

SOBOLEV'S THEOREM FOR DOUBLE PHASE FUNCTIONALS

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Abstract. Our aim in this paper is to establish generalizations of Sobolev's theorem for double phase functionals $\Phi(x, t) = t^p + \{b(x)t(\log(e+t))^\tau\}^q$, where $1 < p \leq q < \infty$, $\tau > 0$ and b is a nonnegative bounded function satisfying $|b(x) - b(y)| \leq C|x - y|^\theta (\log(e + |x - y|^{-1}))^{-\tau}$ for $0 \leq \theta < 1$.

1. Introduction

In harmonic analysis, the maximal operator is a classical tool when studying Sobolev functions and partial differential equations. This also plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on ([5, 21, 25], etc.). It is well known that the maximal operator is bounded on the Lebesgue space $L^p(\mathbf{R}^N)$ if $p > 1$ ([25]).

One of important applications of the boundedness of the maximal operator is Sobolev's inequality; in the classical case,

$$\|I_\alpha * f\|_{p^*} \leq C\|f\|_p$$

for $f \in L^p(\mathbf{R}^N)$, $0 < \alpha < N$ and $1 < p < N/\alpha$, where I_α is the Riesz kernel of order α and $1/p^* = 1/p - \alpha/N$ (see, e.g. [2, Theorem 3.1.4]).

There has been a considerable amount of studies on the variable exponent Lebesgue spaces and Sobolev spaces; see [11, 13] for a survey. We refer to [1, 24] for the study of elasticity and fluid mechanics, [7, 19] for the study of image processing, and [9, 10] for double phase variational problems. Capone, Cruz-Uribe and Fiorenza [6] studied a Sobolev type inequality for Riesz potentials in the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{R}^N)$. For Sobolev's theorem for Riesz potentials, see also [12], [14], [20] etc..

Recently, regarding regularity theory of differential equations, Baroni, Colombo and Mingione [3, 4, 9, 10] studied a double phase functional $\Phi(x, t) = t^p + a(x)t^q$, $x \in \mathbf{R}^N$, $t \geq 0$ where $1 < p < q$, $a(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0, 1]$. In [9], the minimization problem of the double phase functional was discussed under the assumption $q < (1 + \theta/N)p$. Regularity for general functionals

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was studied under the condition $q \leq (1 + \theta/N)p$ in [4]. In [3], the border-line ($p = q$) double phase functional

$$\Phi(x, t) = t^p + a(x)t^p(\log(e + t))$$

was considered. In [18], Harjulehto, Hästö and Karppinen studied local higher integrability of the gradient of a quasiminimizer of the double phase functional $\Phi(x, t) = t^p + a(x)t^q$. See Colasuonno and Squassina [8] for the eigenvalue problem for the double phase functional $\Phi(x, t) = t^p + a(x)t^q$. See also [16].

In the present paper, for $1 < p \leq q < \infty$ and $\tau \geq 0$, let us consider the double phase functional

$$\Phi(x, t) = t^p + \{b(x)t(\log(e + t))^\tau\}^q,$$

where b is a nonnegative bounded function satisfying

$$|b(x) - b(y)| \leq C|x - y|^\theta (\log(e + |x - y|^{-1}))^{-\tau}$$

for $0 \leq \theta < 1$. Hästö [15, Theorem 4.7] showed the boundedness of the maximal operator on $L^\Phi(G)$ when $a(x) = b(x)^q$ is θ -Hölder continuous and $\tau = 0$. See also [17, Proposition 7.2.3].

Our first aim in this paper is to give the boundedness of the maximal operator for the double phase functional $\Phi(x, t)$ (Theorem 1), as an extension of [15, Theorem 4.7]. To show this, we apply [23, Corollary 3.2]. Our strategy is to check all the conditions required in [23, Corollary 3.2] as in the proof of [23, Corollary 5.3]. Next, we give a Sobolev type inequality for $\Phi(x, t)$ (Theorem 2) by applying [23, Theorem 4.9] as in the proof of [23, Corollary 5.9].

For reader's convenience we give direct proofs of Theorems 1 and 2 in the Appendix, by applying the boundedness of the maximal operator on $L^p(\mathbf{R}^N)$ and $L^q(\mathbf{R}^N)$.

Finally, we discuss the continuity of the fractional maximal functions and Riesz potentials for the double phase functional $\Phi(x, t)$ (see Theorems 3-5). The result is new even for the case $\tau = 0$.

Throughout this paper, let C denote various constants independent of the variables in question and $C(a, b, \dots)$ be a constant that depends on a, b, \dots . The symbol $g \sim h$ means that $C^{-1}h \leq g \leq Ch$ for some constant $C > 0$.

2. Preliminaries

In this paper, we consider the following double phase functional

$$\Phi(x, t) = t^p + \{b(x)t(\log(e + t))^\tau\}^q,$$

for $1 < p \leq q < \infty$ and $\tau \geq 0$, where b is a nonnegative bounded function satisfying

$$|b(x) - b(y)| \leq C|x - y|^\theta (\log(e + |x - y|^{-1}))^{-\tau} \quad (1)$$

for $0 \leq \theta < 1$.

The Musielak-Orlicz space $L^\Phi(\mathbf{R}^N)$ is defined by

$$L^\Phi(\mathbf{R}^N) = \left\{ f \in L^1_{\text{loc}}(\mathbf{R}^N) : \int_{\mathbf{R}^N} \Phi\left(y, \frac{|f(y)|}{\lambda}\right) dy < \infty \text{ for some } \lambda > 0 \right\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(\mathbf{R}^N)} = \inf \left\{ \lambda > 0 : \int_{\mathbf{R}^N} \Phi\left(y, \frac{|f(y)|}{\lambda}\right) dy \leq 1 \right\}.$$

For later use, we prepare the following result.

LEMMA 1. *Let $1 \leq q < \infty$ and $\tau \geq 0$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} \frac{1}{|E|} \int_E |f(y)| dy &\leq C(\log(e + r^{-1}))^{-\tau} \\ &\quad \times \frac{1}{|E|} \int_E |f(y)| (\log(e + |f(y)|))^\tau dy + r^{-N/q} (\log(e + r^{-1}))^{-\tau} \end{aligned}$$

for all $r > 0$ and measurable sets $E \subset \mathbf{R}^N$ of positive measure.

Proof. Set $R = r^{-N/q} (\log(e + r^{-1}))^{-\tau}$. We have

$$\begin{aligned} \frac{1}{|E|} \int_E |f(y)| dy &\leq \frac{1}{|E|} \int_E |f(y)| \left(\frac{\log(e + |f(y)|)}{\log(e + R)} \right)^\tau dy + R \\ &\leq C(\log(e + r^{-1}))^{-\tau} \frac{1}{|E|} \int_E |f(y)| (\log(e + |f(y)|))^\tau dy \\ &\quad + r^{-N/q} (\log(e + r^{-1}))^{-\tau}, \end{aligned}$$

as required. \square

COROLLARY 1. *Let $1 \leq q < \infty$ and $\tau \geq 0$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} \frac{1}{|E|} \int_E b(y)|f(y)| dy &\leq C \left\{ r^{-N/q} (\log(e + r^{-1}))^{-\tau} + (\log(e + r^{-1}))^{-\tau} \right. \\ &\quad \left. \times \left(\frac{1}{|E|} \int_E (b(y)|f(y)| (\log(e + |f(y)|))^\tau)^q dy \right)^{1/q} \right\} \end{aligned}$$

for all $r > 0$ and measurable sets $E \subset \mathbf{R}^N$ of positive measure.

In fact, Lemma 1, the boundedness of b and Jensen's inequality give

$$\frac{1}{|E|} \int_E b(y)|f(y)| dy$$

$$\begin{aligned}
&\leq C \left\{ r^{-N/q} (\log(e+r^{-1}))^{-\tau} + (\log(e+r^{-1}))^{-\tau} \frac{1}{|E|} \int_E b(y) |f(y)| (\log(e+|f(y)|))^\tau dy \right\} \\
&\leq C \left\{ r^{-N/q} (\log(e+r^{-1}))^{-\tau} + (\log(e+r^{-1}))^{-\tau} \right. \\
&\quad \left. \times \left(\frac{1}{|E|} \int_E (b(y) |f(y)| (\log(e+|f(y)|))^\tau)^q dy \right)^{1/q} \right\}.
\end{aligned}$$

3. Maximal operator

For a locally integrable function f on \mathbf{R}^N , the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

where $B(x,r)$ is the ball in \mathbf{R}^N with center x and of radius $r > 0$ and $|B(x,r)|$ denotes its Lebesgue measure. The mapping $f \mapsto Mf$ is called the maximal operator.

Recall that

$$\Phi(x,t) = t^p + \{b(x)t(\log(e+t))^\tau\}^q,$$

where $1 < p \leq q < \infty$, $\tau \geq 0$ and b is a nonnegative bounded function satisfying (1) with $0 \leq \theta < 1$.

In [23], we consider the following conditions for $\Phi(x,t)$. It is easy to check the following conditions on Φ required in [23]:

- (Φ1) $\Phi(\cdot, t)$ is measurable on \mathbf{R}^N for each $t \geq 0$ and $\Phi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in \mathbf{R}^N$;
- (Φ2) there exists a constant $A_1 \geq 1$ such that
$$A_1^{-1} \leq \Phi(x, 1) \leq A_1 \quad \text{for all } x \in \mathbf{R}^N;$$
- (Φ3; 0; p) $t \mapsto t^{-p}\Phi(x, t)$ is increasing on $(0, 1]$ for each $x \in \mathbf{R}^N$;
- (Φ3; ∞ ; p) $t \mapsto t^{-p}\Phi(x, t)$ is increasing on $[1, \infty)$ for each $x \in \mathbf{R}^N$.

LEMMA 2. $\Phi(x, t)$ satisfies

- (Φ5; v) for every $\gamma > 0$, there exists a constant $B_{\gamma, v} \geq 1$ such that

$$\Phi(x, t) \leq B_{\gamma, v} \Phi(y, t)$$

whenever $x, y \in \mathbf{R}^N$, $|x - y| \leq \gamma t^{-v}$ and $t \geq 1$

for $v \geq (q - p)/(q\theta)$; and

($\Phi 6; \omega$) there exist a function g on \mathbf{R}^N and a constant $B_\infty \geq 1$ such that $0 \leq g(x) \leq 1$ for all $x \in \mathbf{R}^N$, $g \in L^\omega(\mathbf{R}^N)$ and

$$B_\infty^{-1} \Phi(x, t) \leq \Phi(x', t) \leq B_\infty \Phi(x, t)$$

whenever $x, x' \in \mathbf{R}^N$, $|x'| \geq |x|$ and $g(x) \leq t \leq 1$,

for every $\omega > 0$.

Proof. Let $\nu \geq (q-p)/(q\theta)$. If $|x-y| \leq \gamma t^{-\nu}$ and $t \geq 1$, then

$$\begin{aligned} \Phi(x, t) &= t^p + \{b(x)t(\log(e+t))^\tau\}^q \\ &\leq t^p + \left\{ b(y)t(\log(e+t))^\tau + C|x-y|^\theta (\log(e+|x-y|^{-1}))^{-\tau} t(\log(e+t))^\tau \right\}^q \\ &\leq C \left\{ \Phi(y, t) + t^{(1-\theta\nu)q} \right\} \leq C \left\{ \Phi(y, t) + t^p \right\} \leq C \Phi(y, t). \end{aligned}$$

Hence $\Phi(x, t)$ satisfies ($\Phi 5; \nu$).

Let $g \in L^\omega(\mathbf{R}^N)$ for $\omega > 0$. If $g(x) \leq t \leq 1$, then

$$\Phi(x, t) \leq (1 + C\|b\|_{L^\infty(\mathbf{R}^N)}^q) t^p \leq (1 + C\|b\|_{L^\infty(\mathbf{R}^N)}^q) \Phi(x', t)$$

for every $x, x' \in \mathbf{R}^N$ and $|x'| \geq |x|$. Therefore $\Phi(x, t)$ satisfies ($\Phi 6; \omega$). \square

As an extension of [15, Theorem 4.7], we obtain the following result by Lemma 2 and [23, Corollary 3.2] (see also [17, Proposition 7.2.3] and [22]).

THEOREM 1. Suppose $1 < p \leq q < \infty$, $\tau \geq 0$ and $1/p - 1/q \leq \theta/N$. Then there is a constant $C > 0$ such that

$$\int_{\mathbf{R}^N} \Phi(x, Mf(x)) dx < C$$

for all $f \in L^\Phi(\mathbf{R}^N)$ with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq 1$.

REMARK 1. When $\tau = 0$, $\Phi(x, t) = t^p + a(x)t^q$, $1 < p < q$, $G \subset \mathbf{R}^N$ is bounded, $a \in C^\theta(\bar{G})$ is non-negative and $q \leq (1 + \theta/N)p$, Hästö showed Theorem 1 in [15, Theorem 4.7].

4. Sobolev's inequality

For $0 < \alpha < N$, we define the Riesz potential of order α of a function $f \in L_{\text{loc}}^1(\mathbf{R}^N)$ by

$$I_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y) dy.$$

In this section, let $g(x) = (1 + |x|)^{-(N+1)}$. Then

$$g^*(x) = \min\{g(x), Mg(x)\} \leq C(1 + |x|)^{-N}.$$

Further set

$$\Phi_\infty(t) = t^p$$

as in [23]. Then it satisfies the following conditions:

($\Phi_\infty 0$) $\Phi_\infty(t)$ is continuous, $\Phi_\infty(t) > 0$ for $t > 0$ and $\Phi_\infty(t)/t$ is increasing on $[0, \infty)$;

($\Phi_\infty 1$) there exists a constant $\tilde{B}_\infty \geq 1$ such that

$$\tilde{B}_\infty^{-1} \Phi(x, t) \leq \Phi_\infty(t) \leq \tilde{B}_\infty \Phi(x, t) \quad \text{whenever } g(x) \leq t \leq 1$$

for $g(x)$ in condition ($\Phi 6; \omega$);

($\Phi_\infty 2$) there exists a constant $c_\infty \geq 1$ such that

$$\Phi_\infty(g^*(x)) \leq c_\infty(1 + |x|)^{-N}$$

for all $x \in \mathbf{R}^N$;

($\Phi_\infty N$) $r \mapsto r^\gamma \Phi_\infty^{-1}(r^{-N})$ is increasing on $(1, \infty)$ for some $0 < \gamma < N$.

LEMMA 3. (1) If $q < N/\alpha$, then

($\Phi N \alpha$) $r \mapsto r^{\alpha+\varepsilon} \Phi^{-1}(x, r^{-N})$ is uniformly almost decreasing on $(0, \infty)$ for some $\varepsilon > 0$, that is, there exists a constant $C > 0$ such that

$$r^{\alpha+\varepsilon} \Phi^{-1}(x, r^{-N}) \leq C s^{\alpha+\varepsilon} \Phi^{-1}(x, s^{-N})$$

for all $0 < s < r$ and $x \in \mathbf{R}^N$.

(2) If $p < N/\alpha$, then

($\Phi_\infty N \alpha$) $r \mapsto r^{\alpha+\varepsilon} \Phi_\infty^{-1}(r^{-N})$ is increasing on $(1, \infty)$ for some $\varepsilon > 0$.

Proof.

(1) Since

$$\Phi^{-1}(x, s) \sim \min \left\{ s^{1/p}, (b(x)^{-q}s)^{1/q} (\log(e + b(x)^{-q}s))^{-\tau} \right\}$$

for $x \in \mathbf{R}^N$, we have

$$\begin{aligned} & r^{\alpha+\varepsilon} \Phi^{-1}(x, r^{-N}) \\ & \sim \min \left\{ r^{\alpha-N/p+\varepsilon}, b(x)^{-(\alpha+\varepsilon)q/N} (b(x)^q r^N)^{(\alpha-N/q+\varepsilon)/N} (\log(e + b(x)^{-q} r^{-N}))^{-\tau} \right\}. \end{aligned}$$

Choose $\varepsilon > 0$ such that $N/q - \alpha > \varepsilon$. Then we obtain that $r \mapsto r^{\alpha+\varepsilon} \Phi^{-1}(x, r^{-N})$ is uniformly almost decreasing.

(2) Since $\Phi_\infty^{-1}(r^{-N}) = r^{-N/p}$ for $r \geq 1$, we find $r^\alpha \Phi_\infty^{-1}(r^{-N}) = r^{\alpha-N/p}$. Thus, ($\Phi_\infty N \alpha$) holds if $p < N/\alpha$. \square

Now, consider the double phase functional

$$\Psi(x, t) = t^{p^*} + \{b(x)t(\log(e+t))^\tau\}^{q^*}$$

and

$$\Psi_1(x, t) = t^{p^*} + \left\{b(x)t \left(\log \left(e + b(x)^{\alpha q/N} t\right)\right)^\tau\right\}^{q^*},$$

where $1/p^* = 1/p - \alpha/N > 0$ and $1/q^* = 1/q - \alpha/N > 0$. Then there is $C > 1$ such that

$$C^{-1}\Psi(x, t) \leq \Psi_1(x, t) \leq C\Psi(x, t) \quad (2)$$

for all $x \in \mathbf{R}^N$ and $t > 0$. In fact, the right inequality is clear since b is bounded. To show the left inequality for $t > 1$, take ε such that $0 < \varepsilon < p^*/q^*$. If $b(x)t \leq t^\varepsilon$ and $t > 1$, then

$$\{b(x)t(\log(e+t))^\tau\}^{q^*} \leq Ct^{p^*}$$

and if $b(x)t \geq t^\varepsilon > 1$, then $b(x)^{\alpha q/N} t \geq t^{1-(\alpha q/N)(1-\varepsilon)}$, so that

$$\Psi_1(x, t) \geq t^{p^*} + \left\{b(x)t \left(\log \left(e + t^{1-(\alpha q/N)(1-\varepsilon)}\right)\right)^\tau\right\}^{q^*} \geq C\Psi(x, t)$$

since $\alpha q/N < 1$, which proves (2).

Further we see that both Ψ and Ψ_1 satisfy

($\Psi 1$) $\Psi(\cdot, t)$ is measurable on \mathbf{R}^N for each $t \geq 0$ and $\Psi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in \mathbf{R}^N$;

($\Psi 2$) there exists a constant $\widehat{Q}_1 \geq 1$ such that

$$\widehat{Q}_1^{-1} \leq \Psi(x, 1) \leq \widehat{Q}_1 \quad \text{for all } x \in \mathbf{R}^N;$$

($\Psi 3$) $t \mapsto \Psi(x, t)/t$ is increasing on $(0, \infty)$ for all $x \in \mathbf{R}^N$.

LEMMA 4. Both Ψ and Ψ_1 satisfy

($\Psi 4$) there exists a constant $\widehat{Q}_3 \geq 1$ such that

$$\Psi\left(x, t\Phi(x, t)^{-\alpha/N}\right) \leq \widehat{Q}_3\Phi(x, t)$$

for all $x \in \mathbf{R}^N$ and $t > 0$.

Proof. Since $\Phi(x, t) \geq \max\{t^p, \{b(x)t(\log(e+t))^\tau\}^q\}$, we see that

$$t\Phi(x, t)^{-\alpha/N} \leq \min\left\{t^{p/p^*}, b(x)^{-\alpha q/N} t^{q/q^*} (\log(e+t))^{-\alpha q\tau/N}\right\}.$$

Hence

$$\Psi_1(x, t\Phi(x, t)^{-\alpha/N}) \leq C\left[\left\{t^{p/p^*}\right\}^{p^*} + \left\{b(x)^{q/q^*} t^{q/q^*} (\log(e+t))^{-\alpha q\tau/N}\right\}^{q^*}\right]$$

$$\begin{aligned} & \times \left(\log \left(e + t^{q/q^*} (\log(e+t))^{-\alpha q \tau / N} \right) \right)^\tau \Big\}^{q^*} \\ & \leq C [t^p + \{b(x)t(\log(e+t))^\tau\}^q] = C\Phi(x, t), \end{aligned}$$

as required. \square

Consequently we apply [23, Theorem 4.9] to obtain the following result.

THEOREM 2. *Suppose $1 < p \leq q < N/\alpha$, $\tau \geq 0$ and $1/p - 1/q \leq \theta/N$. Then there is a constant $C > 0$ such that*

$$\int_{\mathbf{R}^N} \Psi(x, |I_\alpha f(x)|) dx < C$$

for all $f \in L^\Phi(\mathbf{R}^N)$ with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq 1$.

REMARK 2. If $1/p - 1/q \leq \theta/N$, then take θ_1 such that

$$1/p - 1/q = \theta_1/N.$$

Then b is θ_1 -Hölder continuous when it is θ -Hölder continuous and bounded, and thus we may assume from the beginning that $1/p - 1/q = \theta/N$.

5. Continuity

For $0 < \sigma < N$ and a function $f \in L^1_{\text{loc}}(\mathbf{R}^N)$ we define the fractional maximal function by

$$M_\sigma f(x) = \sup_{r>0} \frac{r^\sigma}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy.$$

THEOREM 3. *Suppose $1 < p \leq q < \infty$, $\tau \geq 0$, $\sigma - N/p \leq 0$, $0 \leq \sigma + \theta - N/p < 1/p'$ and $1/p - 1/q = \theta/N \geq 0$. Then there is a constant $C > 0$ such that*

$$|b(x)M_\sigma f(x) - b(z)M_\sigma f(z)| \leq C|x-z|^{\sigma+\theta-N/p} (\log(e+|x-z|^{-1}))^{-\tau} \quad (3)$$

for all $x, z \in \mathbf{R}^N$ with $0 < |x-z| < 1/2$ and $f \in L^\Phi(\mathbf{R}^N)$ with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq 1$.

Proof. Let f be a measurable function on \mathbf{R}^N with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq 1$. For $x \in \mathbf{R}^N$ and $r > 0$ set

$$I(x, r) = b(x) \frac{r^\sigma}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy.$$

First we consider the case $0 < r < 2|x-z| < 1$. Then we have

$$I(x, r) = \frac{r^\sigma}{|B(x, r)|} \int_{B(x, r)} (b(x) - b(y)) |f(y)| dy + \frac{r^\sigma}{|B(x, r)|} \int_{B(x, r)} b(y) |f(y)| dy$$

$$\begin{aligned} &\leq Cr^{\sigma+\theta}(\log(e+r^{-1}))^{-\tau} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy + \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} b(y)|f(y)| dy \\ &= I_1 + I_2. \end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned} I_1 &\leq Cr^{\sigma+\theta}(\log(e+r^{-1}))^{-\tau} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p} \\ &\leq Cr^{\sigma+\theta-N/p}(\log(e+r^{-1}))^{-\tau}. \end{aligned}$$

By Corollary 1 with $E = B(x, r)$, we have

$$I_2 \leq Cr^{\sigma-N/q}(\log(e+r^{-1}))^{-\tau}.$$

Therefore

$$\begin{aligned} I(x, r) &\leq C \left\{ r^{\sigma+\theta-N/p}(\log(e+r^{-1}))^{-\tau} + r^{\sigma-N/q}(\log(e+r^{-1}))^{-\tau} \right\} \\ &\leq Cr^{\sigma+\theta-N/p}(\log(e+r^{-1}))^{-\tau} \leq C|x-z|^{\sigma+\theta-N/p}(\log(e+|x-z|^{-1}))^{-\tau}, \end{aligned}$$

since $\sigma + \theta - N/p \geq 0$.

Next we consider the case $0 < 2|x-z| < r < 1$. We have

$$\begin{aligned} I(x, r) - I(z, r) &= b(x) \frac{r^\sigma}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy - b(z) \frac{r^\sigma}{|B(z, r)|} \int_{B(z, r)} |f(y)| dy \\ &\leq |b(x) - b(z)| \frac{r^\sigma}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy \\ &\quad + b(z) \frac{r^\sigma}{|B(z, r)|} \int_{B(x, r) \Delta B(z, r)} |f(y)| dy \\ &\leq C|x-z|^\theta (\log(e+|x-z|^{-1}))^{-\tau} \frac{r^\sigma}{|B(z, r)|} \int_{B(z, r)} |f(y)| dy \\ &\quad + \frac{r^\sigma}{|B(x, r)|} \int_{B(x, r) \Delta B(z, r)} |b(z) - b(y)| |f(y)| dy \\ &\quad + \frac{r^\sigma}{|B(z, r)|} \int_{B(x, r) \Delta B(z, r)} b(y) |f(y)| dy \\ &\leq C \left\{ |x-z|^\theta (\log(e+|x-z|^{-1}))^{-\tau} \frac{r^\sigma}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy \right. \\ &\quad + r^\theta (\log(e+|x-z|^{-1}))^{-\tau} \frac{r^\sigma}{|B(z, r)|} \int_{B(x, r) \Delta B(z, r)} |f(y)| dy \\ &\quad \left. + \frac{r^\sigma}{|B(z, r)|} \int_{B(x, r) \Delta B(z, r)} b(y) |f(y)| dy \right\}. \end{aligned}$$

By Hölder's inequality and Corollary 1 with $E = B(x, r) \Delta B(z, r)$, we establish

$$I(x, r) - I(z, r) \leq C \left\{ |x-z|^\theta (\log(e+|x-z|^{-1}))^{-\tau} r^{\sigma-N/p} \right.$$

$$\begin{aligned}
& + |B(x, r)\Delta B(z, r)|^{1/p'} r^{\sigma+\theta-N} (\log(e+r^{-1}))^{-\tau} \\
& + |B(x, r)\Delta B(z, r)| r^{\sigma-N-N/q} (\log(e+r^{-1}))^{-\tau} \\
& + |B(x, r)\Delta B(z, r)|^{1/q'} r^{\sigma-N} (\log(e+r^{-1}))^{-\tau} \} \\
& \leq C \left\{ |x-z|^\theta (\log(e+|x-z|^{-1}))^{-\tau} r^{\sigma-N/p} \right. \\
& \quad + (r^{N-1}|x-z|)^{1/p'} r^{\sigma+\theta-N} (\log(e+r^{-1}))^{-\tau} \\
& \quad + r^{N-1}|x-z| r^{\sigma-N-N/q} (\log(e+r^{-1}))^{-\tau} \\
& \quad \left. + (r^{N-1}|x-z|)^{1/q'} r^{\sigma-N} (\log(e+r^{-1}))^{-\tau} \right\} \\
& \leq C \left\{ |x-z|^\theta (\log(e+|x-z|^{-1}))^{-\tau} r^{\sigma-N/p} \right. \\
& \quad + |x-z|^{1/p'} r^{\sigma+\theta-N/p-1/p'} (\log(e+r^{-1}))^{-\tau} \\
& \quad + |x-z| r^{\sigma-1-N/q} (\log(e+r^{-1}))^{-\tau} \\
& \quad \left. + |x-z|^{1/q'} r^{\sigma-N/q-1/q'} (\log(e+r^{-1}))^{-\tau} \right\} \\
& \leq C |x-z|^{\sigma+\theta-N/p} (\log(e+|x-z|^{-1}))^{-\tau},
\end{aligned}$$

since $\sigma - N/p \leq 0$ and $\sigma - N/q = \sigma + \theta - N/p < 1/p' \leq 1/q' < 1$.

Therefore

$$I(x, r) \leq I(z, r) + C|x-z|^{\sigma+\theta-N/p} (\log(e+|x-z|^{-1}))^{-\tau},$$

which gives the theorem. \square

THEOREM 4. *Suppose $1 < p < q < \infty$, $\tau > 1/q'$, $\alpha + \theta = N/p$ and $1/p - 1/q = \theta/N > 0$. Then there is a constant $C > 0$ such that*

$$|b(x)I_\alpha f(x) - b(z)I_\alpha f(z)| \leq C(\log(e+|x-z|^{-1}))^{-\tau+1/q'}$$

for all $x, z \in \mathbf{R}^N$ with $0 < |x-z| < 1/2$ and $f \in L^\Phi(\mathbf{R}^N)$ with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq 1$.

Proof. Let f be a measurable function on \mathbf{R}^N with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq 1$. For $x, z \in \mathbf{R}^N$, write

$$\begin{aligned}
& b(x)I_\alpha f(x) - b(z)I_\alpha f(z) \\
& = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} (b(x) - b(y))f(y) dy + \int_{\mathbf{R}^N} |x-y|^{\alpha-N} b(y)f(y) dy \\
& \quad - \int_{\mathbf{R}^N} |z-y|^{\alpha-N} (b(z) - b(y))f(y) dy - \int_{\mathbf{R}^N} |z-y|^{\alpha-N} b(y)f(y) dy \\
& = J_1(x) + J_2(x) - J_1(z) - J_2(z).
\end{aligned}$$

For $\delta = 2|x-z| < 1$, we have by Hölder's inequality

$$J_{11}(x) = \int_{B(x, \delta)} |x-y|^{\alpha-N} (b(x) - b(y))f(y) dy$$

$$\begin{aligned}
&\leq C \int_{B(x,\delta)} |x-y|^{\alpha-N+\theta} (\log(e+|x-y|^{-1}))^{-\tau} |f(y)| dy \\
&\leq C \left(\int_{B(x,\delta)} (|x-y|^{\alpha-N+\theta} (\log(e+|x-y|^{-1}))^{-\tau})^{p'} dy \right)^{1/p'} \left(\int_{B(x,\delta)} |f(y)|^p dy \right)^{1/p} \\
&\leq C \left(\int_0^\delta t^N (t^{\alpha-N+\theta} (\log(e+t^{-1}))^{-\tau})^{p'} \frac{dt}{t} \right)^{1/p'} \leq C (\log(e+\delta^{-1}))^{-\tau+1/p'},
\end{aligned}$$

since $\alpha + \theta - N/p = 0$ and $\tau > 1/p'$. Further, we obtain by Hölder's inequality for $0 < \beta < \alpha$

$$\begin{aligned}
J_{21}(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-N} b(y) f(y) dy \\
&\leq C \int_{B(x,\delta)} |x-y|^{\alpha-N} \left(\frac{\log(e+|f(y)|)}{\log(e+|x-y|^{-\beta})} \right)^\tau b(y) |f(y)| dy \\
&\quad + \int_{B(x,\delta)} |x-y|^{\alpha-N} b(y) |x-y|^{-\beta} dy \\
&\leq C \left\{ \left(\int_{B(x,\delta)} (|x-y|^{\alpha-N} (\log(e+|x-y|^{-1}))^{-\tau})^{q'} dy \right)^{1/q'} \right. \\
&\quad \left. \times \left(\int_{B(x,\delta)} (b(y)|f(y)| (\log(e+|f(y)|))^\tau)^q dy \right)^{1/q} + \delta^{\alpha-\beta} \right\} \\
&\leq C \left\{ \left(\int_0^\delta t^N (t^{\alpha-N} (\log(e+t^{-1}))^{-\tau})^{q'} \frac{dt}{t} \right)^{1/q'} + \delta^{\alpha-\beta} \right\} \\
&\leq C (\log(e+\delta^{-1}))^{-\tau+1/q'},
\end{aligned}$$

since $\alpha - N/q = 0$ and $\tau > 1/q'$. Similarly,

$$J_{11}(z) = \int_{B(x,\delta)} |z-y|^{\alpha-N} (b(z) - b(y)) f(y) dy \leq C (\log(e+\delta^{-1}))^{-\tau+1/p'}$$

and

$$J_{21}(z) = \int_{B(x,\delta)} |z-y|^{\alpha-N} b(y) f(y) dy \leq C (\log(e+\delta^{-1}))^{-\tau+1/q'}.$$

Noting that

$$||x-y|^{\alpha-N} - |z-y|^{\alpha-N}| \leq C|x-z||x-y|^{\alpha-N-1}$$

when $|x-y| > 2|x-z|$, we have by Hölder's inequality

$$\begin{aligned}
J_{31} &= \int_{\mathbf{R}^N \setminus B(x,\delta)} (|x-y|^{\alpha-N} - |z-y|^{\alpha-N}) (b(x) - b(y)) f(y) dy \\
&\leq C|x-z| \int_{\mathbf{R}^N \setminus B(x,\delta)} |x-y|^{\alpha-N-1+\theta} (\log(e+|x-y|^{-1}))^{-\tau} |f(y)| dy
\end{aligned}$$

$$\begin{aligned}
&\leq C|x-z| \left(\int_{\mathbf{R}^N \setminus B(x,\delta)} (|x-y|^{\alpha-N-1+\theta} (\log(e+|x-y|^{-1}))^{-\tau})^{p'} dy \right)^{1/p'} \\
&\quad \times \left(\int_{\mathbf{R}^N \setminus B(x,\delta)} |f(y)|^p dy \right)^{1/p} \\
&\leq C|x-z| \left(\int_{\delta}^{\infty} t^N (t^{\alpha-N-1+\theta} (\log(e+t^{-1}))^{-\tau})^{p'} \frac{dt}{t} \right)^{1/p'} \leq C(\log(e+\delta^{-1}))^{-\tau}
\end{aligned}$$

and

$$\begin{aligned}
J_{32} &= \int_{\mathbf{R}^N \setminus B(x,\delta)} |z-y|^{\alpha-N} (b(x) - b(z)) f(y) dy \\
&\leq C|x-z|^\theta (\log(e+|x-z|^{-1}))^{-\tau} \int_{\mathbf{R}^N \setminus B(x,\delta)} |z-y|^{\alpha-N} f(y) dy \\
&\leq C|x-z|^\theta (\log(e+|x-z|^{-1}))^{-\tau} \left(\int_{\mathbf{R}^N \setminus B(x,\delta)} |z-y|^{(\alpha-N)p'} dy \right)^{1/p'} \\
&\quad \times \left(\int_{\mathbf{R}^N \setminus B(x,\delta)} |f(y)|^p dy \right)^{1/p} \\
&\leq C|x-z|^\theta (\log(e+|x-z|^{-1}))^{-\tau} \delta^{\alpha-N/p} \leq C(\log(e+\delta^{-1}))^{-\tau}.
\end{aligned}$$

Therefore

$$J_{31} + J_{32} \leq C(\log(e+\delta^{-1}))^{-\tau}.$$

Similarly, we have for $\max\{0, \alpha-1\} < \beta < \alpha$

$$\begin{aligned}
J_{33} &= \int_{\mathbf{R}^N \setminus B(x,\delta)} (|x-y|^{\alpha-N} - |z-y|^{\alpha-N}) b(y) f(y) dy \\
&\leq C|x-z| \int_{\mathbf{R}^N \setminus B(x,\delta)} |x-y|^{\alpha-N-1} b(y) |f(y)| dy \\
&\leq C|x-z| \left\{ \int_{\mathbf{R}^N \setminus B(x,\delta)} |x-y|^{\alpha-N-1} \left(\frac{\log(e+|f(y)|)}{\log(e+|x-y|^{-\beta})} \right)^\tau b(y) |f(y)| dy \right. \\
&\quad \left. + \int_{\mathbf{R}^N \setminus B(x,\delta)} |x-y|^{\alpha-N-1} b(y) |x-y|^{-\beta} dy \right\} \\
&\leq C|x-z| \left\{ \left(\int_{\mathbf{R}^N \setminus B(x,\delta)} (|x-y|^{\alpha-N-1} (\log(e+|x-y|^{-1}))^{-\tau})^{q'} dy \right)^{1/q'} \right. \\
&\quad \left. \times \left(\int_{\mathbf{R}^N \setminus B(x,\delta)} (b(y) |f(y)| (\log(e+|f(y)|))^\tau)^q dy \right)^{1/q} + \delta^{\alpha-\beta-1} \right\} \\
&\leq C \left\{ |x-z| \left(\int_{\delta}^{\infty} t^N (t^{\alpha-N-1} (\log(e+t^{-1}))^{-\tau})^{q'} \frac{dt}{t} \right)^{1/q'} + \delta^{\alpha-\beta} \right\} \\
&\leq C(\log(e+\delta^{-1}))^{-\tau}.
\end{aligned}$$

Now we establish

$$\begin{aligned} J(x) - J(z) &= J_1(x) + J_2(x) - J_1(z) - J_2(z) \\ &= J_{11}(x) + J_{11}(z) + J_{21}(x) + J_{21}(z) + J_{31} + J_{32} + J_{32} \\ &\leq C(\log(e + \delta^{-1}))^{-\tau+1/q'}, \end{aligned}$$

which gives the theorem. \square

In the same way as above, we obtain the following result.

THEOREM 5. *Suppose $1 < p < q < \infty$, $\tau \geq 0$, $0 < \alpha + \theta - N/p < \theta$ and $1/p - 1/q = \theta/N > 0$. Then there is a constant $C > 0$ such that*

$$|b(x)I_\alpha f(x) - b(z)I_\alpha f(z)| \leq C|x-z|^{\alpha+\theta-N/p}(\log(e+|x-z|^{-1}))^{-\tau}$$

for all $x, z \in \mathbf{R}^N$ with $0 < |x-z| < 1/2$ and $f \in L^\Phi(\mathbf{R}^N)$ with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq 1$.

6. Appendix

For reader's convenience, we shall give direct proofs of Theorems 1 and 2 by the boundedness of the maximal operator on $L^p(\mathbf{R}^N)$ and $L^q(\mathbf{R}^N)$.

THEOREM 6. *Suppose $1 < p \leq q < \infty$, $\tau \geq 0$ and $1/p - 1/q = \theta/N \geq 0$. Then there is a constant $C > 0$ such that*

$$\int_{\mathbf{R}^N} \Phi(x, Mf(x)) dx < C$$

for all $f \in L^\Phi(\mathbf{R}^N)$ with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq 1$.

Proof. Let f be a measurable function on \mathbf{R}^N with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq 1$. For $x \in \mathbf{R}^N$ and $r > 0$, we have

$$\begin{aligned} I &= b(x) \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy \\ &= \frac{1}{|B(x,r)|} \int_{B(x,r)} (b(x) - b(y)) |f(y)| dy + \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y) |f(y)| dy \\ &\leq Cr^\theta (\log(e + r^{-1}))^{-\tau} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy + \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y) |f(y)| dy \\ &= I_1 + I_2. \end{aligned}$$

For $0 < r < \delta$

$$I_1 \leq Cr^\theta (\log(e + r^{-1}))^{-\tau} Mf(x) \leq C\delta^\theta (\log(e + \delta^{-1}))^{-\tau} Mf(x)$$

and for $0 < \delta \leq r$ by Hölder's inequality

$$\begin{aligned} I_1 &\leq Cr^\theta (\log(e+r^{-1}))^{-\tau} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p} \leq Cr^{\theta-N/p} (\log(e+r^{-1}))^{-\tau} \\ &\leq C\delta^{\theta-N/p} (\log(e+\delta^{-1}))^{-\tau}, \end{aligned}$$

since $\theta - N/p = -N/q < 0$. Thus

$$I_1 \leq C \left\{ \delta^\theta (\log(e+\delta^{-1}))^{-\tau} Mf(x) + \delta^{\theta-N/p} (\log(e+\delta^{-1}))^{-\tau} \right\}.$$

Now, letting $\delta^{-N/p} = Mf(x)$, we obtain

$$I_1 \leq CMf(x)^{1-\theta p/N} (\log(e+Mf(x)))^{-\tau} = CMf(x)^{p/q} (\log(e+Mf(x)))^{-\tau}.$$

Moreover, for $\delta > 0$ we find from Lemma 1 with $E = B(x,r)$ and $r = \delta$ and the boundedness of b

$$I_2 \leq C \left\{ (\log(e+\delta^{-1}))^{-\tau} Mh(x) + \delta^{-N/q} (\log(e+\delta^{-1}))^{-\tau} \right\},$$

where $h(y) = b(y)|f(y)|(\log(e+|f(y)|))^\tau$. Now, letting $\delta^{-N/q} = Mh(x)$, we obtain

$$I_2 \leq CMh(x) (\log(e+Mh(x)))^{-\tau}.$$

Now we establish

$$b(x)Mf(x) \leq C \left\{ Mf(x)^{p/q} (\log(e+Mf(x)))^{-\tau} + Mh(x) (\log(e+Mh(x)))^{-\tau} \right\}.$$

When $Mf(x)^{p/q} \geq Mh(x)$, we have

$$\begin{aligned} &\left\{ b(x)Mf(x) (\log(e+Mf(x)))^\tau \right\}^q \\ &\leq C(Mf(x))^p (\log(e+Mf(x)))^{-\tau q} (\log(e+Mf(x)))^{\tau q} \leq CMf(x)^p \end{aligned}$$

and when $Mf(x)^{p/q} \leq Mh(x)$, we have

$$\begin{aligned} &\left\{ b(x)Mf(x) (\log(e+Mf(x)))^\tau \right\}^q \\ &\leq C(Mh(x))^q (\log(e+Mh(x)))^{-\tau q} (\log(e+Mf(x)))^{\tau q} \leq CMh(x)^q. \end{aligned}$$

Hence we obtain

$$\left\{ b(x)Mf(x) (\log(e+Mf(x)))^\tau \right\}^q \leq C \{ Mf(x)^p + Mh(x)^q \}.$$

Therefore, the boundedness of the maximal operator on $L^p(\mathbf{R}^N)$ and $L^q(\mathbf{R}^N)$ gives the theorem. \square

Recall that

$$\Psi(x,t) = t^{p^*} + \left\{ b(x)t (\log(e+t))^\tau \right\}^{q^*}.$$

THEOREM 7. Suppose $1 < p \leq q < \infty$, $\tau \geq 0$, $\alpha + \theta < N/p$ and $1/p - 1/q = \theta/N \geq 0$. Then there is a constant $C > 0$ such that

$$\int_{\mathbf{R}^N} \Psi(x, |I_\alpha f(x)|) dx < C$$

for all $f \in L^\Phi(\mathbf{R}^N)$ with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq 1$.

Proof. Let f be a measurable function on \mathbf{R}^N with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq 1$. For $x \in \mathbf{R}^N$ and $r > 0$, we have

$$\begin{aligned} & b(x) \int_{\mathbf{R}^N} |x-y|^{\alpha-N} |f(y)| dy \\ &= \int_{\mathbf{R}^N} |x-y|^{\alpha-N} (b(x) - b(y)) |f(y)| dy + \int_{\mathbf{R}^N} |x-y|^{\alpha-N} b(y) |f(y)| dy \\ &\leq C \int_{\mathbf{R}^N} |x-y|^{\alpha-N+\theta} (\log(e + |x-y|^{-1}))^{-\tau} |f(y)| dy + \int_{\mathbf{R}^N} |x-y|^{\alpha-N} b(y) |f(y)| dy \\ &= J_1 + J_2. \end{aligned}$$

For $\delta > 0$, we have

$$\int_{B(x,\delta)} |x-y|^{\alpha-N+\theta} (\log(e + |x-y|^{-1}))^{-\tau} |f(y)| dy \leq C \delta^{\alpha+\theta} (\log(e + \delta^{-1}))^{-\tau} Mf(x)$$

and by Hölder's inequality

$$\begin{aligned} & \int_{\mathbf{R}^N \setminus B(x,\delta)} |x-y|^{\alpha-N+\theta} (\log(e + |x-y|^{-1}))^{-\tau} |f(y)| dy \\ &\leq C \int_\delta^\infty r^{\alpha+\theta} (\log(e + r^{-1}))^{-\tau} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy \right) \frac{dr}{r} \\ &\leq C \int_\delta^\infty r^{\alpha+\theta} (\log(e + r^{-1}))^{-\tau} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p} \frac{dr}{r} \\ &\leq C \int_\delta^\infty r^{\alpha+\theta-N/p} (\log(e + r^{-1}))^{-\tau} \frac{dr}{r} \leq C \delta^{\alpha+\theta-N/p} (\log(e + \delta^{-1}))^{-\tau}, \end{aligned}$$

since $\alpha + \theta - N/p < 0$. Hence

$$J_1 \leq C \left\{ \delta^{\alpha+\theta} (\log(e + \delta^{-1}))^{-\tau} Mf(x) + \delta^{\alpha+\theta-N/p} (\log(e + \delta^{-1}))^{-\tau} \right\}.$$

Now, letting $\delta^{-N/p} = Mf(x)$, we obtain

$$J_1 \leq CMf(x)^{1-(\alpha+\theta)p/N} (\log(e + Mf(x)))^{-\tau} = CMf(x)^{p/q^*} (\log(e + Mf(x)))^{-\tau}.$$

Moreover, for $\delta > 0$,

$$\int_{B(x,\delta)} |x-y|^{\alpha-N} b(y) |f(y)| dy$$

$$\begin{aligned}
&\leq \int_{B(x,\delta)} |x-y|^{\alpha-N} b(y) |f(y)| \left(\frac{\log(e+f(y))}{\log(e+\delta^{-N/q}(\log(e+\delta^{-1}))^{-\tau})} \right)^{\tau} dy \\
&\quad + C\delta^{-N/q}(\log(e+\delta^{-1}))^{-\tau} \int_{B(x,\delta)} |x-y|^{\alpha-N} dy \\
&\leq C \left\{ \delta^{\alpha} (\log(e+\delta^{-1}))^{-\tau} Mh(x) + \delta^{\alpha-N/q} (\log(e+\delta^{-1}))^{-\tau} \right\},
\end{aligned}$$

where $h(y) = b(y)|f(y)|(\log(e+|f(y)|))^{\tau}$. Similarly, we have by Corollary 1 with $E = B(x, r)$

$$\begin{aligned}
\int_{\mathbf{R}^N \setminus B(x,\delta)} |x-y|^{\alpha-N} [b(y)|f(y)|] dy &\leq C \int_{\delta}^{\infty} r^{\alpha} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} [b(y)|f(y)|] dy \right) \frac{dr}{r} \\
&\leq C \int_{\delta}^{\infty} r^{\alpha-N/q} (\log(e+r^{-1}))^{-\tau} \frac{dr}{r} \\
&\leq C\delta^{\alpha-N/q} (\log(e+\delta^{-1}))^{-\tau}
\end{aligned}$$

since $\alpha - N/q < 0$. Thus

$$J_2 \leq C \left\{ \delta^{\alpha} (\log(e+\delta^{-1}))^{-\tau} Mh(x) + \delta^{\alpha-N/q} (\log(e+\delta^{-1}))^{-\tau} \right\}.$$

Now, letting $\delta^{-N/q} = Mh(x)$, we obtain

$$J_2 \leq CMh(x)^{1-q\alpha/N} (\log(e+Mh(x)))^{-\tau} = CMh(x)^{q/q^*} (\log(e+Mh(x)))^{-\tau}.$$

Now we establish

$$b(x)|I_{\alpha}f(x)| \leq C \left\{ Mf(x)^{p/q^*} (\log(e+Mf(x)))^{-\tau} + Mh(x)^{q/q^*} (\log(e+Mh(x)))^{-\tau} \right\}.$$

As in the final discussions of the previous proof, we have

$$\left\{ b(x)|I_{\alpha}f(x)| (\log(e+|I_{\alpha}f(x)|))^{\tau} \right\}^{q^*} \leq C \{ Mf(x)^p + Mh(x)^q \}.$$

Hence we obtain the required result by the boundedness of the maximal operator on $L^p(\mathbf{R}^N)$ and $L^q(\mathbf{R}^N)$. \square

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