A NEW IMPROVED FORM OF THE HILBERT INEQUALITY AND ITS APPLICATIONS

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Abstract. In this paper, it shows a new improved form of the Hilbert inequality by introducing a proper weight function $\Omega(\lambda, x)$ with a parameter $\lambda (\lambda > \frac{1}{2})$. As applications, a new refinement of Widder’s inequality and an extension of Hardy-Littlewood’s inequality are given.

1. Introduction

If $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(y)dy < \infty$, then we have the following Hilbert’s integral inequality (see [14]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left( \int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right)^{\frac{1}{2}}, \quad (1.1)$$

where the constant factor $\pi$ is the best possible. In 1925, by introducing one pair of conjugate exponents $(p, q)$, Hardy [3] gave an extension of (1.1) as follows:

For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(y)dy < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \quad (1.2)$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequalities (1.1) and (1.2) are important in analysis and its applications (see [3, 13]).

In 1934, Hardy gave an extension of (1.2) as follows:

If $k_1(x, y)$ is a non-negative homogeneous function of degree $-1$,

$$k_p = \int_0^\infty k_1(u, 1)u^{\frac{1}{p}}du \in R_+ = (0, +\infty),$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(y)dy < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \quad (1.3)$$

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In 1934, Hardy gave an extension of (1.3) as follows:

If $k_1(x, y)$ is a non-negative homogeneous function of degree $-1$,

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$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \quad (1.4)$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequalities (1.1) and (1.4) are important in analysis and its applications (see [3, 13]).
\[
\int_0^\infty \int_0^\infty k_1(x,y)f(x)g(y)\,dx\,dy < k_p \left( \int_0^\infty f^p(x)\,dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)\,dy \right)^{\frac{1}{q}},
\] (1.3)

where the constant factor \( k_p \) is the best possible (see [3, Theorem 319]). Additionally, a Hilbert-type integral inequality with the non-homogeneous kernel is proveded (see [3, Theorem 350]) as follows:

If \( h(u) > 0, \phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in R_+ \), then

\[
\int_0^\infty \int_0^\infty h(x,y)f(x)g(y)\,dx\,dy < \phi \left( \frac{1}{p} \right) \left( \int_0^\infty x^{p-2}f^p(x)\,dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)\,dy \right)^{\frac{1}{q}},
\] (1.4)

where the constant factor \( \phi(\frac{1}{p}) \) is still the best possible.

By introducing an independent parameter \( \lambda \in (0,\infty) \) and the beta function, in 1998, Yang [1] gave an extension of (1.1) as follows:

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,dx\,dy < B \left( \frac{\lambda}{2}, \frac{\lambda}{2} \right) \left( \int_0^\infty x^{1-\lambda}f^2(x)\,dx \right)^{\frac{1}{2}} \left( \int_0^\infty y^{1-\lambda}g^2(y)\,dy \right)^{\frac{1}{2}},
\] (1.5)

where the constant factor \( B(\frac{\lambda}{2}, \frac{\lambda}{2}) \) is the best possible, and

\[
B(u,v) := \int_0^\infty \frac{t^{v-1}}{(1+t)^{u+v}}\,dt, \quad (u,v > 0)
\]

is the Beta function.

In 1999, by introducing matrix method, Gao [10] gave another extension of (1.1) as follows:

If \( 0 < \int_0^\infty f^2(x)\,dx < \infty \) and \( 0 < \int_0^\infty g^2(y)\,dy < \infty \), the inner product \((f,g)\) and the norm \(||f||\) of \( f \) are defined \((f,g) = \int_0^\infty f(t)g(t)\,dt\) and \(||f|| = \left( \int_0^\infty f^2(t)\,dt \right)^{\frac{1}{2}}\), respectively. Then

\[
\left( \int_0^\infty \int_0^\infty \frac{f(s)g(t)}{s+t}\,ds\,dt \right)^2 < \pi^2(1-A)||f||^2||g||^2,
\] (1.6)

where \( A = \frac{1}{\pi} \left( \frac{x}{||g||} - \frac{y}{||f||} \right)^2 \) with \( x = (\frac{2}{\pi})^{\frac{1}{2}}(g,e) \) and \( y = (2\pi)^{\frac{1}{2}}(f,e^{-s}) \), \( e \) is exponential integral with parameter.

In recent years, there has been increasing interest in extending of (1.1) and (1.2); see, for example, [2, 4, 5, 7, 8, 9, 11, 12, 15] and the references cited therein.

In this paper, by using a new method with a parameter \( \lambda (\lambda > \frac{1}{2}) \), applying the weight functions and the technique of real analysis, we establish a improvement of the Hilbert inequality. As application, we give a new refinement of Widder’s inequality and an extension of Hardy-Littlewood’s inequality to illustrate the main results.
2. Some Lemmas

In what follows, we use the following notation for the convenience of the reader.

\[ H(\lambda, x, y) = \frac{f(x)f(y)}{x^\lambda + y^\lambda}, \quad (1) \]

\[ F(x, y) = 1 - c(x) + c(y), \quad (2) \]

and

\[ J_1 = \int_0^\infty \int_0^\infty \frac{f^2(x)}{x^\lambda + y^\lambda} \left( \frac{x}{y} \right)^{\frac{1}{2}} F(x, y) \, dx \, dy, \]
\[ J_2 = \int_0^\infty \int_0^\infty \frac{f^2(y)}{x^\lambda + y^\lambda} \left( \frac{y}{x} \right)^{\frac{1}{2}} F(x, y) \, dx \, dy, \quad (3) \]

where \( \lambda > \frac{1}{2} \), \( f : [0, \infty) \to \mathbb{R}^+ \) is a real function and \( c : [0, +\infty) \to \mathbb{R} \) is a non-negative function.

In order to prove the main results, we need the following four lemmas.

**Lemma 2.1.** Let \( H(\lambda, x, y) \) and \( F(x, y) \) be defined as (1) and (2), respectively. Then

\[ \int_0^\infty \int_0^\infty H(\lambda, x, y) \, dx \, dy = \int_0^\infty \int_0^\infty H(\lambda, x, y) \, F(x, y) \, dx \, dy. \]

**Proof.** It is obvious that

\[ \int_0^\infty \int_0^\infty H(\lambda, x, y) \, dx \, dy = \int_0^\infty \int_0^\infty H(\lambda, x, y) \, F(x, y) \, dx \, dy. \]

Hence, Lemma 2.1 holds. \( \square \)

**Lemma 2.2.** Let \( \lambda > \frac{1}{2} \), then

\[ \int_0^\infty \frac{t^{m-1}}{1 + t^\lambda} \, dt = \frac{\pi}{\lambda \sin \frac{m\pi}{\lambda}}, \quad (2.1) \]

**Proof.** By the integral formula (see, [6, page 591])

\[ \int_0^\infty \frac{t^{m-1}}{\left(1 + bt^a\right)^{m+n}} \, dt = a^{-1}b^{-\frac{n}{a}}B\left(\frac{m}{a}, m+n-m\right), \]

where \( a, b > 0, \ m \) and \( n \) are real numbers, \( B(p, q) \) is Beta function. We have equality (2.1) immediately. \( \square \)

**Lemma 2.3.** Let \( J_1, J_2 \) be defined as (3). Then

\[ J_1J_2 = \left( \frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \right)^2 \left( \int_0^\infty x^{1-\lambda} f^2(x) \, dx \right)^2 - \left( \int_0^\infty \Omega(\lambda, x) f^2(x) \, dx \right)^2, \quad (2.2) \]
where the weight function $\Omega(\lambda, x)$ is defined by

$$\Omega(\lambda, x) = x^{1-\lambda} \left( \frac{\pi c(x)}{\lambda \sin \frac{\pi}{2\lambda}} - \int_0^\infty \frac{c(xu)}{1 + u^\lambda} \left( \frac{1}{u} \right)^{\frac{1}{2}} du \right). \quad (2.3)$$

**Proof.** Let $u = \frac{y}{x}$. Then

$$J_1 = \int_0^\infty \int_0^\infty \frac{f^2(x)}{x^{\lambda} + y^{\lambda}} \left( \frac{x}{y} \right)^{\frac{1}{2}} F(x, y) dxdy$$

$$= \int_0^\infty \left( x^{1-\lambda} \int_0^\infty \frac{1}{1 + u^\lambda} \left( \frac{1}{u} \right)^{\frac{1}{2}} (1 - c(x) + c(xu)) du \right) f^2(x) dx.$$

By Lemma 2.2, we have

$$J_1 = \int_0^\infty \left( x^{1-\lambda} \pi \frac{\sin \frac{\pi}{2\lambda}}{\lambda \sin \frac{\pi}{2\lambda}} - x^{1-\lambda} \left( \frac{c(x)}{\lambda \sin \frac{\pi}{2\lambda}} - \int_0^\infty \frac{c(xu)}{1 + u^\lambda} \left( \frac{1}{u} \right)^{\frac{1}{2}} du \right) \right) f^2(x) dx$$

$$= \int_0^\infty x^{1-\lambda} \frac{f^2(x) dx}{\lambda \sin \frac{\pi}{2\lambda}} - \int_0^\infty x^{1-\lambda} \left( \frac{\pi c(x)}{\lambda \sin \frac{\pi}{2\lambda}} - \int_0^\infty \frac{c(xu)}{1 + u^\lambda} \left( \frac{1}{u} \right)^{\frac{1}{2}} du \right) f^2(x) dx$$

$$= \frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \int_0^\infty x^{1-\lambda} f^2(x) dx - \int_0^\infty \Omega(\lambda, x) f^2(x) dx.$$

where the weight function $\Omega(\lambda, x)$ is defined by (2.3).

Similarly, we have

$$J_2 = \int_0^\infty \int_0^\infty \frac{f^2(y)}{x^{\lambda} + y^{\lambda}} \left( \frac{y}{x} \right)^{\frac{1}{2}} F(x, y) dxdy$$

$$= \frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \int_0^\infty x^{1-\lambda} f^2(x) dx + \int_0^\infty \Omega(\lambda, x) f^2(x) dx \quad (2.5)$$

It follows from (2.4) and (2.5) that (2.2) holds. □

**Remark 2.1.** Let $\lambda = 1$, we get from (2.2)

$$J_1 J_2 = \pi^2 \left( \int_0^\infty f^2(x) dx \right)^2 - \left( \int_0^\infty \tilde{\Omega}(x) f^2(x) dx \right)^2, \quad (2.6)$$

where \[ \tilde{\Omega}(x) = c(x) - \int_0^\infty \frac{c(xu)}{1 + u} \left( \frac{1}{u} \right)^{\frac{1}{2}} du. \]

Let $u = t^2$, we have

$$\tilde{\Omega}(x) = c(x) - 2 \int_0^\infty \frac{c(xu^2)}{1 + t^2} dt. \quad (2.7)$$
**Lemma 2.4.** Let \( c(x) = \frac{1}{1+x} (x \geq 0) \), then (2.7) becomes
\[
\tilde{\Omega}(x) = \frac{\pi}{1 + x} - \frac{\pi}{1 + \sqrt{x}}. \tag{2.8}
\]

**Proof.** By the integral formula (see, [16, page 158, formula 97])
\[
\int_0^\infty \frac{dx}{(a^2 + x^2)(b^2 + x^2)} = \frac{\pi}{2ab(a+b)},
\]
it is ease to deduce that
\[
\int_0^\infty \frac{c(xt^2)}{1+t^2} \, dt = \int_0^\infty \frac{dt}{(1+t^2)(1+xt^2)} = \frac{\pi}{1 + \sqrt{x}}. \tag{2.9}
\]
Substituting (2.9) into (2.7), we obtain (2.8). \( \square \)

3. Main results

We may now state and prove our main results.

**Theorem 3.1.** Let \( H(\lambda,x,y) \) and \( F(x,y) \) be defined as (1) and (2), respectively. Then
\[
\left( \int_0^\infty \int_0^\infty f(x)f(y) \frac{1}{x^\lambda + y^\lambda} \, dx \, dy \right)^2 < \left( \frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \right)^2 \left( \int_0^\infty x^{1-\lambda} f^2(x) \, dx \right)^2 - \left( \int_0^\infty \Omega(\lambda,x) f^2(x) \, dx \right)^2, \tag{3.1}
\]
where the weight function \( \Omega(\lambda,x) \) is defined by (2.3).

**Proof.** By Schwarz’s inequality and Lemma 2.1, we have
\[
\left( \int_0^\infty \int_0^\infty f(x)f(y) \frac{1}{x^\lambda + y^\lambda} \, dx \, dy \right)^2 = \left( \int_0^\infty \int_0^\infty H(\lambda,x,y) \, dx \, dy \right)^2
\]
\[
= \left( \int_0^\infty \int_0^\infty H(\lambda,x,y)F(x,y) \right)^2
\]
\[
= \left\{ \int_0^\infty \int_0^\infty \left( \frac{f(x)}{(x^\lambda + y^\lambda)\frac{1}{2}} \left( \frac{x}{y} \right)^{\frac{\lambda}{2}} \frac{F(x,y)}{2} \right) \left( \frac{f(y)}{(x^\lambda + y^\lambda)\frac{1}{2}} \left( \frac{y}{x} \right)^{\frac{\lambda}{2}} \frac{F(x,y)}{2} \right) \, dx \, dy \right\}^2
\]
\[
\leq \int_0^\infty \int_0^\infty \frac{f^2(x)}{x^\lambda + y^\lambda} \left( \frac{x}{y} \right)^{\frac{\lambda}{2}} F(x,y) \, dx \, dy \int_0^\infty \int_0^\infty \frac{f^2(y)}{x^\lambda + y^\lambda} \left( \frac{y}{x} \right)^{\frac{\lambda}{2}} F(x,y) \, dx \, dy. \tag{3.2}
\]
Since \( f(x) \neq 0 \), it is impossible to take equality in (3.2), we obtain
\[
\left( \int_0^\infty \int_0^\infty f(x)f(y) \frac{1}{x^\lambda + y^\lambda} \, dx \, dy \right)^2 < J_1 J_2.
\]
By Lemma 2.3, we get (3.1). This completes the proof. \( \square \)
THEOREM 3.2. Let \( F(x,y) \) be defined as (2) and let \( f, g : [0, +\infty) \to \mathbb{R}^+ \) be real function. Then

\[
\left( \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} \, dx \, dy \right)^4 < \left\{ \left( \frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \right)^2 \left( \int_0^\infty x^{1-\lambda} f(x) \, dx \right)^2 \right. \\
- \left( \int_0^\infty \Omega(\lambda, x) f^2(x) \, dx \right)^2 \right\} \times \\
\left. \times \left\{ \left( \frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \right)^2 \left( \int_0^\infty x^{1-\lambda} g^2(x) \, dx \right)^2 - \left( \int_0^\infty \Omega(\lambda, x) g^2(x) \, dx \right)^2 \right\}, \quad (3.3)\]

where the weight function \( \Omega(\lambda, x) \) is defined by (2.3).

Proof. By Schwarz’s inequality, we have

\[
\left( \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} \, dx \, dy \right)^4 = \left\{ \int_0^1 \left( \int_0^\infty t^{\lambda - \frac{1}{2}} f(x) \, dx \int_0^\infty t^{\lambda - \frac{1}{2}} g(y) \, dy \right) \, dt \right\}^2 \\
\leq \left\{ \int_0^1 \left( \int_0^\infty f(x) \, dx \right)^2 \, dt \right\} \left\{ \int_0^1 \left( \int_0^\infty g(y) \, dy \right)^2 \, dt \right\} \\
= \left( \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} \, dx \, dy \right)^2 \left( \int_0^\infty \frac{g(x)g(y)}{x^\lambda + y^\lambda} \, dx \, dy \right)^2. \quad (3.4)\]

Similarly to the proof of Theorem 3.1, we obtain

\[
\left( \int_0^\infty \int_0^\infty \frac{g(x)g(y)}{x^\lambda + y^\lambda} \, dx \, dy \right)^2 \\
\leq \left( \frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \right)^2 \left( \int_0^\infty x^{1-\lambda} g^2(x) \, dx \right)^2 - \left( \int_0^\infty \Omega(\lambda, x) g^2(x) \, dx \right)^2. \quad (3.5)\]

Since \( g(x) \neq 0 \), it is impossible to take equality in (3.5).

Substituting (3.1) and (3.5) into (3.4), we get (3.3). This completes the proof. \( \Box \)

In particular, let \( \lambda = 1 \), we have a new improvement of the Hilbert inequality as following corollary.

COROLLARY 3.1. Let \( F(x,y) \) be defined as (2) and let \( f, g : [0, +\infty) \to \mathbb{R}^+ \) be real function. Then

\[
\left( \int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y} \, dx \, dy \right)^4 < \left\{ \pi^2 \left( \int_0^\infty f^2(x) \, dx \right)^2 - \left( \int_0^\infty \tilde{\Omega}(x) f^2(x) \, dx \right)^2 \right\} \times \\
\times \left\{ \pi^2 \left( \int_0^\infty g^2(x) \, dx \right)^2 - \left( \int_0^\infty \tilde{\Omega}(x) g^2(x) \, dx \right)^2 \right\}, \quad (3.6)\]

where the weight function \( \tilde{\Omega}(x) \) is defined by (2.7).
If we choose \( c(x) = \frac{1}{1+x} (x \geq 0) \), by (2.8), then (3.6) becomes
\[
\left( \int_0^\infty \int_0^\infty f(x)f(y) \frac{dy}{x+y} \right)^4 < \pi^4 \left\{ \left( \int_0^\infty f^2(x)dx \right)^2 - \left( \int_0^\infty \tilde{\Omega}(x)f^2(x)dx \right)^2 \right\} \times \\
\times \left\{ \left( \int_0^\infty g^2(x)dx \right)^2 - \left( \int_0^\infty \tilde{\Omega}(x)g^2(x)dx \right)^2 \right\},
\]
where \( \tilde{\Omega}(x) = \frac{1}{1+x} - \frac{1}{1+\sqrt{x}} \).

**Remark 3.1.** The non-negative function \( c(x) \) is chosen for maximum flexibility, because it only satisfies condition: \( F(x,y) = 1 - c(x) + c(y) \geq 0 \). Therefore, if we choose \( c(x) = \frac{1}{2} \cos \sqrt{x} \), then by the integral formula(see, [16, page 189, formula 534])
\[
\int_0^\infty \frac{\cos ax}{b^2 + x^2} \, dx = \frac{\pi}{2b} e^{-ab}, \ (a \geq 0, \text{Re} \ b > 0).
\]
It is easy to calculate that
\[
2 \int_0^\infty c(\sqrt{x}) \, dx = \int_0^\infty \frac{\cos \sqrt{x}}{1+t^2} \, dt = \frac{\pi}{2} e^{-\sqrt{x}},
\]
hence, we get
\[
\tilde{\Omega}(x) = \frac{1}{2} \left( \cos \sqrt{x} - e^{-\sqrt{x}} \right).
\]

**Corollary 3.2.** With the same assumptions as Corollary 3.1, let \( \lambda = 2 \), then
\[
\left( \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^2 + y^2} \, dxdy \right)^4 < \left\{ \frac{\pi^2}{2} \left( \int_0^\infty \frac{1}{x} f^2(x)dx \right)^2 - \left( \int_0^\infty \tilde{\Omega}(2,x) f^2(x)dx \right)^2 \right\} \times \\
\times \left\{ \frac{\pi^2}{2} \left( \int_0^\infty \frac{1}{x} g^2(x)dx \right)^2 - \left( \int_0^\infty \tilde{\Omega}(2,x) g^2(x)dx \right)^2 \right\},
\]
where the weight function \( \tilde{\Omega}(2,x) \) is defined by
\[
\tilde{\Omega}(2,x) = \frac{1}{x} \left( \frac{\pi c(x)}{\sqrt{2}} - \int_0^\infty \frac{c(xu)}{1+u^2} \left( \frac{1}{u} \right)^{\frac{1}{2}} \, du \right).
\]

4. Applications

In this section, we give a new refinement of Widder’s inequality and an extension of Hardy-Littlewood’s inequality as follows.
4.1. A new refinement of Widder’s inequality

The following inequality is Widder’s inequality (see [15]):

Let \( a_n \geq 0 (n = 0, 1, 2, \ldots) \), \( A(x) = \sum_{n=0}^{\infty} a_n x^n \), \( A^*(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \). If \( A(x) \neq 0 \), then

\[
\int_0^1 A^2(x) \, dx < \pi \int_0^\infty \left( e^{-x} A^*(x) \right)^2 \, dx
\] (4.1)

We give a new refinement of (4.1) as follows:

**Theorem 4.1.** Let \( a_n \geq 0 (n = 0, 1, 2, \ldots) \), \( A(x) = \sum_{n=0}^{\infty} a_n x^n \), \( A(x) \neq 0 \) and \( A^*(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \). Let \( F(x, y) \) be defined as (3). then

\[
\left( \int_0^1 A^2(x) \, dx \right)^2 < \pi^2 \left( \int_0^\infty \left( e^{-x} A^*(x) \right)^2 \, dx \right)^2 - \left( \int_0^\infty \Omega(x) \left( e^{-x} A^*(x) \right)^2 \, dx \right)^2
\] (4.2)

where the weight function \( \tilde{\Omega}(x) \) is defined by (2.7).

**Proof.** First, we have following relation:

\[
\int_0^\infty e^{-t} A^*(tx) \, dt = \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{a_n(x)^n}{n!} \, dt = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \int_0^\infty t^n e^{-t} \, dt = \sum_{n=0}^{\infty} a_n x^n = A(x).
\]

Squaring and integrating both sides of aforementioned equality from 0 to 1, we obtain

\[
\int_0^1 A^2(x) \, dx = \int_0^1 \left( \int_0^\infty e^{-t} A^*(tx) \, dt \right)^2 \, dx.
\]

Let \( tx = s \), from right side of aforementioned equality, we have

\[
\int_0^1 A^2(x) \, dx = \int_0^1 \left( \int_0^\infty e^{-t} A^*(tx) \, dt \right)^2 \, dx = \int_0^1 \left( \int_0^\infty e^{-\frac{s}{x}} A^*(s) \, ds \right)^2 \frac{1}{x^2} \, dx.
\]

Let \( y = \frac{1}{x} \), we get

\[
\int_0^1 A^2(x) \, dx = \int_1^\infty \left( \int_0^\infty e^{-sy} A^*(s) \, ds \right)^2 \, dy.
\]

Let \( u = y - 1 \), then

\[
\int_0^1 A^2(x) \, dx = \int_0^\infty \left( \int_0^\infty e^{-su-s} A^*(s) \, ds \right)^2 \, du
\]
\[
\int_0^\infty \left( \int_0^\infty e^{-su} f(s) ds \right)^2 du = \int_0^\infty \int_0^\infty \frac{f(s)f(t)}{s+t} ds dt, \tag{4.3}
\]

where \( f(x) = e^{-x}A^x(x) \).

Thus, by Theorem (3.1) (choose \( \lambda = 1 \)), we get (4.2) from (4.3). This completes the proof. \( \square \)

4.2. An extension of Hardy-Littlewood’s inequality

Hardy-Littlewood proved the following inequality (see [3]):

Let \( f(x), x \in [0, 1) \), be a non-negative real function, \( a_n = \int_0^1 x^n f(x) dx, n = 0, 1, 2, \ldots \)

Then

\[
\sum_{n=0}^\infty a_n^2 < \pi \int_0^1 f^2(x) dx, \tag{4.4}
\]

where \( \pi \) is the best constant.

In 1997, Gao [9] extended the inequality (4.4) and established the following inequality, named the Hardy-Littlewood integral inequality.

\[
\int_0^\infty f^2(x) dx < \pi \int_0^1 h^2(x) dx, \tag{4.5}
\]

where \( f(x) = \int_0^1 t^x h(x) dx, x \in [0, +\infty) \).

In 1999, Gao [11] refined (4.5) as following inequality:

\[
\int_0^\infty f^2(x) dx < \pi \int_0^1 t h^2(t) dt. \tag{4.6}
\]

In this paper, we will further extend the inequality (4.6). For notational simplicity, define

\[
f(x) = \int_0^1 t^x |h(t)| dt, (x \geq 0, \lambda > \frac{1}{2}) \tag{4.7}
\]

where \( h(t) \neq 0, (t \in [0, 1]) \), is a real function.

We are in position to state and show the following theorem.

**Theorem 4.2.** Suppose that \( f(x) \) be defined as (4.7), and \( F(x,y) \) be defined as (2). Then

\[
\left( \int_0^\infty f^2(x) dx \right)^4 < \left\{ \frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \int_0^\infty x^{1-\lambda} f^2(x) dx \right\}^2 \left( \int_0^\infty \Omega(\lambda, x) f^2(x) dx \right)^2 \times \left( \int_0^1 t h^2(t) dt \right)^2, \tag{4.8}
\]

where the weight function \( \Omega(\lambda, x) \) is defined by (2.3).
Proof. In view of definition of $f(x)$, we rewrite $f^2(x)$ as

$$f^2(x) = \int_0^1 f(x)t^{x^2} |h(t)|dt.$$ 

By Schwarz’s inequality and Theorem 3.1, we have

$$\left( \int_0^\infty f^2(x)dx \right)^2 = \left\{ \int_0^1 \left( \int_0^\infty f(x)t^{x^2 - \frac{1}{2}} dx \right)^2 t^{\frac{1}{2}} |h(t)|dt \right\}^2$$

$$\leq \int_0^1 \left( \int_0^\infty f(x)t^{x^2 - \frac{1}{2}} dx \right)^2 dt \int_0^1 t h^2(t)dt$$

$$= \int_0^1 \left( \int_0^\infty f(x) f(y)t^{x^2 + y^2 - 1} dx dy \right) dt \int_0^1 t h^2(t)dt$$

$$= \left( \int_0^\infty \int_0^\infty f(x) f(y) x^{\lambda} y^{\lambda} dx dy \right) \int_0^1 t h^2(t)dt$$

$$\leq \left\{ \left( \frac{\pi}{\lambda \sin \frac{\pi}{2\lambda}} \int_0^\infty x^{1-\lambda} f^2(x)dx \right)^2 - \left( \int_0^\infty \Omega(\lambda,x)f^2(x)dx \right) \right\}^{\frac{1}{2}} \int_0^1 t h^2(t)dt,$$ 

(4.9)

where the weight function $\Omega(\lambda,x)$ is defined by (2.3).

Since $h(t) \neq 0$, $f(x) \neq 0$, it is impossible to take equality in (4.9). Thus, we get inequality (4.8). This completes the proof. □

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REFERENCES

A NEW IMPROVED FORM OF THE HILBERT INEQUALITY AND ITS APPLICATIONS


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