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EXTENSION OF AN OPPENHEIM TYPE DETERMINANTAL INEQUALITY FOR THE BLOCK HADAMARD PRODUCT

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Abstract. Günther and Klotz [Linear Algebra Appl. 437 (2012) 948-956] conjectured an Oppenheim type determinantal inequality for the block Hadamard product of two block commuting positive semidefinite matrices, which is proved by Lin [Linear Algebra Appl. 452 (2014) 1-6]. In this note, we extend the inequality to more block commuting positive semidefinite matrices. And the extension of this type inequality for the Hadamard product is also derived.

1. Introduction

Let $\mathbb{M}_n$ be the set of $n \times n$ complex matrices. If X is positive definite, we put $X > 0$. If X is positive semidefinite, we put $X \geqslant 0$, for two Hermitian matrices $X, Y \in \mathbb{M}_n, X \geqslant Y$ means $X - Y$ is positive semidefinite. The Hadamard product of $X, Y \in \mathbb{M}_n$ is denoted by $X \circ Y$.

The set of all complex matrices partitioned into $p \times p$ blocks with each block $q \times q$ is denoted by $\mathbb{M}_p(\mathbb{M}_q)$. Let $\mathbf{A} = (A_{ij}), \mathbf{B} = (B_{ij}) \in \mathbb{M}_p(\mathbb{M}_q)$. The block Hadamard product of $\mathbf{A}$ and $\mathbf{B}$ is given by $\mathbf{A} \square \mathbf{B} := (A_{ij}B_{ij})$. If every block of $\mathbf{A}$ commutes with every block of $\mathbf{B}$, we say that $\mathbf{A}, \mathbf{B}$ block commute. The block Hadamard product of $\mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathbb{M}_p(\mathbb{M}_q)$ is denoted by $\Pi_{i=1}^m \square \mathbf{A}_i$. For more information on the block Hadamard product, please refer to [2, 4, 5].

Let $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{M}_n$ be positive semidefinite matrices. Oppenheim [3, p. 509] proved that,

$$\det(A \circ B) \geqslant \det A \Big(\prod_{i=1}^n b_{ii}\Big). \tag{1}$$

The following inequality [6, Theorem 3.7] improves inequality (1),

$$\det(A \circ B) + \det(AB) \geqslant \det A \cdot \Big(\prod_{i=1}^n b_{ii}\Big) + \det B \cdot \Big(\prod_{i=1}^n a_{ii}\Big). \tag{2}$$

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Let $\mathbf{A}=(A_{ij}),\mathbf{B}=(B_{ij})\in\mathbb{M}_p(\mathbb{M}_q)$ be positive semidefinite and block commute. Günther and Klotz [2] extended Oppenheim's inequality (1) as follows

$$\det(\mathbf{A}\square\mathbf{B})=\det(\mathbf{B}\square\mathbf{A})\geqslant\det\mathbf{A}\cdot(\prod_{\mu=1}^{p}\det B_{\mu\mu}). \tag{3}$$

Under the same assumptions, the authors conjectured the following Oppenheim type determinantal inequality of two block commuting positive semidefinite matrices,

$$\det(\mathbf{A}\square\mathbf{B})+\det(\mathbf{AB})\geqslant\det(\mathbf{A})\cdot\prod_{\mu=1}^{p}\det B_{\mu\mu}+\det(\mathbf{B})\cdot\prod_{\mu=1}^{p}\det A_{\mu\mu}. \tag{4}$$

In [5], Lin resolved the conjecture by proving

$$\begin{aligned}\det(\mathbf{A}\square\mathbf{B})\geqslant&\det(\mathbf{AB})\\&\times\prod_{\mu=2}^{p}\det\left(\frac{\det A_{\mu\mu}\det\mathbf{A}_{\mu-1}}{\det\mathbf{A}_{\mu}}+\frac{\det B_{\mu\mu}\det\mathbf{B}_{\mu-1}}{\det\mathbf{B}_{\mu}}-1\right),\end{aligned} \tag{5}$$

where $\mathbf{A},\mathbf{B}\in\mathbb{M}_p(\mathbb{M}_q)$ are positive definite and block commute, $\mathbf{A}_\mu,\mathbf{B}_\mu$ are the $\mu\times\mu$ leading principal block submatrices of $\mathbf{A},\mathbf{B}$, respectively.

In this paper, we first extend Lin's inequality (5) to more block commuting positive definite matrices, then give the extension of the Oppenheim type determinantal inequality (4). And the extension of inequality (2) is also derived.

2. Main results

We begin with two lemmas.

LEMMA 1. [3, Corollary 7.7.4] *If* $A,B\in\mathbb{M}_n$ *such that* $A\geqslant B>0$*, then* $\det A\geqslant\det B$.

LEMMA 2. [2, Corollary 3.3] *Let* $\boldsymbol{A},\boldsymbol{B}\in\mathbb{M}_p(\mathbb{M}_q)$*. If* $\boldsymbol{A},\boldsymbol{B}$ *are positive semidefinite and block commute, then* $\boldsymbol{A}\square\boldsymbol{B}\geqslant0$

We give the following extension of Lin's inequality (5).

THEOREM 1. *Let* $\mathbf{A}_i=A_i^{hl}\in\mathbb{M}_p(\mathbb{M}_q),h,l=1,\dots,p,i=1,\dots,n$*, and let* $\boldsymbol{A}_i^{(k)}$ *be the* $k\times k$ *leading principal block submatrices of* $\boldsymbol{A}_i$*. If* $\mathbf{A}_i,i=1,\dots,n$*, are positive definite and block commute, then*

$$\det\left(\prod_{i=1}^{n}\square\boldsymbol{A}_i\right)\geqslant\det\left(\prod_{i=1}^{n}\boldsymbol{A}_i\right)\times\prod_{k=2}^{p}\left(\sum_{i=1}^{n}\frac{\det A_i^{kk}\det\boldsymbol{A}_i^{(k-1)}}{\det\boldsymbol{A}_i^{(k)}}-(n-1)\right). \tag{6}$$

Proof. We use induction with respect to n. When $n=2$, the result is (5). Assume that (6) is true for $n=m-1$, that is

$$\det\left(\prod_{i=1}^{m-1}\square\mathbf{A}_i\right)\geqslant\det\big(\prod_{i=1}^{m-1}\mathbf{A}_i\big)\times\prod_{k=2}^{p}\left(\sum_{i=1}^{m-1}\frac{\det A_i^{kk}\det\mathbf{A}_i^{(k-1)}}{\det\mathbf{A}_i^{(k)}}-(m-2)\right).$$

For $n=m$, by (5), we have

$$\begin{aligned}
&\det\left(\prod_{i=1}^{m}\square\mathbf{A}_i\right)\\
&=\det\left(\big(\prod_{i=1}^{m-1}\square\mathbf{A}_i\big)\square\mathbf{A}_m\right)\\
&\geqslant\det\left(\big(\prod_{i=1}^{m-1}\square\mathbf{A}_i\big)\mathbf{A}_m\right)\\
&\quad\times\prod_{k=2}^{p}\det\left(\frac{\det\big(\prod_{i=1}^{m-1}\square\mathbf{A}_i\big)^{kk}\det(\prod_{i=1}^{m-1}\square\mathbf{A}_i)^{(k-1)}}{\det(\prod_{i=1}^{m-1}\square\mathbf{A}_i)^{(k)}}+\frac{\det A_m^{kk}\det(\mathbf{A}_m)^{(k-1)}}{\det(\mathbf{A}_m)^{(k)}}-1\right).
\end{aligned}$$

Then, by applying the induction hypothesis we get

$$\begin{aligned}
&\det\left(\prod_{i=1}^{m}\square\mathbf{A}_i\right)\\
&\geqslant\det\big(\prod_{i=1}^{m-1}\mathbf{A}_i\big)\det\mathbf{A}_m\\
&\quad\times\prod_{k=2}^{p}\left(\sum_{i=1}^{m-1}\frac{\det A_i^{kk}\det\mathbf{A}_i^{(k-1)}}{\det\mathbf{A}_i^{(k)}}-(m-2)\right)\\
&\quad\times\prod_{k=2}^{p}\det\left(\frac{\det\big(\prod_{i=1}^{m-1}\square\mathbf{A}_i\big)^{kk}\det(\prod_{i=1}^{m-1}\square\mathbf{A}_i)^{(k-1)}}{\det(\prod_{i=1}^{m-1}\square\mathbf{A}_i)^{(k)}}+\frac{\det A_m^{kk}\det(\mathbf{A}_m)^{(k-1)}}{\det(\mathbf{A}_m)^{(k)}}-1\right).
\end{aligned}$$

Let

$$a_k=\sum_{i=1}^{m-1}\frac{\det A_i^{kk}\det\mathbf{A}_i^{(k-1)}}{\det\mathbf{A}_i^{(k)}}-(m-2),$$

$$b_k=\frac{\det\big(\prod_{i=1}^{m-1}\square\mathbf{A}_i\big)^{kk}\det(\prod_{i=1}^{m-1}\square\mathbf{A}_i)^{(k-1)}}{\det(\prod_{i=1}^{m-1}\square\mathbf{A}_i)^{(k)}}+\frac{\det A_m^{kk}\det(\mathbf{A}_m)^{(k-1)}}{\det(\mathbf{A}_m)^{(k)}}-1.$$

By Fischer's inequality [3, p. 506], we have

$$\frac{\det A_i^{kk}\det\mathbf{A}_i^{(k-1)}}{\det\mathbf{A}_i^{(k)}}\geqslant 1,$$

$$\frac{\det\big(\prod_{i=1}^{m-1}\square\mathbf{A}_i\big)^{kk}\det(\prod_{i=1}^{m-1}\square\mathbf{A}_i)^{(k-1)}}{\det(\prod_{i=1}^{m-1}\square\mathbf{A}_i)^{(k)}}\geqslant 1.$$

Then

$$a_k \geqslant 1, b_k \geqslant 1,$$

and

$$a_k b_k \geqslant a_k + b_k - 1.$$

We get

$$\begin{aligned}
&\det\Big(\prod_{i=1}^{m} \square \mathbf{A}_i\Big) \\
&\geqslant \det\big(\prod_{i=1}^{m-1} \mathbf{A}_i\big) \det \mathbf{A}_m \times \prod_{k=2}^{p} a_k \times \prod_{k=2}^{p} b_k \\
&\geqslant \det\big(\prod_{i=1}^{m} \mathbf{A}_i\big) \times \prod_{k=2}^{p} (a_k + b_k - 1) \\
&\geqslant \det\big(\prod_{i=1}^{m} \mathbf{A}_i\big) \times \prod_{k=2}^{p} \left(\sum_{i=1}^{m} \frac{\det A_i^{kk} \det \mathbf{A}_i^{(k-1)}}{\det \mathbf{A}_i^{(k)}} - (m-1) \right).
\end{aligned}$$

This completes the proof. □

We show a numerical inequality which is going to be needed in the proof of our following main result.

LEMMA 3. *Let* $a_{i,k} \geqslant 1, i = 1, \ldots, q, k = 1, \ldots, n$. *Then*

$$\prod_{k=1}^{n} \Big(\sum_{i=1}^{q} a_{i,k} - (q-1) \Big) \geqslant \sum_{i=1}^{q} \prod_{k=1}^{n} a_{i,k} - (q-1) \tag{7}$$

Proof. Use induction with respect to n. The base case $n = 1$ is trivial. Assume that (7) is true for $n = m$. Then, for $n = m+1$, we have

$$\begin{aligned}
&\prod_{k=1}^{m+1} \Big(\sum_{i=1}^{q} a_{i,k} - (q-1) \Big) \\
&= \Big(\sum_{i=1}^{q} a_{i,m+1} - (q-1) \Big) \prod_{k=1}^{m} \Big(\sum_{i=1}^{q} a_{i,k} - (q-1) \Big) \\
&\geqslant \Big(\sum_{i=1}^{q} a_{i,m+1} - (q-1) \Big) \Big(\sum_{i=1}^{q} \prod_{k=1}^{m} a_{i,k} - (q-1) \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{q} \prod_{k=1}^{m+1} a_{i,k} + (q-1)^2 \\
&\quad + \sum_{i=1}^{q} a_{i,m+1} \Big(\sum_{j \neq i}^{q} \prod_{k=1}^{m} a_{j,k} - (q-1) \Big) - (q-1) \sum_{i=1}^{q} \prod_{k=1}^{m} a_{i,k}. \\
&= \sum_{i=1}^{q} \prod_{k=1}^{m+1} a_{i,k} - (q-1) \\
&\quad + \sum_{i=1}^{q} (a_{i,m+1} - 1) \Big(\sum_{j \neq i}^{q} \prod_{k=1}^{m} a_{j,k} - (q-1) \Big) \\
&\geqslant \sum_{i=1}^{q} \prod_{k=1}^{m+1} a_{i,k} - (q-1).
\end{aligned}
$$

This proves the Lemma 3. □

Now we are ready to present the extension of the Oppenheim type determinantal inequality (4), which is implied by (6).

THEOREM 2. *Let $\mathbf{A}_i = A_i^{hl} \in \mathbb{M}_p(\mathbb{M}_q), h, l = 1, \ldots, p, i = 1, \ldots, m,$ and let $\mathbf{A}_i^{(k)}$ be the $k \times k$ leading principal block submatrices of $\mathbf{A}_i$. If $\mathbf{A}_i, i = 1, \ldots, m,$ are positive semidefinite and block commute, then*

$$
\det\Big(\prod_{i=1}^{m} \square \mathbf{A}_i\Big) + (m-1)\det\big(\prod_{i=1}^{m} \mathbf{A}_i\big) \geqslant \sum_{i=1}^{m} \prod_{j=1, j\neq i}^{m} \det \mathbf{A}_j \prod_{k=1}^{p} A_i^{kk}. \tag{8}
$$

Proof. If any of A_i^{kk} in (8) is singular, then so is $\mathbf{A}_i$. In this case the right hand side of (8) vanishes. So without loss of generality, we can assume that $\mathbf{A}_i$ are positive definite.

We may rewrite (8) as

$$
\det\Big(\prod_{i=1}^{m} \square \mathbf{A}_i\Big) \geqslant \det\big(\prod_{i=1}^{m} \mathbf{A}_i\big)\Big(\sum_{i=1}^{m} \frac{\prod_{k=1}^{p} A_i^{kk}}{\det \mathbf{A}_i} - (m-1)\Big). \tag{9}
$$

In (6), for $k = 2, \ldots, p, i = 1, \ldots, m,$ by Fischer's inequality [3, p. 506], we have

$$
\frac{\det A_i^{kk} \det \mathbf{A}_i^{(k-1)}}{\det \mathbf{A}_i^{(k)}} \geqslant 1.
$$

Then, by (6) and (7), we get

$$
\begin{aligned}
\det\Big(\prod_{i=1}^{m} \square \mathbf{A}_i\Big) &\geqslant \det\big(\prod_{i=1}^{m} \mathbf{A}_i\big) \times \prod_{k=2}^{p} \Big(\sum_{i=1}^{m} \frac{\det A_i^{kk} \det \mathbf{A}_i^{(k-1)}}{\det \mathbf{A}_i^{(k)}} - (m-1) \Big) \\
&\geqslant \det\big(\prod_{i=1}^{m} \mathbf{A}_i\big) \Big(\sum_{i=1}^{m} \prod_{k=2}^{p} \frac{\det A_i^{kk} \det \mathbf{A}_i^{(k-1)}}{\det \mathbf{A}_i^{(k)}} - (m-1) \Big).
\end{aligned}
$$

Note that

$$\prod_{k=2}^{p}\frac{\det A_i^{kk}\det\mathbf{A}_i^{(k-1)}}{\det\mathbf{A}_i^{(k)}}=\left(\prod_{k=2}^{p}\det A_i^{kk}\right)\left(\prod_{k=2}^{p}\frac{\det\mathbf{A}_i^{(k-1)}}{\det\mathbf{A}_i^{(k)}}\right)$$
$$=\left(\prod_{k=2}^{p}\det A_i^{kk}\right)\left(\frac{\det\mathbf{A}_i^{(1)}}{\det\mathbf{A}_i^{(p)}}\right)$$
$$=\frac{\prod_{k=1}^{p}\det A_i^{kk}}{\det\mathbf{A}_i}.$$

Then, (9) is obtained, and the proof is completed. □

For the Hadamard product, Fu and Liu [1] proved the following inequality.

THEOREM 3. [1, Theorem 7] *Let $A_i=(a_i^{hl})\in\mathbb{M}_n, h,l=1,\ldots,n, i=1,\ldots,m,$ be positive definite matrices, and let $A_i^{(k)}$ be the $k\times k$ leading principal submatrix of A_i. Then*

$$\det\left(\prod_{i=1}^{m}\circ A_i\right)\geqslant\det\big(\prod_{i=1}^{m}A_i\big)\times\prod_{k=2}^{n}\left(\sum_{i=1}^{m}\frac{\det a_i^{kk}\det A_i^{(k-1)}}{\det A_i^{(k)}}-(m-1)\right). \tag{10}$$

Similar to the proof of Theorem 2, the following extension of inequality (2) can be obtained by (10) and (7). So we can say that (10) implies the extension.

THEOREM 4. *Let $A_i=(a_i^{hl})\in\mathbb{M}_n, h,l=1,\ldots,n, i=1,\ldots,m,$ be positive semidefinite matrices. Then*

$$\det\left(\prod_{i=1}^{m}\circ A_i\right)+(m-1)\det\big(\prod_{i=1}^{m}A_i\big)\geqslant\sum_{i=1}^{m}\prod_{j=1,j\neq i}^{m}\det A_j\prod_{k=1}^{n}a_i^{kk}. \tag{11}$$

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