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A NEW IMPROVED FORM OF THE HILBERT INEQUALITY AND ITS APPLICATIONS

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Abstract. In this paper, it shows a new improved form of the Hilbert inequality by introducing a proper weight function $\Omega(\lambda,x)$ with a parameter $\lambda(\lambda>\frac{1}{2})$. As applications, a new refinement of Widder's inequality and an extension of Hardy-Littlewood's inequality are given.

1. Introduction

If $0<\int_0^\infty f^2(x)\mathrm{d}x<\infty$ and $0<\int_0^\infty g^2(y)\mathrm{d}y<\infty$, then we have the following Hilberts integral inequality(see [14]):

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y<\pi\left(\int_0^\infty f^2(x)\mathrm{d}x\int_0^\infty g^2(y)\mathrm{d}y\right)^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor π is the best possible. In 1925, by introducing one pair of conjugate exponents (p,q), Hardy [3] gave an extension of (1.1) as follows:

For $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f(x),g(y)\geqslant 0$, $0<\int_0^\infty f^p(x)\mathrm{d}x<\infty$ and $0<\int_0^\infty g^q(x)\mathrm{d}x<\infty$, then

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y<\frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)\mathrm{d}y\right)^{\frac{1}{q}}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequalities (1.1) and (1.2) are important in analysis and its applications (see [3, 13]).

In 1934, Hardy gave an extension of (1.2) as follows:

If $k_1(x,y)$ is a non-negative homogeneous function of degree -1,

$$k_p=\int_0^\infty k_1(u,1)u^{\frac{-1}{p}}\mathrm{d}u\in R_+=(0,+\infty),$$

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then

$$\int_0^{\infty}\int_0^{\infty} k_1(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y < k_p\left(\int_0^{\infty} f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_0^{\infty} g^q(y)\mathrm{d}y\right)^{\frac{1}{q}}, \qquad (1.3)$$

where the constant factor k_p is the best possible (see [3, Theorem 319]). Additionally, a Hilbert-type integral inequality with the non-homogeneous kernel is proveded (see [3, Theorem 350]) as follows:

if $h(u) > 0$, $\phi(\sigma) = \int_0^{\infty} h(u)u^{\sigma-1}\mathrm{d}u \in R_+$, then

$$\int_0^{\infty}\int_0^{\infty} h(x,y)f(x)g(y)\mathrm{d}x\mathrm{d}y < \phi\left(\frac{1}{p}\right)\left(\int_0^{\infty} x^{p-2}f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_0^{\infty} g^q(y)\mathrm{d}y\right)^{\frac{1}{q}}, \qquad (1.4)$$

where the constant factor $\phi(\frac{1}{p})$ is still the best possible.

By introducing an independent parameter $\lambda \in (0,\infty)$ and the beta function, in 1998, Yang [1] gave an extension of (1.1) as follows:

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{(x+y)^{\lambda}}\mathrm{d}x\mathrm{d}y < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_0^{\infty} x^{1-\lambda}f^2(x)\mathrm{d}x\int_0^{\infty} y^{1-\lambda}g^2(y)\mathrm{d}y\right)^{\frac{1}{2}}, \qquad (1.5)$$

where the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ is the best possible, and

$$B(u,v) := \int_0^{\infty}\frac{t^{v-1}}{(1+t)^{u+v}}\mathrm{d}t,\ (u,v>0)$$

is the Beta function.

In 1999, by introducing matrix method, Gao [10] gave another extension of (1.1) as follows:

If $0 < \int_0^{\infty} f^2(x)\mathrm{d}x < \infty$ and $0 < \int_0^{\infty} g^2(y)\mathrm{d}y < \infty$, the inner product (f,g) and the norm $||f||$ of f are defined $(f,g) = \int_0^{\infty} f(t)g(t)\mathrm{d}t$ and $||f|| = \left(\int_0^{\infty} f^2(t)\mathrm{d}t\right)^{\frac{1}{2}}$, respectively. Then

$$\left(\int_0^{\infty}\int_0^{\infty}\frac{f(s)g(t)}{s+t}\mathrm{d}s\mathrm{d}t\right)^2 < \pi^2(1-A)||f||^2||g||^2, \qquad (1.6)$$

where $A = \frac{1}{\pi}\left(\frac{x}{||g||} - \frac{y}{||f||}\right)^2$ with $x = (\frac{2}{\pi})^{\frac{1}{2}}(g,e)$ and $y = (2\pi)^{\frac{1}{2}}(f,e^{-s})$, e is exponential integral with parameter.

In recent years, there has been increasing interest in extending of (1.1) and (1.2); see, for example, [2, 4, 5, 7, 8, 9, 11, 12, 15] and the references cited therein.

In this paper, by using a new method with a parameter $\lambda(\lambda > \frac{1}{2})$, applying the weight functions and the technique of real analysis, we establish a improvement of the Hilbert inequality. As application, we give a new refinement of Widder's inequality and an extension of Hardy-Littlewood's inequality to illustrate the main results.

2. Some Lemmas

In what follows, we use the following notation for the convenience of the reader.

$$H(\lambda,x,y)=\frac{f(x)f(y)}{x^{\lambda}+y^{\lambda}}, \tag{1}$$

$$F(x,y)=1-c(x)+c(y), \tag{2}$$

and

$$J_1=\int_0^{\infty}\int_0^{\infty}\frac{f^2(x)}{x^{\lambda}+y^{\lambda}}\left(\frac{x}{y}\right)^{\frac{1}{2}}F(x,y)\mathrm{d}x\mathrm{d}y,\ J_2=\int_0^{\infty}\int_0^{\infty}\frac{f^2(y)}{x^{\lambda}+y^{\lambda}}\left(\frac{y}{x}\right)^{\frac{1}{2}}F(x,y)\mathrm{d}x\mathrm{d}y, \tag{3}$$

where $\lambda>\frac{1}{2}$, $f:[0,\infty)\to R^{+}$ is a real function and $c:[0,+\infty)\to R$ is a non-negative function.

In order to prove the main results, we need the following four lemmas.

LEMMA 2.1. *Let $H(\lambda,x,y)$ and $F(x,y)$ be defined as (1) and (2),respectively. Then*

$$\int_0^{\infty}\int_0^{\infty}H(\lambda,x,y)\mathrm{d}x\mathrm{d}y=\int_0^{\infty}\int_0^{\infty}H(\lambda,x,y)F(x,y)\mathrm{d}x\mathrm{d}y.$$

Proof. It is obvious that

$$\int_0^{\infty}\int_0^{\infty}H(\lambda,x,y)\mathrm{d}x\mathrm{d}y=\int_0^{\infty}\int_0^{\infty}H(\lambda,x,y)F(x,y)\mathrm{d}x\mathrm{d}y.$$

Hence, Lemma 2.1 holds. □

LEMMA 2.2. *Let $\lambda>\frac{1}{2}$, then*

$$\int_0^{\infty}\frac{t^{m-1}}{1+t^{\lambda}}\mathrm{d}t=\frac{\pi}{\lambda\sin\frac{m\pi}{\lambda}}. \tag{2.1}$$

Proof. By the integral formula(see, [6, page 591])

$$\int_0^{\infty}\frac{t^{m-1}}{\left(1+bt^{a}\right)^{m+n}}\mathrm{d}t=a^{-1}b^{-\frac{m}{a}}B\left(\frac{m}{a},m+n-\frac{m}{a}\right),$$

where $a,b>0$, m and n are real numbers, $B(p,q)$ is Beta function. We have equality(2.1) immediately. □

LEMMA 2.3. *Let J_1, J_2 be defined as (3). Then*

$$J_1J_2=\left(\frac{\pi}{\lambda\sin\frac{\pi}{2\lambda}}\right)^2\left(\int_0^{\infty}x^{1-\lambda}f^2(x)\mathrm{d}x\right)^2-\left(\int_0^{\infty}\Omega(\lambda,x)f^2(x)\mathrm{d}x\right)^2, \tag{2.2}$$

where the weight function $\Omega(\lambda,x)$ is defined by

$$\Omega(\lambda,x)=x^{1-\lambda}\left(\frac{\pi c(x)}{\lambda\sin\frac{\pi}{2\lambda}}-\int_0^\infty\frac{c(xu)}{1+u^\lambda}\left(\frac{1}{u}\right)^{\frac{1}{2}}\mathrm{d}u\right). \tag{2.3}$$

Proof. Let $u=\frac{y}{x}$. Then

$$\begin{aligned}J_1&=\int_0^\infty\int_0^\infty\frac{f^2(x)}{x^\lambda+y^\lambda}\left(\frac{x}{y}\right)^{\frac{1}{2}}F(x,y)\mathrm{d}x\mathrm{d}y\\&=\int_0^\infty\left(x^{1-\lambda}\int_0^\infty\frac{1}{1+u^\lambda}\left(\frac{1}{u}\right)^{\frac{1}{2}}(1-c(x)+c(xu))\mathrm{d}u\right)f^2(x)\mathrm{d}x.\end{aligned}$$

By Lemma2.2, we have

$$\begin{aligned}J_1&=\int_0^\infty\left(x^{1-\lambda}(1-c(x))\frac{\pi}{\lambda\sin\frac{\pi}{2\lambda}}+x^{1-\lambda}\int_0^\infty\frac{c(xu)}{1+u^\lambda}\left(\frac{1}{u}\right)^{\frac{1}{2}}\mathrm{d}u\right)f^2(x)\mathrm{d}x\\&=\int_0^\infty\left(\frac{x^{1-\lambda}\pi}{\lambda\sin\frac{\pi}{2\lambda}}-x^{1-\lambda}\left(\frac{c(x)\pi}{\lambda\sin\frac{\pi}{2\lambda}}-\int_0^\infty\frac{c(xu)}{1+u^\lambda}\left(\frac{1}{u}\right)^{\frac{1}{2}}\mathrm{d}u\right)\right)f^2(x)\mathrm{d}x\\&=\int_0^\infty\frac{x^{1-\lambda}\pi}{\lambda\sin\frac{\pi}{2\lambda}}f^2(x)\mathrm{d}x-\int_0^\infty x^{1-\lambda}\left(\frac{\pi c(x)}{\lambda\sin\frac{\pi}{2\lambda}}-\int_0^\infty\frac{c(xu)}{1+u^\lambda}\left(\frac{1}{u}\right)^{\frac{1}{2}}\mathrm{d}u\right)f^2(x)\mathrm{d}x\\&=\frac{\pi}{\lambda\sin\frac{\pi}{2\lambda}}\int_0^\infty x^{1-\lambda}f^2(x)\mathrm{d}x-\int_0^\infty\Omega(\lambda,x)f^2(x)\mathrm{d}x,\end{aligned} \tag{2.4}$$

where the weight function $\Omega(\lambda,x)$ is defined by (2.3).

Similarly, we have

$$\begin{aligned}J_2&=\int_0^\infty\int_0^\infty\frac{f^2(y)}{x^\lambda+y^\lambda}\left(\frac{y}{x}\right)^{\frac{1}{2}}F(x,y)\mathrm{d}x\mathrm{d}y\\&=\frac{\pi}{\lambda\sin\frac{\pi}{2\lambda}}\int_0^\infty x^{1-\lambda}f^2(x)\mathrm{d}x+\int_0^\infty\Omega(\lambda,x)f^2(x)\mathrm{d}x.\end{aligned} \tag{2.5}$$

It follows from (2.4) and (2.5) that (2.2) holds. □

REMARK 2.1. let $\lambda=1$, we get from (2.2)

$$J_1J_2=\pi^2\left(\int_0^\infty f^2(x)\mathrm{d}x\right)^2-\left(\int_0^\infty\widetilde{\Omega}(x)f^2(x)\mathrm{d}x\right)^2, \tag{2.6}$$

where

$$\widetilde{\Omega}(x)=\pi c(x)-\int_0^\infty\frac{c(xu)}{1+u}\left(\frac{1}{u}\right)^{\frac{1}{2}}\mathrm{d}u.$$

Let $u=t^2$, we have

$$\widetilde{\Omega}(x)=\pi c(x)-2\int_0^\infty\frac{c(xt^2)}{1+t^2}\mathrm{d}t. \tag{2.7}$$

LEMMA 2.4. *Let* $c(x) = \frac{1}{1+x}\ (x \geqslant 0)$*, then* (2.7) *becomes*

$$\widetilde{\Omega}(x) = \frac{\pi}{1+x} - \frac{\pi}{1+\sqrt{x}}. \tag{2.8}$$

Proof. By the integral formula(see, [16, page 158, formula 97])

$$\int_0^\infty \frac{\mathrm{d}x}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)},$$

it is ease to deduce that

$$\int_0^\infty \frac{c(xt^2)}{1+t^2}\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}t}{(1+t^2)(1+xt^2)} = \frac{\pi}{1+\sqrt{x}}. \tag{2.9}$$

Substituting (2.9) into (2.7), we obtain (2.8). □

3. Main results

We may now state and prove our main results.

THEOREM 3.1. *Let* $H(\lambda,x,y)$ *and* $F(x,y)$ *be defined as (1) and (2),respectively. Then*

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x^\lambda+y^\lambda}\mathrm{d}x\mathrm{d}y\right)^2 < \left(\frac{\pi}{\lambda\sin\frac{\pi}{2\lambda}}\right)^2\left(\int_0^\infty x^{1-\lambda}f^2(x)\mathrm{d}x\right)^2 - \left(\int_0^\infty \Omega(\lambda,x)f^2(x)\mathrm{d}x\right)^2, \tag{3.1}$$

where the weight function $\Omega(\lambda,x)$ *is defined by* (2.3).

Proof. By Schwarz's inequality and Lemma 2.1, we have

$$\begin{aligned}
&\left(\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x^\lambda+y^\lambda}\mathrm{d}x\mathrm{d}y\right)^2 = \left(\int_0^\infty\int_0^\infty H(\lambda,x,y)\mathrm{d}x\mathrm{d}y\right)^2\\
&= \left(\int_0^\infty\int_0^\infty H(\lambda,x,y)F(x,y)\right)^2\\
&= \left\{\int_0^\infty\int_0^\infty \left(\frac{f(x)}{(x^\lambda+y^\lambda)^{\frac12}}\left(\frac{x}{y}\right)^{\frac14}(F(x,y))^{\frac12}\right)\left(\frac{f(y)}{(x^\lambda+y^\lambda)^{\frac12}}\left(\frac{y}{x}\right)^{\frac14}(F(x,y))^{\frac12}\right)\mathrm{d}x\mathrm{d}y\right\}^2\\
&\leqslant \int_0^\infty\int_0^\infty \frac{f^2(x)}{x^\lambda+y^\lambda}\left(\frac{x}{y}\right)^{\frac12}F(x,y)\mathrm{d}x\mathrm{d}y\int_0^\infty\int_0^\infty \frac{f^2(y)}{x^\lambda+y^\lambda}\left(\frac{y}{x}\right)^{\frac12}F(x,y)\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.2}$$

Since $f(x) \neq 0$, it is impossible to take equality in (3.2), we obtain

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x^\lambda+y^\lambda}\mathrm{d}x\mathrm{d}y\right)^2 < J_1J_2.$$

By Lemma 2.3, we get (3.1). This completes the proof. □

THEOREM 3.2. *Let $F(x,y)$ be defined as (2) and let $f,g:[0,+\infty)\to R^+$ be real function. Then*

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}\mathrm{d}x\mathrm{d}y\right)^4 < \left\{\left(\frac{\pi}{\lambda\sin\frac{\pi}{2\lambda}}\right)^2\left(\int_0^\infty x^{1-\lambda}f^2(x)\mathrm{d}x\right)^2 - \left(\int_0^\infty \Omega(\lambda,x)f^2(x)\mathrm{d}x\right)^2\right\}\times$$
$$\times\left\{\left(\frac{\pi}{\lambda\sin\frac{\pi}{2\lambda}}\right)^2\left(\int_0^\infty x^{1-\lambda}g^2(x)\mathrm{d}x\right)^2 - \left(\int_0^\infty \Omega(\lambda,x)g^2(x)\mathrm{d}x\right)^2\right\}, \quad (3.3)$$

where the weight function $\Omega(\lambda,x)$ is defined by (2.3).

Proof. By Schwarz's inequality, we have

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^\lambda+y^\lambda}\mathrm{d}x\mathrm{d}y\right)^4 = \left\{\left[\int_0^1\left(\int_0^\infty t^{x^\lambda-\frac12}f(x)\mathrm{d}x\int_0^\infty t^{y^\lambda-\frac12}g(y)\mathrm{d}y\right)\mathrm{d}t\right]^2\right\}^2$$
$$\leqslant \left\{\int_0^1\left(\int_0^\infty t^{x^\lambda-\frac12}f(x)\mathrm{d}x\right)^2\mathrm{d}t\right\}^2\left\{\int_0^1\left(\int_0^\infty t^{y^\lambda-\frac12}g(y)\mathrm{d}y\right)^2\mathrm{d}t\right\}^2$$
$$= \left(\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x^\lambda+y^\lambda}\mathrm{d}x\mathrm{d}y\right)^2\left(\int_0^\infty\int_0^\infty \frac{g(x)g(y)}{x^\lambda+y^\lambda}\mathrm{d}x\mathrm{d}y\right)^2. \quad (3.4)$$

Similarly to the proof of Theorem 3.1, we obtain

$$\left(\int_0^\infty\int_0^\infty \frac{g(x)g(y)}{x^\lambda+y^\lambda}\mathrm{d}x\mathrm{d}y\right)^2 \leqslant \left(\frac{\pi}{\lambda\sin\frac{\pi}{2\lambda}}\right)^2\left(\int_0^\infty x^{1-\lambda}g^2(x)\mathrm{d}x\right)^2 - \left(\int_0^\infty \Omega(\lambda,x)g^2(x)\mathrm{d}x\right)^2. \quad (3.5)$$

Since $g(x)\neq 0$, it is impossible to take equality in (3.5).

Substituting (3.1) and (3.5) into (3.4), we get (3.3). This completes the proof. □

In particular, let $\lambda=1$, we have a new improvement of the Hilbert inequality as following corollary.

COROLLARY 3.1. *Let $F(x,y)$ be defined as (2) and let $f,g:[0,+\infty)\to R^+$ be real function. Then*

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x+y}\right)^4 < \left\{\pi^2\left(\int_0^\infty f^2(x)\mathrm{d}x\right)^2 - \left(\int_0^\infty \widetilde{\Omega}(x)f^2(x)\mathrm{d}x\right)^2\right\}\times$$
$$\times\left\{\pi^2\left(\int_0^\infty g^2(x)\mathrm{d}x\right)^2 - \left(\int_0^\infty \widetilde{\Omega}(x)g^2(x)\mathrm{d}x\right)^2\right\}, \quad (3.6)$$

where the weight function $\widetilde{\Omega}(x)$ is defined by (2.7).

If we choose $c(x)=\frac{1}{1+x}\,(x\geqslant 0)$, by (2.8), then (3.6) becomes

$$\left(\int_0^\infty\int_0^\infty\frac{f(x)f(y)}{x+y}\right)^4<\pi^4\left\{\left(\int_0^\infty f^2(x)\mathrm{d}x\right)^2-\left(\int_0^\infty\widehat{\Omega}(x)f^2(x)\mathrm{d}x\right)^2\right\}\times$$
$$\times\left\{\left(\int_0^\infty g^2(x)\mathrm{d}x\right)^2-\left(\int_0^\infty\widehat{\Omega}(x)g^2(x)\mathrm{d}x\right)^2\right\},$$

where $\widehat{\Omega}(x)=\frac{1}{1+x}-\frac{1}{1+\sqrt{x}}$.

REMARK 3.1. The non-negative function $c(x)$ is chosen for maximum flexibility, because it only satisfies condition: $F(x,y)=1-c(x)+c(y)\geqslant 0$. Therefore, if we choose $c(x)=\frac{1}{2}\cos\sqrt{x}$, then by the integral formula(see, [16, page 189, formula 534])

$$\int_0^\infty\frac{\cos ax}{b^2+x^2}\mathrm{d}t=\frac{\pi}{2b}e^{-ab},\ (a\geqslant 0,Re\,b>0).$$

It is easy to calculate that

$$2\int_0^\infty\frac{c(xt^2)}{1+t^2}\mathrm{d}t=\int_0^\infty\frac{\cos\sqrt{x}t}{1+t^2}\mathrm{d}t=\frac{\pi}{2}e^{-\sqrt{x}},$$

hence, we get

$$\widehat{\Omega}(x)=\frac{1}{2}\left(\cos\sqrt{x}-e^{-\sqrt{x}}\right).$$

COROLLARY 3.2. *With the same assumptions as Corollary 3.1, let $\lambda=2$, then*

$$\left(\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x^2+y^2}\mathrm{d}x\mathrm{d}y\right)^4<\left\{\frac{\pi^2}{2}\left(\int_0^\infty\frac{1}{x}f^2(x)\mathrm{d}x\right)^2-\left(\int_0^\infty\bar{\Omega}(2,x)f^2(x)\mathrm{d}x\right)^2\right\}\times$$
$$\times\left\{\frac{\pi^2}{2}\left(\int_0^\infty\frac{1}{x}g^2(x)\mathrm{d}x\right)^2-\left(\int_0^\infty\bar{\Omega}(2,x)g^2(x)\mathrm{d}x\right)^2\right\},$$

where the weight function $\bar{\Omega}(2,x)$ is defined by

$$\bar{\Omega}(2,x)=\frac{1}{x}\left(\frac{\pi c(x)}{\sqrt{2}}-\int_0^\infty\frac{c(xu)}{1+u^2}\left(\frac{1}{u}\right)^{\frac{1}{2}}\mathrm{d}u\right).$$

4. Applications

In this section, we give a new refinement of Widder's inequality and an extension of Hardy-Littlewood's inequality as follows.

4.1. A new refinement of Widder's inequality

The following inequality is Widder's inequality (see [15]):

Let $a_n \geqslant 0\,(n=0,1,2,\ldots)$, $A(x)=\sum_{n=0}^{\infty} a_n x^n$, $A^*(x)=\sum_{n=0}^{\infty}\frac{a_n x^n}{n!}$. If $A(x)\neq 0$, then

$$\int_0^1 A^2(x)\mathrm{d}x < \pi\int_0^{\infty}\left(e^{-x}A^*(x)\right)^2\mathrm{d}x \tag{4.1}$$

We give a new refinement of (4.1) as follows:

THEOREM 4.1. *Let $a_n \geqslant 0\,(n=0,1,2,\ldots)$, $A(x)=\sum_{n=0}^{\infty} a_n x^n$, $A(x)\neq 0$ and $A^*(x)=\sum_{n=0}^{\infty}\frac{a_n x^n}{n!}$. Let $F(x,y)$ be defined as (3). then*

$$\left(\int_0^1 A^2(x)\mathrm{d}x\right)^2 < \pi^2\left(\int_0^{\infty}\left(e^{-x}A^*(x)\right)^2\mathrm{d}x\right)^2-\left(\int_0^{\infty}\widetilde{\Omega}(x)\left(e^{-x}A^*(x)\right)^2\mathrm{d}x\right)^2 \tag{4.2}$$

where the weight function $\widetilde{\Omega}(x)$ is defined by (2.7).

Proof. First, we have following relation:

$$\int_0^{\infty} e^{-t}A^*(tx)\mathrm{d}t=\int_0^{\infty} e^{-t}\sum_{n=0}^{\infty}\frac{a_n(xt)^n}{n!}\mathrm{d}t=\sum_{n=0}^{\infty}\frac{a_n x^n}{n!}\int_0^{\infty} t^n e^{-t}\mathrm{d}t$$
$$=\sum_{n=0}^{\infty} a_n x^n = A(x).$$

Squaring and integrating both sides of aforementioned equality from 0 to 1, we obtain

$$\int_0^1 A^2(x)\mathrm{d}x=\int_0^1\left(\int_0^{\infty} e^{-t}A^*(tx)\mathrm{d}t\right)^2\mathrm{d}x.$$

Let $tx=s$, from right side of aforementioned equality, we have

$$\int_0^1 A^2(x)\mathrm{d}x=\int_0^1\left(\int_0^{\infty} e^{-t}A^*(tx)\mathrm{d}t\right)^2\mathrm{d}x$$
$$=\int_0^1\left(\int_0^{\infty} e^{-\frac{s}{x}}A^*(s)\mathrm{d}s\right)^2\frac{1}{x^2}\mathrm{d}x.$$

Let $y=\frac{1}{x}$, we get

$$\int_0^1 A^2(x)\mathrm{d}x=\int_1^{\infty}\left(\int_0^{\infty} e^{-sy}A^*(s)\mathrm{d}s\right)^2\mathrm{d}y.$$

Let $u=y-1$, then

$$\int_0^1 A^2(x)\mathrm{d}x=\int_0^{\infty}\left(\int_0^{\infty} e^{-su-s}A^*(s)\mathrm{d}s\right)^2\mathrm{d}u$$

$$= \int_0^\infty \left(\int_0^\infty e^{-su} f(s) \mathrm{d}s \right)^2 \mathrm{d}u$$

$$= \int_0^\infty \int_0^\infty \frac{f(s)f(t)}{s+t} \mathrm{d}s\mathrm{d}t, \tag{4.3}$$

where $f(x) = e^{-x}A^*(x)$.
Thus, by Theorem (3.1) (choose $\lambda = 1$), we get (4.2) from (4.3). This completes the proof. □

4.2. An extension of Hardy-Littlewood's inequality

Hardy-Littlewood proved the following inequality(see [3]):

Let $f(x), x \in [0,1)$, be a non-negative real function, $a_n = \int_0^1 x^n f(x)\mathrm{d}x$, $n = 0,1,2,\ldots$. Then

$$\sum_{n=0}^\infty a_n^2 < \pi \int_0^1 f^2(x)\mathrm{d}x, \tag{4.4}$$

where π is the best constant.

In 1997, Gao [9] extended the inequality (4.4) and established the following inequality, named the Hardy-Littlewood integral inequality.

$$\int_0^\infty f^2(x)\mathrm{d}x < \pi \int_0^1 h^2(x)\mathrm{d}x, \tag{4.5}$$

where $f(x) = \int_0^1 t^x h(x)\mathrm{d}x$, $x \in [0,+\infty)$.

In 1999, Gao [11] refined (4.5) as following inequality:

$$\int_0^\infty f^2(x)\mathrm{d}x < \pi \int_0^1 t\,h^2(t)\mathrm{d}t. \tag{4.6}$$

In this paper, we will further extend the inequality (4.6). For notational simplicity, define

$$f(x) = \int_0^1 t^{x^\lambda} |h(t)|\mathrm{d}t, \ (x \geqslant 0, \lambda > \frac{1}{2}) \tag{4.7}$$

where $h(t) \neq 0, (t \in [0,1))$, is a real function.

We are in position to state and show the following theorem.

THEOREM 4.2. *Suppose that $f(x)$ be defined as (4.7), and $F(x,y)$ be defined as (2). Then*

$$\left(\int_0^\infty f^2(x)\mathrm{d}x \right)^4 < \left\{ \left(\frac{\pi}{\lambda \sin\frac{\pi}{2\lambda}} \int_0^\infty x^{1-\lambda} f^2(x)\mathrm{d}x \right)^2 - \left(\int_0^\infty \Omega(\lambda,x) f^2(x)\mathrm{d}x \right)^2 \right\} \times$$

$$\times \left(\int_0^1 t\,h^2(t)\mathrm{d}t \right)^2, \tag{4.8}$$

where the weight function $\Omega(\lambda,x)$ is defined by (2.3).

Proof. In view of definition of $f(x)$, we rewrite $f^2(x)$ as

$$f^2(x) = \int_0^1 f(x)t^{x^\lambda} |h(t)| \mathrm{d}t.$$

By Schwarz's inequality and Theorem 3.1, we have

$$\begin{aligned}
\left(\int_0^\infty f^2(x)\mathrm{d}x\right)^2 &= \left\{\int_0^\infty \left(\int_0^1 f(x)t^{x^\lambda} |h(t)|\mathrm{d}t\right)\mathrm{d}x\right\}^2 \\
&= \left\{\int_0^1 \left(\int_0^\infty f(x)t^{x^\lambda - \frac{1}{2}}\,\mathrm{d}x\right) t^{\frac{1}{2}} |h(t)|\mathrm{d}t\right\}^2 \\
&\leqslant \int_0^1 \left(\int_0^\infty f(x)t^{x^\lambda - \frac{1}{2}}\,\mathrm{d}x\right)^2 \mathrm{d}t \int_0^1 t\,h^2(t)\mathrm{d}t \\
&= \int_0^1 \left(\int_0^\infty f(x)t^{x^\lambda - \frac{1}{2}}\,\mathrm{d}x\right)\left(\int_0^\infty f(y)t^{y^\lambda - \frac{1}{2}}\,\mathrm{d}y\right)\mathrm{d}t \int_0^1 t\,h^2(t)\mathrm{d}t \\
&= \int_0^1 \left(\int_0^\infty \int_0^\infty f(x)f(y)t^{x^\lambda + y^\lambda - 1}\,\mathrm{d}x\mathrm{d}y\right)\mathrm{d}t \int_0^1 t\,h^2(t)\mathrm{d}t \\
&= \left(\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x^\lambda + y^\lambda}\,\mathrm{d}x\mathrm{d}y\right)\int_0^1 t\,h^2(t)\mathrm{d}t \\
&\leqslant \left\{\left(\frac{\pi}{\lambda \sin\frac{\pi}{2\lambda}}\int_0^\infty x^{1-\lambda} f^2(x)\mathrm{d}x\right)^2 - \left(\int_0^\infty \Omega(\lambda,x)f^2(x)\mathrm{d}x\right)^2\right\}^{\frac{1}{2}} \int_0^1 t\,h^2(t)\mathrm{d}t,
\end{aligned} \tag{4.9}$$

where the weight function $\Omega(\lambda,x)$ is defined by (2.3).

Since $h(t) \neq 0$, $f(x) \neq 0$, it is impossible to take equality in (4.9). Thus, we get inequality (4.8). This completes the proof. □

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