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A WEIGHTED ESTIMATE FOR GENERALIZED HARMONIC EXTENSIONS

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Abstract. We prove some weighted L_p estimates for generalized harmonic extensions in the half-space.

Let $u=u(x)$ be a "good" function in $\mathbb{R}^n$. Denote by $\mathbb{P}u=(\mathbb{P}u)(x,y)$ its harmonic extension to the half-space $\mathbb{R}^{n+1}_+\equiv\mathbb{R}^n\times(0,\infty)$,

$$(\mathbb{P}u)(x,y)=\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\int\limits_{\mathbb{R}^n}u(\xi)\cdot\frac{y\,d\xi}{\left(|x-\xi|^2+y^2\right)^{\frac{n+1}{2}}},\qquad x\in\mathbb{R}^n, y>0.$$

By elementary convolution estimates, the linear mapping $\mathbb{P}:u\mapsto(\mathbb{P}u)(\,\cdot\,,y)$ is non-expanding in $L_p(\mathbb{R}^n)$ for any $p\in[1,\infty]$, that is, $\|(\mathbb{P}u)(\,\cdot\,,y)\|_p\leqslant\|u\|_p$ for any $y>0$.

In the breakthrough paper [1], Caffarelli and Silvestre introduced, for any $s\in(0,1)$, the following generalized s*-harmonic extension* $u\mapsto\mathbb{P}_s u$,

$$(\mathbb{P}_s u)(x,y)=c_{n,s}\int\limits_{\mathbb{R}^n}u(\xi)\cdot\frac{y^{2s}\,d\xi}{\left(|x-\xi|^2+y^2\right)^{\frac{n+2s}{2}}},\quad c_{n,s}=\frac{\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(s)}.$$

One of the main results in [1] states that the L_2-norm of $(-\Delta)^{\frac{s}{2}}u=\mathscr{F}^{-1}\left[|\xi|^s\mathscr{F}[u]\right]$ on $\mathbb{R}^n$ (here $\mathscr{F}$ is the Fourier transorm in $\mathbb{R}^n$) coincides, up to a constant that depends only on s, with some weighted L_2-norm of $|\nabla(\mathbb{P}_s u)|$ on $\mathbb{R}^{n+1}_+$.

Notice that for arbitrary $y>0$, the kernel

$$\mathscr{P}_s(x,y)=\frac{\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(s)}\,\frac{y^{2s}}{\left(|x|^2+y^2\right)^{\frac{n+2s}{2}}}\tag{1}$$

has unitary L_1-norm, thus the linear mapping $u\mapsto(\mathbb{P}_s u)(\cdot,y)$ is non-expanding in $L_p(\mathbb{R}^n)$ as well. In particular, we have

$$\int\limits_{\mathbb{R}^n}|(\mathbb{P}_s u)(\,\cdot\,,y)|^p\,dx\leqslant\int\limits_{\mathbb{R}^n}|u|^p\,dx\quad\text{for any } s\in(0,1)\,,\ \ y>0\,,\ \ p\in[1,\infty).$$

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We are interested in similar results for weighted L_p-norms. More precisely, we deal with inequalities of the form

$$\int\limits_{\mathbb{R}^n} \frac{|(\mathbb{P}_s u)(x,y)|^p}{\big(|x|^2+y^2\big)^\alpha}\,dx \leqslant C_p \int\limits_{\mathbb{R}^n} \frac{|u(x)|^p}{\big(|x|^2+y^2\big)^\alpha}\,dx \tag{2}$$

where $C_p > 0$ does not depend on y, u. These inequalities seems to be new even in the classical case $s = \frac{1}{2}$.

The next statement is crucially used in [2].

THEOREM 1. *Let* $s \in (0,1)$, $\alpha \geqslant 0$.

i) *If* $p = 1$, *The inequality* (2) *holds if and only if* $\alpha \leqslant \frac{n}{2} + s$.

ii) *For arbitrary* $1 < p < \infty$, *the inequality* (2) *holds if and only if* $\alpha < \frac{n}{2} + sp$.

Proof. Take a measurable function u, an arbitrary $y > 0$, and put $u^y(x) = u(yx)$. By dilation, we have $(\mathbb{P}_s u)(x,y) = (\mathbb{P}_s u^y)(\frac{x}{y}, 1)$. Thus it suffices to prove (2) for $y = 1$.

In case $p = 1$, we rewrite the inequality

$$\int\limits_{\mathbb{R}^n} \frac{|(\mathbb{P}_s u)(x,1)|}{\big(|x|^2+1\big)^\alpha}\,dx \leqslant C_1 \int\limits_{\mathbb{R}^n} \frac{|u(x)|}{\big(|x|^2+1\big)^\alpha}\,dx \tag{3}$$

in the form

$$\int\limits_{\mathbb{R}^n} \Big| \int\limits_{\mathbb{R}^n} \frac{u(\xi)}{\big(|\xi|^2+1\big)^\alpha} \, \frac{c_{n,s}}{(|x-\xi|^2+1)^{\frac{n+2s}{2}}} \, \frac{\big(|\xi|^2+1\big)^\alpha}{\big(|x|^2+1\big)^\alpha}\,d\xi \Big| dx \leqslant C_1 \int\limits_{\mathbb{R}^n} \frac{|u(\xi)|}{\big(|\xi|^2+1\big)^\alpha}\,d\xi, \tag{4}$$

to make evident that we are indeed estimating the norm of the transform

$$v \mapsto \mathbb{L}v, \quad (\mathbb{L}v)(x) = \int\limits_{\mathbb{R}^n} v(\xi) \frac{c_{n,s}}{(|x-\xi|^2+1)^{\frac{n+2s}{2}}} \frac{\big(|\xi|^2+1\big)^\alpha}{\big(|x|^2+1\big)^\alpha}\,d\xi$$

as a linear operator $L_1(\mathbb{R}^n) \to L_1(\mathbb{R}^n)$. We use the duality $L_1(\mathbb{R}^n)' = L_\infty(\mathbb{R}^n)$, that gives

$$\|\mathbb{L}\|_{L_1 \to L_1} = \sup_{\xi \in \mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{c_{n,s}}{(|x-\xi|^2+1)^{\frac{n+2s}{2}}} \frac{\big(|\xi|^2+1\big)^\alpha}{\big(|x|^2+1\big)^\alpha}\,dx. \tag{5}$$

If $\alpha > \frac{n}{2} + s$, then the supremum in (5) is evidently infinite. If $\alpha \leqslant \frac{n}{2} + s$ then easily

$$\int\limits_{|x| \geqslant |\xi|/2} \frac{c_{n,s}}{(|x-\xi|^2+1)^{\frac{n+2s}{2}}} \frac{\big(|\xi|^2+1\big)^\alpha}{\big(|x|^2+1\big)^\alpha}\,dx \leqslant \int\limits_{\mathbb{R}^n} 2^{2\alpha} \mathscr{P}_s(x-\xi,1)\,dx = 2^{2\alpha}.$$

Further, $|x|\leqslant|\xi|/2$ implies $|x-\xi|\geqslant|\xi|/2$ and $|x-\xi|\geqslant|x|$. Therefore,

$$\int\limits_{|x|\leqslant|\xi|/2}\frac{c_{n,s}}{(|x-\xi|^2+1)^{\frac{n+2s}{2}}}\frac{\big(|\xi|^2+1\big)^{\alpha}}{\big(|x|^2+1\big)^{\alpha}}\,dx\leqslant\int\limits_{|x|\leqslant|\xi|/2}\frac{2^{2\alpha}c_{n,s}\,dx}{(|x-\xi|^2+1)^{\frac{n+2s}{2}-\alpha}\big(|x|^2+1\big)^{\alpha}}$$

$$\leqslant\int\limits_{\mathbb{R}^n}2^{2\alpha}\mathscr{P}_s(x,1)\,dx=2^{2\alpha}.$$

We can conclude that $C_1=\|\mathbb{L}\|_{L_1\to L_1}<\infty$, and $i)$ is proved.

Next, we take $p>1$. To handle the case $\alpha\geqslant\frac{n}{2}+sp$ we notice that the function

$$\overline{u}(x):=\frac{(|x|^2+1)^{\frac{2\alpha-n}{2p}}}{\log(|x|^2+2)}$$

satisfies

$$\int\limits_{\mathbb{R}^n}\frac{|\overline{u}(x)|^p}{\big(|x|^2+1\big)^{\alpha}}\,dx=\int\limits_{\mathbb{R}^n}\frac{dx}{(|x|^2+1)^{\frac{n}{2}}\log^p(|x|^2+2)}<\infty.$$

On the other hand, for any arbitrary $x\in\mathbb{R}^n$ we have

$$\int\limits_{\mathbb{R}^n}\mathscr{P}_s(x-\xi,1)\overline{u}(\xi)\,d\xi>\int\limits_{\mathbb{R}^n}\frac{C(x)\,d\xi}{(|\xi|^2+1)^{\frac{n}{2}}\log(|\xi|^2+2)},$$

and the last integral diverges. Thus, for $p>1$ and $\alpha\geqslant\frac{n}{2}+sp$ the inequality (2) does not hold with a finite constant C in the right hand side.

If $\alpha<\frac{n}{2}+sp$, we use Hölder's inequality to get

$$|(\mathbb{P}_s u)(x,1)|\leqslant\Big(\int\limits_{\mathbb{R}^n}\mathscr{P}_s(x-\xi,1)\,\frac{|u(\xi)|^p}{\big(|\xi|^2+1\big)^{\beta}}\,d\xi\Big)^{\frac{1}{p}}$$

$$\times\Big(\int\limits_{\mathbb{R}^n}\mathscr{P}_s(x-\xi,1)\big(|\xi|^2+1\big)^{\frac{\beta}{p-1}}\,d\xi\Big)^{\frac{p-1}{p}},\quad(6)$$

where $\beta:=\max\{\alpha-\frac{n}{2}-s,0\}<s(p-1)$.

If $\alpha\leqslant\frac{n}{2}+s$ then $\beta=0$ and the last integral equals 1. In this case we obtain

$$\int\limits_{\mathbb{R}^n}\frac{|(\mathbb{P}_s u)(x,1)|^p}{\big(|x|^2+1\big)^{\alpha}}\,dx\leqslant\int\limits_{\mathbb{R}^n}\mathbb{L}\Big[\frac{|u(\cdot)|^p}{\big(|\cdot|^2+1\big)^{\alpha}}\Big](x)\,dx\leqslant\|\mathbb{L}\|_{L_1\to L_1}\int\limits_{\mathbb{R}^n}\frac{|u(x)|^p}{\big(|x|^2+1\big)^{\alpha}}\,dx,\quad(7)$$

and (2) follows from the first part of the proof.

If $\frac{n}{2}+s<\alpha<\frac{n}{2}+sp$, we estimate

$$\int\limits_{\mathbb{R}^n}\frac{|(\mathbb{P}_s u)(x,1)|^p}{\big(|x|^2+1\big)^{\alpha}}\,dx\leqslant\Big(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\mathscr{P}_s(x-\xi,1)\;\frac{\big(|\xi|^2+1\big)^{\frac{n+2s}{2}}}{\big(|x|^2+1\big)^{\frac{n+2s}{2}}}\,\frac{|u(\xi)|^p}{\big(|\xi|^2+1\big)^{\alpha}}\,d\xi\,dx\Big)$$

$$\times\Big(\sup_{x\in\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\frac{c_{n,s}}{(|x-\xi|^2+1)^{\frac{n+2s}{2}}}\;\frac{\big(|\xi|^2+1\big)^{\frac{\beta}{p-1}}}{\big(|x|^2+1\big)^{\frac{\beta}{p-1}}}\,d\xi\Big)^{p-1}.$$

If we prove that the last supremum is finite then (2) again follows from the first statement of the present theorem. We have

$$\int\limits_{|\xi|\leqslant 2|x|}\frac{c_{n,s}}{(|x-\xi|^2+1)^{\frac{n+2s}{2}}}\,\frac{\big(|\xi|^2+1\big)^{\frac{\beta}{p-1}}}{\big(|x|^2+1\big)^{\frac{\beta}{p-1}}}\,d\xi\leqslant 2^{\frac{2\beta}{p-1}}\int\limits_{\mathbb{R}^n}\mathscr{P}_s(x-\xi,1)\,d\xi=2^{\frac{2\beta}{p-1}}.$$

Further, $|\xi|\geqslant 2|x|$ implies $|x-\xi|\geqslant|\xi|/2$. Therefore, from $\beta<s(p-1)$ we get

$$\int\limits_{|\xi|\geqslant 2|x|}\frac{c_{n,s}}{(|x-\xi|^2+1)^{\frac{n+2s}{2}}}\,\frac{\big(|\xi|^2+1\big)^{\frac{\beta}{p-1}}}{\big(|x|^2+1\big)^{\frac{\beta}{p-1}}}\,d\xi$$
$$\leqslant\int\limits_{|\xi|\geqslant 2|x|}\frac{2^{\frac{2\beta}{p-1}}c_{n,s}\,d\xi}{(|x-\xi|^2+1)^{\frac{n}{2}+s-\frac{\beta}{p-1}}}\leqslant C(n,s,\beta,p),\qquad(8)$$

and the proof of (2) is complete.

The following statement partially solves the problem whether the mapping $u\mapsto(\mathbb{P}_s u)(\cdot,y)$ is non-expanding in weighted L_p spaces.

THEOREM 2. *Let* $s\in(0,1)$.

i) *If* $0\leqslant\alpha\leqslant\frac{n}{2}-s$ *then for arbitrary* $1\leqslant p<\infty$ *the best constant in (2) is* $C_p=1$.

ii) *If* $\alpha>\frac{n}{2}$ *then the best constant* C_p *in (2) is greater than* 1*, at least for* p *close to* 1^+.

REMARK 1. We conjecture that the statement *ii)* holds for all $1\leqslant p<\infty$. The value of C_p for $\frac{n}{2}-s<\alpha\leqslant\frac{n}{2}$ is a completely open problem.

Proof. We again suppose $y=1$.

Firstly, we prove *i)* in case $p=1$. It has been proved in [1] that the function

$$\omega(\xi,y)=\int\limits_{\mathbb{R}^n}\mathscr{P}_s(\xi-x,y)\,\frac{dx}{\big(|x|^2+1\big)^{\alpha}}\qquad(9)$$

solves the following boundary value problem in $\mathbb{R}^{n+1}_+$,

$$-\operatorname{div}(y^{1-2s}\nabla\omega)=0;\qquad\omega(\xi,0)=\big(|\xi|^2+1\big)^{-\alpha}.\qquad(10)$$

Consider the barrier function $\widetilde{\omega}(\xi,y) = \big(|\xi|^2+y^2+1\big)^{-\alpha}$. A direct computation gives

$$-\mathrm{div}(y^{1-2s}\nabla\widetilde{\omega}) = 2\alpha y^{1-2s}\widetilde{\omega}^{1+\frac{2}{\alpha}}\big((n-2s+2)+(n-2s-2\alpha)(|\xi|^2+y^2)\big) \geqslant 0$$

because of the assumption on α. Since $\widetilde{\omega}(\xi,0) = \omega(\xi,0)$, we have that $\omega \leqslant \widetilde{\omega}$ in $\mathbb{R}^{n+1}_+$ by the maximum principle. In particular,

$$(|\xi|^2+1)^{\alpha}\omega(\xi,1) < \Big(\frac{|\xi|^2+1}{|\xi|^2+2}\Big)^{\alpha} < 1.$$

Therefore, the supremum in (5) does not exceed 1, and thus the best constant in (3) is $C_1 = \|\mathbb{L}\|_{L_1\to L_1} \leqslant 1$.

Since $\|\mathbb{L}\|_{L_1\to L_1} \leqslant 1$, the inequalities in (7) readily give $C_p \leqslant 1$, for any $p \geqslant 1$.

Finally, to prove that $C_p = 1$ if $\alpha \leqslant \frac{n}{2} - s$, it suffices to consider the sequence $u(\varepsilon x)$, where $u \in \mathscr{C}_0^\infty(\mathbb{R}^n)$, $u \geqslant 0$, is a fixed nontrivial function, and then to push ε to 0. The proof of $i)$ is complete.

To prove $ii)$ consider the function $v(x) = \big(|x|^2+1\big)^{-\alpha}$. Clearly $v \in L_1(\mathbb{R}^n)$ and

$$\int\limits_{\mathbb{R}^n} (\mathbb{P}v)(x)\,dx = \int\limits_{\mathbb{R}^n} \mathscr{P}_s(\xi - x,1)\,dx \int\limits_{\mathbb{R}^n} v(\xi)\,d\xi = \int\limits_{\mathbb{R}^n} v(\xi)\,d\xi.$$

Since

$$(\mathbb{P}v)(0) = \int\limits_{\mathbb{R}^n} \mathscr{P}_s(\xi,1)v(\xi)\,d\xi < \max v(\xi) = v(0),$$

there exists a point ξ such that $(\mathbb{P}v)(\xi) > v(\xi)$. Therefore, the supremum in (5) is greater then 1, and the best constant in (3) is $C_1 = \|\mathbb{L}\|_{L_1\to L_1} > 1$. By continuity, the best constant in (2) is greater than 1 for p sufficiently close to 1.

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