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A NONEXISTENCE RESULT FOR DISCRETE SYSTEMS RELATED TO THE REVERSED HARDY–LITTLEWOOD–SOBOLEV INEQUALITY

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Abstract. In this paper, we study the Euler-Lagrange system related to the extremal sequences of the discrete reversed Hardy-Littlewood-Sobolev inequality. This system comes into play in estimating bounds of the Coulomb energy and is associated with the study of conformal geometry. By estimating the infinite series, we prove that the nonexistence of super-solutions of the Euler-Lagrange system satisfied by the extremal sequences of this discrete reversed Hardy-Littlewood-Sobolev inequality.

1. Introduction

The Hardy-Littlewood-Sobolev (HLS) inequality is read as (cf. [17])

$$\left|\int_{R^n}\int_{R^n}\frac{f(x)g(y)dxdy}{|x-y|^{\lambda}}\right| \leqslant C\|f\|_{L^r}\|g\|_{L^s}, \quad \forall (f,g)\in L^r(R^n)\times L^s(R^n), \tag{1}$$

where $0<\lambda<n$, $\min\{r,s\}>1$ and $\frac{1}{r}+\frac{1}{s}+\frac{\lambda}{n}=2$. In 1983, Lieb proved the existence of extremal functions of (1), and obtained the Euler-Lagrange system (cf. [15])

$$\begin{cases} u(x)=\displaystyle\int_{R^n}\frac{v^q(y)dy}{|x-y|^{\lambda}}, & u>0 \quad in \quad R^n,\\ v(x)=\displaystyle\int_{R^n}\frac{u^p(y)dy}{|x-y|^{\lambda}}, & v>0 \quad in \quad R^n, \end{cases} \tag{2}$$

where

$$u=c_1f^{r-1}, v=c_2g^{s-1}, p=\frac{1}{r-1}, q=\frac{1}{s-1}, \tag{3}$$

and c_1,c_2 are proper constants. Now, $\frac{1}{r}+\frac{1}{s}+\frac{\lambda}{n}=2$ becomes the critical condition of Sobolev type

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{\lambda}{n}. \tag{4}$$

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In 2015, Dou and Zhu [6] proved a reversed Hardy-Littlewood-Sobolev (RHLS) inequality (see also [16])

$$\left|\int_{R^n}\int_{R^n}\frac{f(x)g(y)dxdy}{|x-y|^\lambda}\right| \geqslant C\|f\|_{L^r}\|g\|_{L^s}, \quad \forall (f,g)\in L^r(R^n)\times L^s(R^n), \tag{5}$$

and obtained the existence of extremal functions, where $n \geqslant 1$, $\lambda < 0$, $\frac{n}{n-\lambda} < r,s < 1$ and $\frac{1}{r}+\frac{1}{s}+\frac{\lambda}{n} = 2$. The Euler-Lagrange system which the extremal functions satisfy is still (2) with (3).

When $f \equiv g$ and $r = s$ in (1), Lieb classified the extremal functions and thus obtained the best constant (cf. [15]). Meanwhile, when $p = q$ and $u \equiv v$, (2) is reduced to

$$u(x) = \int_{R^n}\frac{u^p(y)dy}{|x-y|^\lambda}, \quad u > 0 \quad in \quad R^n. \tag{6}$$

This equation is related to the study of the conformal geometry and the nonlinear elliptic PDEs. Lieb posed a problem in [15] - whether the solutions of the integral equation (6) still has the classification result. When $0 < \lambda < n$ and $p = 2n/\lambda - 1$, Chen, Li, Ou [4] and Li [13] classified the positive solution $u \in L^{2n/\lambda}_{loc}(R^n)$ and thus answered independently the question. Namely, u must be of the form

$$u(x) = a(b^2+|x-x_0|^2)^{-\lambda/2} \tag{7}$$

with $a,b > 0$ and $x_0 \in R^n$. When $0 < \lambda < n$ and $p > 0$, Chen, Li and Ou obtained the results of the existence/nonexistence, the radial symmetry, the integrability of the solutions (cf. [2]), and the symmetry of the components (cf. [1]). When $\lambda < 0$, Li studied (6) with the negative exponent $\frac{2n-\lambda}{\lambda} \leqslant p < 0$ (cf. [13]), and proved that $p = \frac{2n-\lambda}{\lambda}$ and u is classified as the form (7). A problem posed by Li is whether or not does (6) admit any positive (regular) solutions for all $n \geqslant 1$, $\lambda < 0$ and $p < (2n-\lambda)/\lambda$. Xu answered this question and obtained the following results (cf. [20]).

(Ri) Let $\lambda < 0$ and $p < 0$. Eq. (6) has a positive solution if and only if $2n-\lambda = p\lambda$. Now, u is given by (7).

(Rii) If $0 < \lambda < n$ and $p < 0$, then (6) has no positive solution.

Let $(u,v) \in L^{p+1}(R^n)\times L^{q+1}(R^n)$ be a pair of positive solutions of (2), $\lambda \in (0,n)$ and $\min\{p,q\} > 1$ satisfy (4). Chen, Li and Ou proved u and v are radially symmetric and decreasing about some point by the method of moving planes (cf. [3]). Afterwards, the optimal integrability intervals of u and v are obtained (cf. [8]). Based on these results, the decay rates of u and v were estimated in [11] and [18].

When $\lambda < 0$ and $\min\{p,q\} < (n-\lambda)/\lambda$, paper [9] studied (2). The author proved that both u^{-1} and v^{-1} decay to zero with rate λ, and $u^{-1}, v^{-1} \in L^s(R^n)$ for all $s > n/\lambda$. Moreover, [9] pointed out that (4) is a necessary condition for the existence of positive solutions of (2). Dou and Zhu [6] obtained the existence of extremal functions of (5), which implies that the (4) is still the sufficient condition for the existence of positive solutions of (2). In addition, they also obtained the classification result of (Ri) by the method of moving spheres developed by Li and Zhu in [14].

Eq. (4) can also be viewed as the necessary and sufficient condition ensuring the conformal invariant. Namely, the system (2) and the energy functionals $\|u\|_{p+1}$

and $\|v\|_{q+1}$ are invariant under the scaling transformation (see [10] for $\lambda \in (0,n)$ and $\min\{p,q\} > 0$, and [9] for $\lambda < 0$ and $\max\{p,q\} < 0$).

In 2015, Huang, Li and Yin employed (1) to prove the discrete Hardy-Littlewood-Sobolev (DHLS) inequality (cf. [7])

$$\left|\sum_{i,j,i\neq j} \frac{f_i g_j}{|i-j|^\lambda}\right| \leqslant C\|f\|_{l^r}\|g\|_{l^s}, \quad \forall (f,g) \in l^r(Z^n)\times l^s(Z^n), \tag{8}$$

where $f=(f_i)_{i\in Z^n}$, $g=(g_j)_{j\in Z^n}$, $n\geqslant 1$, $0<\lambda\leqslant n$, $\min\{r,s\}>1$ and $\frac{1}{r}+\frac{1}{s}+\frac{\lambda}{n}\geqslant 2$. When $n=\lambda=1$, it is the classical Hardy-Littlewood-Polya inequality. They pointed out in [7] that there exist extremal sequences when $\frac{1}{r}+\frac{1}{s}+\frac{\lambda}{n}>2$, and conjectured that it is also true when $\frac{1}{r}+\frac{1}{s}+\frac{\lambda}{n}=2$. More results on the best constants and the extremal sequences of the inequality with finite elements can be seen in [5], [12] and [19].

The following discrete reversed Hardy-Littlewood-Sobolev inequality can be estiblished as the same argument in [7]

$$\|f\|_{l^r}\|g\|_{l^s} \leqslant C\left(\sum_{i,j\in Z^n}\frac{|f_i||g_j|}{|i-j|^\lambda}+\sum_{j\in Z^n}|f_j||g_j|\right), \quad \forall (f,g)\in l^r(Z^n)\times l^s(Z^n), \tag{9}$$

where $f=(f_i)_{i\in Z^n}$, $g=(g_j)_{j\in Z^n}$,

$$\lambda<0, \quad \frac{n}{n-\lambda}<r,s<1, \quad \frac{1}{r}+\frac{1}{s}+\frac{\lambda}{n}\leqslant 2. \tag{10}$$

The Euler-Lagrange system satisfied by the extremal sequences is

$$\begin{cases} u_j = \sum_{k\in Z^n}\frac{v_k^q}{|k-j|^\lambda}+v_j^q, \\ v_j = \sum_{k\in Z^n}\frac{u_k^p}{|k-j|^\lambda}+u_j^p, \end{cases} \tag{11}$$

where $u_j, v_j>0$ $(j\in Z^n)$, and $\lambda<0$, $\max\{p,q\}<0$.

In this paper, we are concerned with nonexistence of positive solution of the following system

$$\begin{cases} u_j = c_1(j)(\sum_{k\in Z^n,k\neq j}\frac{v_k^q}{|k-j|^\lambda}+v_j^q), \\ v_j = c_2(j)(\sum_{k\in Z^n,k\neq j}\frac{u_k^p}{|k-j|^\lambda}+u_j^p), \end{cases} \tag{12}$$

with double bounded sequences $c_i(j)$ $(i=1,2$ and $j\in Z^n)$. A sequence $c(j)$ is called double bounded, if there exists $C>1$ such that $C^{-1}\leqslant c(j)\leqslant C$ for all $j\in Z^n$. Clearly, the nonexistence of positive solution of (12) implies the nonexistence of positive solution of (11).

The following theorem is the corresponding result of (Rii) to some extent.

THEOREM 1. *When $0<\lambda<n$ and $\min\{p,q\}<0$, then for any double bounded sequences $c_1(j)$ and $c_2(j)$, there does not exist solution of (12) satisfying $u_j\simeq|j|^{\theta_1}$ and $v_j\simeq|j|^{\theta_2}$ for all $\theta_1,\theta_2\in R$.*

Here $u_j \simeq |j|^\theta$ means that there exists $C > 0$ such that

$$\frac{|j|^\theta}{C} \leqslant u_j \leqslant C|j|^\theta$$

when $|j| \to \infty$.

2. Proof of Theorem 1

Proof. Without loss of generality, we assume $q \leqslant p$. Thus, $q < 0$. We prove Theorem 1 by contradiction.

Suppose that there exist $\theta_1, \theta_2 \in R$ such that the solutions u, v of (12) satisfy

$$u(x) \simeq |x|^{\theta_1}, \quad v(x) \simeq |x|^{\theta_2}. \tag{13}$$

Therefore, from (12) it follows

$$u_j \geqslant c \sum_{|k| \geqslant 2|j|} \frac{v_k^q}{|k-j|^\lambda} \geqslant c \sum_{|k| \geqslant 2|j|} |k|^{-\lambda + q\theta_2},$$

which implies

$$n - \lambda + q\theta_2 < 0. \tag{14}$$

Similarly, we can also obtain

$$n - \lambda + p\theta_1 < 0. \tag{15}$$

Noting $\lambda \in (0, n)$ and $q < 0$, we see from (14) and (15) that

$$\theta_2 > 0, \tag{16}$$

and

$$p\theta_1 < 0. \tag{17}$$

Clearly, from (12) there holds

$$v_j := \sum_{m=1}^{5} T_m,$$

where

$$T_1 := \sum_{|k-j| \leqslant |j|/2} \frac{u_k^p}{|k-j|^\lambda}$$

$$T_2 := \sum_{|k| \geqslant 2|j|} \frac{u_k^p}{|k-j|^\lambda}$$

$$T_3 := \sum_{|k-j| \geqslant |j|/2, 2 \leqslant |k| \leqslant 2|j|} \frac{u_k^p}{|k-j|^\lambda}$$

$$T_4 := \sum_{|k| \leqslant 2} \frac{u_k^p}{|k-j|^\lambda}$$

$$T_5 := u_j^p.$$

When $|k-j| \leqslant |j|/2$, there holds $|j|/2 \leqslant |k| \leqslant 3|j|/2$. Thus by (13),

$$T_1 \leqslant C|j|^{p\theta} \sum_{|k-j| \leqslant |j|/2} \frac{1}{|k|^{\lambda}} \leqslant C|j|^{n-\lambda+p\theta_1}.$$

When $|k| \geqslant 2|j|$, there holds $|k-j| \geqslant |k|/2$. Thus by (13) and (15), we have

$$T_2 \leqslant C \sum_{|k| \geqslant 2|j|} \frac{1}{|k|^{\lambda-p\theta_1}} \leqslant C|j|^{n-\lambda+p\theta_1}.$$

When $|k-j| \geqslant |j|/2$ and $2 \leqslant |k| \leqslant 2|j|$, we have

$$T_3 \leqslant \frac{C}{|j|^{\lambda}} \sum_{|k-j| \geqslant |j|/2, 2 \leqslant |k| \leqslant 2|j|} |k|^{p\theta_1}.$$

Therefore,

$$T_3 \leqslant \begin{cases} C|j|^{n-\lambda+p\theta_1}, & \textit{when } n+p\theta_1 > 0; \\ C|j|^{-\lambda} \log|j|, & \textit{when } n+p\theta_1 = 0; \\ C|j|^{-\lambda}, & \textit{when } n+p\theta_1 < 0. \end{cases}$$

When $|k| \leqslant 2$,

$$T_4 \leqslant C|j|^{-\lambda}.$$

At last, by (13),

$$T_5 \leqslant C|j|^{p\theta_1}.$$

Therefore,

$$v_j \leqslant C\left(|j|^{n-\lambda+p\theta_1} + |j|^{-\lambda} \log|j| + |j|^{-\lambda} + |j|^{p\theta_1}\right).$$

Comparing this with (13), we get from (15) and (17) that

$$\theta_2 \leqslant \max\{n-\lambda+p\theta_1, -\lambda+\varepsilon, p\theta_1\} < 0.$$

Here $\varepsilon > 0$ is sufficiently small. This contradicts with (16). Theorem 1 is proved.

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