\right) = F\left(\frac{1}{2},\frac{1}{2};2;0^+\right) = 1,$$

$$\sup_{x\in(0,1)} F\left(\frac{1}{2},\frac{1}{2};2;x\right) = F\left(\frac{1}{2},\frac{1}{2};2;1^-\right) = \frac{\Gamma(2)\Gamma(1)}{\Gamma^2(3/2)} = \frac{4}{\pi}. \quad \square$$

THEOREM 3.2. *The function*

$$r \mapsto \frac{r'}{r^2}\left[\mathscr{E}(r) - r'^2\mathscr{K}(r)\right]e^{\mathscr{K}(r)}$$

is strictly increasing from $(0,1)$ *onto* $(\pi e^{\pi/2}/4, 4)$.

Proof. It follows from (1.2), (2.1), (2.3) and (2.5) that

$$\frac{\mathscr{E}(r) - r'^2\mathscr{K}(r)}{rr'^2} = \frac{d\mathscr{K}(r)}{dr} = \frac{\pi r}{4}F(3/2,3/2;2;r^2) = \frac{\pi r}{4r'^2}F(1/2,1/2;2;r^2),$$

$$\begin{aligned}\frac{r'}{r^2}\left[\mathscr{E}(r) - r'^2\mathscr{K}(r)\right]e^{\mathscr{K}(r)} &= \frac{\pi}{4}\sqrt{1-r^2}F(1/2,1/2;2;r^2)e^{\mathscr{K}(r)} \\ &= \frac{\pi}{4}\sqrt{1-r^2}F(1/2,1/2;2;r^2)e^{\frac{\pi}{2}F(1/2,1/2;1;r^2)}.\end{aligned} \tag{3.3}$$

Let $x = r^2$, W_n and u_n be respectively defined by (2.17) and (2.18), and

$$\begin{aligned}G_2(x) &= \frac{\pi}{4}\sqrt{1-x}F(1/2,1/2;2;x)e^{\mathscr{K}(\sqrt{x})} \\ &= \frac{\pi}{4}\sqrt{1-x}F(1/2,1/2;2;x)e^{\frac{\pi}{2}F(1/2,1/2;1;x)},\end{aligned} \tag{3.4}$$

$$G_3(x) = \frac{32\sqrt{1-x}}{\pi}e^{-\mathscr{K}(\sqrt{x})}G_2'(x).$$

Then it follows from (1.1), (2.1), (2.3), (2.17) and (2.18) that

$$\begin{aligned}G_3(x) &= \frac{32\sqrt{1-x}}{\pi}e^{-\mathscr{K}(\sqrt{x})}G_2'(x) \\ &= \pi(1-x)F\left(\frac{1}{2},\frac{1}{2};2;x\right)F\left(\frac{3}{2},\frac{3}{2};2;x\right) - 4F\left(\frac{1}{2},\frac{1}{2};2;x\right) \\ &\quad +(1-x)F\left(\frac{3}{2},\frac{3}{2};3;x\right) \\ &= \pi F^2\left(\frac{1}{2},\frac{1}{2};2;x\right) - 4F\left(\frac{1}{2},\frac{1}{2};2;x\right) + (1-x)F\left(\frac{3}{2},\frac{3}{2};3;x\right) \\ &= \pi\left(\sum_{n=0}^{\infty}\frac{W_n^2}{n+1}x^n\right)^2 - 4\sum_{n=0}^{\infty}\frac{W_n^2}{n+1}x^n - \sum_{n=0}^{\infty}\frac{2(4n-1)W_n^2}{(n+1)(n+2)}x^n \\ &= \pi\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{W_k^2W_{n-k}^2}{(k+1)(n-k+1)}\right)x^n - \sum_{n=0}^{\infty}\frac{6(2n+1)W_n^2}{(n+1)(n+2)}x^n = \sum_{n=0}^{\infty}u_nx^n.\end{aligned} \tag{3.5}$$

Elaborated computations show that

$$u_0 = \pi - 3 > 0, \quad u_1 = \frac{\pi - 3}{4} > 0, \quad u_2 = \frac{14\pi - 45}{128} < 0, \tag{3.6}$$

$$u_3 = \frac{31\pi - 105}{512} < 0, \quad u_4 = \frac{626\pi - 2205}{16384} < 0, \quad u_5 = \frac{1718\pi - 6237}{65536} < 0, \tag{3.7}$$

$$u_6 = \frac{79948\pi - 297297}{4194304} < 0, \quad u_7 = \frac{242659\pi - 920205}{16777216} < 0. \tag{3.8}$$

From (1.1), (2.2), (2.4), (3.4) and (3.5) we get

$$G_2(0^+) = \frac{\pi}{4} F\left(\frac{1}{2}, \frac{1}{2}; 2; 0^+\right) e^{\mathscr{K}(0^+)} = \frac{\pi}{4} e^{\pi/2}, \tag{3.9}$$

$$F\left(\frac{1}{2}, \frac{1}{2}; 2; 1^-\right) = \frac{\Gamma(2)\Gamma(1)}{\Gamma^2(3/2)} = \frac{4}{\pi},$$

$$\begin{aligned} G_2(1^-) &= \lim_{x \to 1^-} \sqrt{1-x} e^{\frac{\pi}{2} F(1/2, 1/2; 1; x)} \\ &= \lim_{x \to 1^-} \sqrt{1-x} e^{2\log 2 - \log\sqrt{1-x} + O((1-x)\log(1-x))} = 4, \end{aligned} \tag{3.10}$$

$$\begin{aligned} G_3(1^-) &= \pi \left(\frac{4}{\pi}\right)^2 - \frac{16}{\pi} + \lim_{x \to 1^-} (1-x) F\left(\frac{3}{2}, \frac{3}{2}; 3; x\right) \\ &= \frac{8}{\pi} \lim_{x \to 1^-} (1-x)\left[4(\log 2 - 1) - \log(1-x) + O((1-x)\log(1-x))\right] = 0. \end{aligned} \tag{3.11}$$

From Lemmas 2.1 and 2.5 together with (3.5)–(3.8) and (3.11) we have

$$G_3(x) > 0 \tag{3.12}$$

for $x \in (0,1)$.

Therefore, Theorem 3.2 follows from (3.3)–(3.5), (3.9), (3.10) and (3.12). □

THEOREM 3.3. *The function*

$$r \mapsto e^{\mathscr{K}(r)} - \frac{p}{r'}$$

is strictly decreasing on $(0,1)$ *if and only if* $p \geqslant 4$ *and strictly increasing on* $(0,1)$ *if and only if* $p \leqslant \pi e^{\pi/2}/4 = 3.7781401\cdots$.

Proof. Let $x = r^2$, $G_2(x)$ be defined by (3.4), and

$$G_4(x) = e^{\mathscr{K}(\sqrt{x})} - \frac{p}{\sqrt{1-x}} = e^{\mathscr{K}(r)} - \frac{p}{r'} = e^{\frac{\pi}{2}F(1/2,1/2;1;x)} - \frac{p}{\sqrt{1-x}}. \tag{3.13}$$

Then (2.1) and (2.3) lead to

$$G_4'(x) = \frac{\pi}{8} F\left(\frac{3}{2}, \frac{3}{2}; 2; x\right) e^{\mathscr{K}(\sqrt{x})} - \frac{p}{2(1-x)^{3/2}} \tag{3.14}$$
$$= \frac{\frac{\pi}{4}\sqrt{1-x} F\left(\frac{1}{2}, \frac{1}{2}; 2; x\right) e^{\mathscr{K}(\sqrt{x})} - p}{2(1-x)^{3/2}} = \frac{G_2(x) - p}{2(1-x)^{3/2}}.$$

It follows from Theorem 3.2 that $G_2(x)$ is strictly increasing from $(0,1)$ onto $(\pi e^{\pi/2}/4, 4)$. Therefore, Theorem 3.3 follows from (3.13) and (3.14) together with the monotonicity of $G_2(x)$ on the interval $(0,1)$. □

From (1.3), (1.4) and Theorem 3.2 we get Corollary 3.4 immediately.

COROLLARY 3.4. *The double inequalities*

$$r'^2 \mathscr{K}(r) + p\frac{r^2}{r'} e^{-\mathscr{K}(r)} < \mathscr{E}(r) < r'^2 \mathscr{K}(r) + q\frac{r^2}{r'} e^{-\mathscr{K}(r)},$$

$$\frac{r^2}{AG(1,r)} + \frac{2pr'^2}{\pi r} e^{-\frac{\pi}{2AG(1,r)}} < T(1,r) < \frac{r^2}{AG(1,r)} + \frac{2qr'^2}{\pi r} e^{-\frac{\pi}{2AG(1,r)}} \tag{3.15}$$

hold for all $r \in (0,1)$ *if and only if* $p \leqslant \pi e^{\pi/2}/4 = 3.7781401\cdots$ *and* $q \geqslant 4$.

REMARK 3.5. Let x, y, z be nonnegative real numbers such that at most one of them is 0. Then the symmetric elliptic integral of the second kind $R_G(x,y,z)$ is defined by

$$R_G(x,y,z) = \frac{1}{4}\int_0^{\infty} [(t+x)(t+y)(t+z)]^{-1/2} \left(\frac{x}{t+x} + \frac{y}{t+y} + \frac{z}{t+z}\right),$$

and the Toader mean $T(a,b)$ can be written as

$$T(a,b) = \frac{4}{\pi} R_G\left(a^2, b^2, 0\right) =: R_E\left(a^2, b^2\right).$$

Using the inequalities for R_E presented in [45, 46] one can obtain bounds for T which are sharper than those in (3.15).

COROLLARY 3.6. *The double inequality*

$$\frac{\pi}{2} + p\log\frac{1}{r'} < \mathscr{K}(r) < \frac{\pi}{2} + q\log\frac{1}{r'} \tag{3.16}$$

is valid for all $r \in (0,1)$ *if and only if* $p \leqslant \pi/4$ *and* $q \geqslant 1$. *Moreover, we have*

$$\log\frac{4}{r'} < \mathscr{K}(r) < \frac{\pi}{2} + \log\frac{1}{r'} \tag{3.17}$$

for all $r \in (0,1)$.

Proof. If $p \leqslant \pi/4$ and $q \geqslant 1$, then inequality (3.16) follows from Theorem 3.1 and the fact that

$$\lim_{r\to 0^+} r'^p e^{\mathscr{K}(r)} = e^{\pi/2}.$$

If the first inequality of (3.16) holds for all $r \in (0,1)$, then (1.2) leads to

$$\lim_{r\to 0^+} \frac{\mathscr{K}(r) - \left(\frac{\pi}{2} + p\log\frac{1}{r'}\right)}{r^2} = \frac{\pi}{8} - \frac{p}{2} \geqslant 0,$$

which leads to $p \leqslant \pi/4$.

If the second inequality of (3.16) holds for all $r \in (0,1)$, then it follows from (2.4) that

$$\lim_{x\to 1^-} \frac{\mathscr{K}(\sqrt{x}) - \left(\frac{\pi}{2} + q\log\frac{1}{\sqrt{1-x}}\right)}{\log(1-x)}$$
$$= \lim_{x\to 1^-} \frac{\log 4 - \frac{\pi}{2} + \frac{q-1}{2}\log(1-x) + O((1-x)\log(1-x))}{\log(1-x)} = \frac{q-1}{2} \geqslant 0,$$

which leads to $q \geqslant 1$.

Inequality (3.17) follows from the second inequality of (3.16) and Theorem 3.1 together with the fact that

$$\lim_{r\to 1^-} r' e^{\mathscr{K}(r)} = \lim_{x\to 1^-} \sqrt{1-x}\, e^{\frac{\pi}{2}F(1/2,1/2;1;x)} = 4. \quad \square$$

COROLLARY 3.7. *The double inequality*

$$\log\left(e^{\pi/2} - p + \frac{p}{r'}\right) < \mathscr{K}(r) < \log\left(e^{\pi/2} - q + \frac{q}{r'}\right) \tag{3.18}$$

holds for all $r \in (0,1)$ *if and only if* $p \leqslant \pi e^{\pi/2}/4 = 3.7781401\cdots$ *and* $q \geqslant 4$. *Moreover, we have*

$$\log\frac{4}{r'} < \mathscr{K}(r) < \log\left(e^{\pi/2} - 4 + \frac{4}{r'}\right) \tag{3.19}$$

for all $r \in (0,1)$.

Proof. If $p \leqslant \pi e^{\pi/2}/4$ and $q \geqslant 4$, then inequality (3.18) follows from Theorem 3.3 and the fact that

$$\lim_{r\to 0^+} \left(e^{\mathscr{K}(r)} - \frac{p}{r'}\right) = e^{\pi/2} - p.$$

If the first inequality of (3.18) holds for all $r \in (0,1)$, then (1.2) leads to

$$\lim_{x\to 0^+} \frac{\sqrt{1-x}\left[e^{\mathscr{K}(\sqrt{x})} + p - e^{\pi/2}\right] - p}{x}$$
$$= \lim_{x\to 0^+} \frac{\sqrt{1-x}\left[e^{\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)^2}{(n!)^2}x^n} + p - e^{\pi/2}\right] - p}{x} = \frac{\pi e^{\pi/2}}{8} - \frac{p}{2} \geqslant 0,$$

which leads to $p \leqslant \pi e^{\pi/2}/4$.

If the second inequality of (3.18) holds for all $r \in (0,1)$, then (1.2) and (2.4) lead to

$$\begin{aligned} q &\geqslant \lim_{x \to 1^-} \sqrt{1-x}\left[e^{\frac{\pi}{2}F(1/2,1/2;1;x)} - e^{\pi/2} + q\right] \\ &= \lim_{x \to 1^-} \sqrt{1-x}\left[e^{\log 4 - \frac{1}{2}\log(1-x) + O((1-x)\log(1-x))} - e^{\pi/2} + q\right] = 4. \end{aligned}$$

Inequality (3.19) follows from the first inequality of (3.17) and the second inequality of (3.18). □

REMARK 3.8. Let

$$R_F(x,y,z) = \frac{1}{2}\int_0^{\infty} [(t+x)(t+y)(t+z)]^{-1/2}\, dt$$

be the symmetric elliptic integral of the first kind. Then the complete elliptic integral of the first kind $\mathscr{K}(r)$ can be expressed by

$$\mathscr{K}(r) = R_F\left(r'^2, 1, 0\right) := \frac{\pi}{2} R_K\left(r'^2, 1\right).$$

Bounds for complete elliptic integral $\mathscr{K}$ included in Corollaries 3.6 and 3.7 are not necessarily simple and sharp. Using the known bounds for R_K given in [39, 40, 43, 46] on can obtain simple and sharp bounds which are sharper than those given in Corollaries 3.6 and 3.7.

From (1.4), Corollary 3.6 and Corollary 3.7 we get Corollary 3.9 immediately.

COROLLARY 3.9. *The double inequalities*

$$\frac{1}{1 - p\log r} < AG(1,r) < \frac{1}{1 - q\log r},$$

$$\frac{\pi}{2\log\left(e^{\pi/2} - \lambda + \frac{\lambda}{r}\right)} < AG(1,r) < \frac{\pi}{2\log\left(e^{\pi/2} - \mu + \frac{\mu}{r}\right)}$$

hold for all $r \in (0,1)$ *if and only if* $p \geqslant 2/\pi$, $q \leqslant 1/2$, $\lambda \geqslant 4$ *and* $\mu \leqslant \pi e^{\pi/2}/4$. *Moreover, one has*

$$\frac{1}{1 - \frac{2}{\pi}\log r} < AG(1,r) < \frac{\pi}{2\log\frac{4}{r}},$$

$$AG(1,r) > \frac{\pi}{2\log\left(e^{\pi/2} - 4 + \frac{4}{r}\right)} \tag{3.20}$$

for all $r \in (0,1)$.

REMARK 3.10. Neuman and Sándor [45, Theorem 3.2] proved that the inequality

$$AG(1,r) > \frac{\frac{1+r}{2} - \sqrt{r}}{\log \frac{1+r}{2\sqrt{r}}} \tag{3.21}$$

holds for all $r \in (0,1)$.

Let

$$I(r) = \frac{\frac{1+r}{2} - \sqrt{r}}{\log \frac{1+r}{2\sqrt{r}}}, \quad J(r) = \frac{\pi}{2\log\left(e^{\pi/2} - 4 + \frac{4}{r}\right)}. \tag{3.22}$$

Then numerical computations lead to

$$I(0.05) = 0.3531\cdots < J(0.05) = 0.3576\cdots, \tag{3.23}$$

$$I(0.1) = 0.4223\cdots < J(0.1) = 0.4235\cdots. \tag{3.24}$$

From (3.22)–(3.24) we know that the lower bound for $AG(1,r)$ given in (3.20) is better than that given in (3.21) for some $r \in (0,1)$.

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REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, 1965.

[2] M. ADIL KHAN, S. BEGUM, Y. KHURSHID AND Y.-M. CHU, *Ostrowski type inequalities involving conformable fractional integrals*, J. Inequal. Appl. **2018** (2018), Article 70, 14 pages.

[3] M. ADIL KLAN, Y.-M. CHU, T. U. KHAN AND J. KHAN, *Some new inequalities of Hermite-Hadamard type of s-convex functions with applications*, Open Math. **15** (2017), 1414–1430.

[4] H. ALZER, *Sharp inequalities for the complete elliptic integral of the first kind*, Math. Proc. Cambridge Philos. Soc. **124**, 2 (1998), 309–314.

[5] H. ALZER AND S.-L. QIU, *Monotonicity theorem and inequalities for the complete elliptic integrals*, J. Comput. Appl. Math., **172**, 2 (2004), 289–312.

[6] H. ALZER AND K. RICHARDS, *On the modulus of the Grötzsch ring*, J. Math. Anal. Appl. **432**, 1 (2015), 134–141.

[7] H. ALZER AND K. RICHARDS, *Inequalities for the ratio of complete elliptic integrals*, Proc. Amer. Math. Soc. **145**, 4 (2017), 1661–1670.

[8] G. D. ANDERSON, S.-L. QIU AND M. K. VAMANAMURTHY, *Elliptic integral inequalities, with applications*, Constr. Approx. **14**, 2 (1998), 195–207.

[9] G. D. ANDERSON, S.-L. QIU, M. K. VAMANAMURTHY AND M. VUORINEN, *Generalized elliptic integrals and modular equations*, Pacific J. Math. **192**, 1 (2000), 1–37.

[10] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Functional inequalities for hypergeometric functions and complete elliptic integrals*, SIAM. J. Math. Anal. **23**, 2 (1992), 512–524.

[11] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.

[12] R. W. BARNARD, K. PEARCE AND K. C. RICHARDS, *An inequality involving the generalized hypergeometric function and the arc length of an ellipse*, SIAM J. Math. Anal. **31**, 3 (2000), 693–699.

[13] J. M. BORWEIN AND P. B. BORWEIN, *Pi and the AGM*, John Wiley & Sons, New York, 1987.

[14] P. BRACKEN, *An arithmetic-geometric mean inequality*, Expo. Math. **19**, 3 (2001), 273–279.

[15] P. F. BYRD AND M. D. FRIEDMAN, *Handbook of Elliptic Integrals for Engineers and Scientists*, Springer-Verlag, New York, 1971.

[16] B. C. CARLSON AND J. L. GUSTAFSON, *Asymptotic expansion of the first elliptic integral*, SIAM J. Math. Anal. **16**, 5 (1985), 1072–1092.

[17] B. C. CARLSON AND M. VUORINEN, *Inequality of the AGM and the logarithmic mean*, SIAM Rev. **33**, 4 (1991), 653–654.

[18] Y.-M. CHU AND Y.-P. LV, *The Schur harmonic convexity of the Hamy symmetric function and its applications*, J. Inequal. Appl. **2009** (2009), Article ID 838529, 10 pages.

[19] Y.-M. CHU, Y.-F. QIU AND M.-K. WANG, *Hölder mean inequalities for the complete elliptic integrals*, Integral Transforms Spec. Funct. **23**, 7 (2012), 521–527.

[20] Y.-M. CHU, S.-L. QIU AND M.-K. WANG, *Sharp inequalities involving the power mean and complete elliptic integral of the first kind*, Rocky Mountain J. Math. **43**, 3 (2013), 1489–1496.

[21] Y.-M. CHU AND T.-C. SUN, *The Schur harmonic convexity for a class of symmetric functions*, Acta Math. Sci. **30B**, 5 (2010), 1501–1506.

[22] Y.-M. CHU AND M.-K. WANG, *Optimal Lehmer mean bounds for the Toader mean*, Results Math. **61**, 3-4 (2012), 223–229.

[23] Y.-M. CHU, M.-K. WANG, Y.-P. JIANG AND S.-L. QIU, *Concavity of the complete elliptic integrals of the second kind with respect to Hölder means*, J. Math. Anal. Appl. **395**, 2 (2012), 637–642.

[24] Y.-M. CHU, M.-K. WANG AND X.-Y. MA, *Sharp bounds for Toader mean in terms of contraharmonic mean with applications*, J. Math. Inequal. **7**, 2 (2013), 161–166.

[25] Y.-M. CHU, M.-K. WANG AND S.-L. QIU, *Optimal combinations bounds of root-square and arithmetic means for Toader mean*, Proc. Indian Acad. Sci. Math. Sci. **122**, 1 (2012), 41–51.

[26] Y.-M. CHU, M.-K. WANG, S.-L. QIU AND Y.-P. JIANG, *Bounds for complete integrals of the second kind with applications*, Comput. Math. Appl. **63**, 7 (2012), 1177–1184.

[27] Y.-M. CHU, M.-K. WANG, S.-L. QIU AND Y.-F. QIU, *Sharp generalized Seiffert mean bounds for Toader mean*, Abstr. Appl. Anal. **2011** (2011), Article ID 605259, 8 pages.

[28] Y.-M. CHU, G.-D. WANG AND X.-H. ZHANG, *Schur convexity and Hadamard's inequality*, Math. Inequal. Appl. **13**, 4 (2010), 725–731.

[29] Y.-M. CHU, G.-D. WANG AND X.-H. ZHANG, *The Schur multiplicative and harmonic convexities of the complete symmetric function*, Math. Nachr. **284**, 5–6 (2011), 653–663.

[30] Y.-M. CHU AND W.-F. XIA, *Two sharp inequalities for power mean, geometric mean, and harmonic mean*, J. Inequal. Appl. **2009** (2009), Article ID 741923, 6 pages.

[31] Y.-M. CHU AND W.-F. XIA, *Inequalities for generalized logarithmic means*, J. Inequal. Appl. **2009** (2009), Article ID 763252, 7 pages.

[32] Y.-M. CHU AND W.-F. XIA, *Solution of an open problem for Schur convexity or concavity of the Gini mean calues*, Sci. China **52A**, 10 (2009), 2099–2106.

[33] Y.-M. CHU AND W.-F. XIA, *Two optimal double inequalities between power mean and logarithmic mean*, Comput. Math. Appl. **60**, 1 (2010), 83–89.

[34] Y.-M. CHU, W.-F. XIA AND X.-H. ZHANG, *The Schur convexity, Schur multiplicative and harmonic convexities of the second dual form of the Hamy symmetric function with applications*, J. Multivariate Anal. **105** (2012), 412–421.

[35] Y.-M. CHU, W.-F. XIA AND T.-H. ZHAO, *Schur convexity for a class of symmetric functions*, Sci. China Math. **53**, 2 (2010), 465–474.

[36] Y.-M. CHU, W.-F. XIA AND T.-H. ZHAO, *Some properties for a class of symmetric functions and applications*, J. Math. Inequal. **5**, 1 (2011), 1–11.

[37] Y.-M. CHU AND X.-M. ZHANG, *Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave*, J. Math. Kyoto Univ. **48**, 1 (2008), 229–238.

[38] Y.-M. CHU, X.-M. ZHANG AND G.-D. WANG, *The Schur geometrical convexity of of the extended mean values*, J. Convex Anal. **15**, 4 (2008), 707–718.

[39] H. KAZI AND E. NEUMAN, *Inequalities and bounds for elliptic integrals*, J. Approx. Theory **146**, 2 (2007), 212–226.

[40] H. KAZI AND E. NEUMAN, *Inequalities and bounds for elliptic integrals II*, in: Special Functions and Orghogonal Polynomials, 127–138, Contemp. Math., 471, Amer. Math. Soc., Providence, RI, 2008.

[41] R. KÜHNAU, *Eine Methode, die Positivität einer Funktion zu prüfen*, Z. Angew. Math. Mech. **74** (1994), 140–143.

[42] Y.-M. LI, W.-F. XIA, Y.-M. CHU AND X.-H. ZHANG, *Optimal lower and upper bounds for the geometric convex combination of the error function*, J. Inequal. Appl. **2015** (2015), Article 382, 8 pages.

[43] E. NEUMAN, *Bounds for symmetric elliptic integrals*, J. Approx. Theory **122**, 2 (2003), 249–259.

[44] E. NEUMAN, *Inequalities for weighted sums of powers and their applications*, Math. Inequal. Appl. **15**, 4 (2012), 995–1005.

[45] E. NEUMAN AND J. SÁNDOR, *On certain means of two arguments and their extension*, Int. J. Math. Sci. **16** (2003), 981–993.

[46] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT AND C. W. CLARK, NIST Digital Library of Mathematical Functions, available onlin at `http://dlmf.nist.gov`.

[47] F. QI, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl. **2010** (2010), Article ID 493058, 84 pages.

[48] W.-M. QIAN AND Y.-M. CHU, *Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters*, J. Inequal. Appl. **2017** (2017), Article 274, 10 pages.

[49] S.-L. QIU AND M. K. VAMANAMURTHY, *Sharp estimates for complete elliptic integrals*, SIAM. J. Math. Anal. **27**, 3 (1996), 823–834.

[50] S.-L. QIU AND M. VUORINEN, *Duplication inequalities for the ratios of hypergeometric functions*, Forum Math. **12**, 1 (2000), 109–133.

[51] J. SÁNDOR, *On certain inequalities for means*, J. Math. Anal. Appl. **189**, 2 (1995), 602–606.

[52] J. SÁNDOR, *On certain inequalities for means II*, J. Math. Anal. Appl. **199**, 2 (1996), 629–635.

[53] J. SÁNDOR, *On certain inequalities for means III*, Arch. Math. **76**, 1 (2001), 34–40.

[54] K. B. STOLARSKY, *Generalizations of the logarithmic mean*, Math. Mag. **48** (1975), 87–92.

[55] GH. TOADER, *Some mean values related to the arithmetic-geometric mean*, J. Math. Anal. Appl. **218**, 2 (1998), 358–368.

[56] M. K. VAMANAMURTHY AND M. VUORINEN, *Inequalities for Means*, J. Math. Anal. Appl. **183**, 1 (1994), 155–166.

[57] M.-K. WANG AND Y.-M. CHU, *Refinements of transformation inequalities for zero-balanced hypergeometric functions*, Acta Math. Sci. **37B**, 3 (2017), 607–622.

[58] M.-K. WANG AND Y.-M. CHU, *Landen inequalities for a class of hypergeometric functions with applications*, Math. Inequal. Appl. **21**, 2(2018), 521–537.

[59] M.-K. WANG, Y.-M. CHU AND Y.-P. JIANG, *Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions*, Rocky Mountain J. Math. **46**, 2 (2016), 679–691.

[60] M.-K. WANG, Y.-M. CHU AND S.-L. QIU, *Some monotonicity properties of generalized elliptic integrals with applications*, Math. Inequal. Appl. **16**, 3 (2013), 671–677.

[61] M.-K. WANG, Y.-M. CHU AND S.-L. QIU, *Sharp bounds for generalized elliptic integrals of the first kind*, J. Math. Anal. Appl. **429**, 2 (2015), 744–757.

[62] M.-K. WANG, Y.-M. CHU, Y.-F. QIU AND S.-L. QIU, *An optimal power mean inequality for the complete elliptic integrals*, Appl. Math. Lett. **24**, 6 (2011), 887–890.

[63] M.-K. WANG, Y.-M. CHU AND Y.-Q. SONG, *Asymptotical formulas for Gaussian and generalized hypergeometric functions*, Appl. Math. Comput. **276** (2016), 44–60.

[64] M.-K. WANG, Y.-M. LI AND Y.-M. CHU, *Inequalities and infinite product formula for Ramanujan generalized modular equation function*, Ramanujan J. **46**, 1 (2018), 189–200.

[65] M.-K. WANG, S.-L. QIU AND Y.-M. CHU, *Infinite series formula for Hübner upper bounds founction with applications to Hersch-Pfluger distortion function*, Math. Inequal. Appl. **21**, 3 (2017), 629–648.

[66] G.-D. WANG, X.-H. ZHANG AND Y.-M. CHU, *A power mean inequality involving the complete elliptic integrals*, Rocky Mountain J. Math. **44**, 5 (2014), 1661–1667.

[67] W.-F. XIA AND Y.-M. CHU, *Schur-convexity for a class of symmetric functions and its applications*, J. Inequal. Appl. **2009** (2009), Article ID 493759, 15 pages.

[68] W.-F. XIA AND Y.-M. CHU, *The Schur convexity of Gini mean values in the sense of harmonic mean*, Acta Math. Sci. **31B**, 3 (2011), 1103–1112.

[69] W.-F. XIA AND Y.-M. CHU, *Optimal inequalities between Neuman-Sándor, centroical and harmonic means*, J. Math. Inequal. **7**, 4 (2013), 593–600.

[70] W.-F. XIA AND Y.-M. CHU, *Optimal inequalities for the convex combination of error function*, J. Math. Inequal. **9**, 1 (2015), 85–99.

[71] W.-F. XIA, Y.-M. CHU AND G.-D. WANG, *The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means*, Abstr. Appl. Anal. **2010** (2010), Article ID 604804, 9 pages.

[72] W.-F. XIA, W. JANOUS AND Y.-M. CHU, *The optimal convex combination bounds of arithmetic and hamronic means in terms of power mean*, J. Math. Inequal. **6**, 2 (2012), 241–248.

[73] W.-F. XIA, X.-H. ZHANG, G.-D. WANG AND Y.-M. CHU, *Some properties for a class of symmetric functions with applications*, Indian J. Pure Appl. Math. **43**, 3 (2012), 227–249.

[74] ZH.-H. YANG AND Y.-M. CHU, *Asymptotic formulas for gamma function with applications*, Appl. Math. Comput. **270** (2015), 665–680.

[75] ZH.-H. YANG AND Y.-M. CHU, *A monotonicity property involving the generalized elliptic integral of the first kind*, Math. Inequal. Appl. **20**, 3(2017), 729–735.

[76] ZH.-H. YANG, Y.-M. CHU AND W. ZHANG, *Accurate approximations for the complete elliptic integrals of the second kind*, J. Math. Anal. Appl. **438**, 2 (2016), 875–888.

[77] ZH.-H. YANG, Y.-M. CHU AND X.-H. ZHANG, *Sharp Stolarsky mean bounds for the complete elliptic integral of the second kind*, J. Nonlinear Sci. Appl. **10**, 3 (2017), 929–936.

[78] ZH.-H. YANG, W.-M. QIAN AND Y.-M. CHU, *On rational bounds for the gamma function*, J. Inequal. Appl. **2017** (2017), Article 210, 17 pages.

[79] ZH.-H. YANG, W.-M. QIAN, Y.-M. CHU AND W. ZHANG, *On approximating the error function*, Math. Inequal. Appl. **21**, 2 (2018), 469–479.

[80] ZH.-H. YANG, W.-M. QIAN, Y.-M. CHU AND W. ZHANG, *On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind*, J. Math. Anal. Appl. **462**, 2 (2018), 1714–1726.

[81] ZH.-H. YANG, Y.-Q. SONG AND Y.-M. CHU, *Sharp bounds for the arithmetic-geometric mean*, J. Inequal. Appl. **2014** (2014), Article 192, 13 pages.

[82] ZH.-H. YANG, W. ZHANG AND Y.-M. CHU, *Sharp Gautschi inequality for parameter* $0 < p < 1$ *with applications*, Math. Inequal. Appl. **20**, 4 (2017), 1107–1120.

[83] X.-H. ZHANG, G.-D. WANG AND Y.-M. CHU, *Convexity with respect to Hölder mean involving zero-balanced hypergeometric functions*, J. Math. Anal. Appl. **353**, 1 (2009), 256–259.

[84] T.-H. ZHAO AND Y.-M. CHU, *A class of logarithmically completely monotonic functions associated with a gamma function*, J. Inequal. Appl. **2010** (2010), Artilce ID 392431, 11 pages.

[85] T.-H. ZHAO, Y.-M. CHU AND Y.-P. JIANG, *Monotonic and logarithmically convex properties of a function involving gamma functions*, J. Inequal. Appl. **2009** (2009), Article ID 72861, 13 pages.

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