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EXTENSIONS OF HIAI TYPE LOG–MAJORIZATION INEQUALITIES

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Abstract. In this paper, we will obtain several log-majorization inequalities via Furuta inequalities with negative powers, which extends the related results before.

1. Introduction

Throughout this paper, a capital letter, such as T, stands for an $n\times n$ complex matrix. We write $T>O$ if T is positive definite.

Recall that for $X,Y>O$, the log-majorization $X\prec_{\log} Y$ means that

$$\begin{cases} \prod_{i=1}^{k}\lambda_i(X)\leqslant \prod_{i=1}^{k}\lambda_i(Y), & k=1,2,\cdots,n-1;\\ \prod_{i=1}^{k}\lambda_i(X)= \prod_{i=1}^{k}\lambda_i(Y), & k=n, \end{cases}$$

where $\lambda_1(X)\geqslant\lambda_2(X)\geqslant\cdots\geqslant\lambda_n(X)$ are the eigenvalues of X in decreasing order counting multiplicities.

Recently, F. Hiai proved two beautiful log-majorization inequalities in [1] as follows.

THEOREM 1.1. ([1]). *Let* $r,z>0$ *with* $\alpha>1$.
Put

$$Q_{\alpha,z}(A,B)=(B^{\frac{1-\alpha}{2z}}A^{\frac{\alpha}{z}}B^{\frac{1-\alpha}{2z}})^z \tag{1.1}$$

and

$$P_{\alpha,r}(A,B)=\{B^{\frac{1}{2r}}(B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{-\frac{1}{2r}})^{\alpha}B^{\frac{1}{2r}}\}^r. \tag{1.2}$$

If $z/r\geqslant\max\{\alpha/2,\alpha-1\}$, *then* $Q_{\alpha,z}(A,B)\prec_{\log}P_{\alpha,r}(A,B)$ *for every* $A,B>0$;
If $0<z/r\leqslant\min\{\alpha/2,\alpha-1\}$, *then* $P_{\alpha,r}(A,B)\prec_{\log}Q_{\alpha,z}(A,B)$ *for every* $A,B>0$.

In this paper, we will obtain several log-majorization inequalities via Furuta inequalities with negative powers, which extends Theroem 1.1.

In order to prove the main result, here, we introduce a famous operator inequality — Furuta inequalities with negative powers as follows.

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THEOREM 1.2. (Furuta inequalities with negative powers, [2]). *If* $A \geqslant B \geqslant O$ *with* $A > O$, $0 < p \leqslant 1$ *and* $0 < q \leqslant 1$, $-1 \leqslant r < 0$, *then*

$$(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \leqslant A^{\frac{p+r}{q}} \tag{1.3}$$

holds as long as real numbers p, q, r *satisfy*

$$-r(1-q) \leqslant p \leqslant q - r(1-q) \tag{1.4}$$

and one of the following two conditions :

$$\frac{1}{2} \leqslant q \leqslant 1 \tag{1.5}$$

or

$$0 < q < \frac{1}{2}, \quad \frac{-r(1-q)-q}{1-2q} \leqslant p \leqslant \frac{-r(1-q)}{1-2q}. \tag{1.6}$$

By further discussion, C. Yang et al showed several forms of Furuta inequality with negative powers in [3]. Here, we list two forms related to the main results as follows.

THEOREM 1.3. ([3]). *For* $A \geqslant B \geqslant 0$ *with* $A > 0$, *the following results hold.*
(Form I) If $1 \geqslant t > p \geqslant 0$, $\frac{1}{2} \geqslant p$, *then* $A^{-t} \geqslant (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{-t}{p-t}}$;
(Form II) If $1 \geqslant t > p \geqslant \frac{1}{2}$, *then* $A^{2p-1-t} \geqslant (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{2p-1-t}{p-t}}$.

Although the proof of Theorem 1.3 is shown in [3], here we give a sketchy introduction of the proof for the convenience of the reader as follows.

Put $r = -t$ and $q = \frac{p-t}{-t}$ in Theorem 1.2. Together with (1.4) and (1.5), which are equivalent to $1 \geqslant t > \frac{t}{2} \geqslant p \geqslant 0$, we can obtain (1.3), which is just $A^{-t} \geqslant (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{-t}{p-t}}$; Together with (1.4) and (1.6), which are equivalent to $1 \geqslant t > p > \frac{t}{2} \geqslant 0$ and $p \leqslant \frac{1}{2}$, we can also obtain (1.3). Therefore, (Form I) in Theorem 1.3 holds.

Put $r = -t$ and $q = \frac{p-t}{2p-1-t}$ in Theorem 1.2. Together with (1.4) and (1.6), which are equivalent to $1 \geqslant t > p \geqslant \frac{1}{2}$, we can obtain (1.3), which is just $A^{2p-1-t} \geqslant (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{2p-1-t}{p-t}}$. Therefore, (Form II) in Theorem 1.3 holds.

2. Main Results

In this section, we will show the extensions of Hiai type log-majorization inequalities, which are equivalent to the two forms of Furuta inequalities with negatives powers.

THEOREM 2.1. *If* $r > 0$, $p \leqslant \frac{1}{2}$, $0 \leqslant p < t \leqslant 1$, $0 < \theta \leqslant 1$ *and* $\alpha > 1$, *then*

$$P_{\alpha,r}(A,B) \succ_{\log} \widetilde{Q}_{\alpha,p,r,t,\theta}(A,B) \tag{2.1}$$

holds for every $A, B > O$, *where*

$$\widetilde{Q}_{\alpha,p,r,t,\theta} = \{B^{-\frac{t\theta}{2r}} [B^{\frac{t}{2r}} (B^{-\frac{1}{2r}} A^{\frac{1}{r}} B^{-\frac{1}{2r}})^{p\alpha} B^{\frac{t}{2r}}]^{-\frac{t\theta}{p-t}} B^{-\frac{t\theta}{2r}}\}^{-\frac{r(p-t)}{pt\theta}}. \tag{2.2}$$

Furthermore, (2.1) is equivalent to Form I of Furuta inequality with negative powers.

Proof. First, we prove that (2.1) can be obtained by Form I of Furuta inequality with negative powers.

We only neeed to prove that $P_{\alpha,r}(A,B) \leqslant I$ ensures that $\widetilde{Q}_{\alpha,p,r,t,\theta}(A,B) \leqslant I$.

$P_{\alpha,r}(A,B) \leqslant I$ means that

$$(B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{-\frac{1}{2r}})^{\alpha} \leqslant B^{-\frac{1}{r}}. \tag{2.3}$$

Put $B_1 = (B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{-\frac{1}{2r}})^{\alpha}$, $A_1 = B^{-\frac{1}{r}}$. By Form I of Furuta inequality with negative powers,

$$A_1^{-t\theta} \geqslant (A_1^{-\frac{t}{2}}B_1^pA_1^{-\frac{t}{2}})^{-\frac{t\theta}{p-t}} \tag{2.4}$$

holds for $p \leqslant \frac{1}{2}$, $0 \leqslant p < t \leqslant 1$ and $0 \leqslant \theta \leqslant 1$.

It follows that

$$B^{\frac{t\theta}{r}} \geqslant [B^{\frac{t}{2r}}(B^{\frac{-1}{2r}}A^{\frac{1}{r}}B^{\frac{-1}{2r}})^{p\alpha}B^{\frac{t}{2r}}]^{-\frac{t\theta}{p-t}}, \tag{2.5}$$

which is equivalent to that

$$B^{-\frac{t\theta}{2r}}[B^{\frac{t}{2r}}(B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{-\frac{1}{2r}})^{p\alpha}B^{\frac{t}{2r}}]^{-\frac{t\theta}{p-t}}B^{-\frac{t\theta}{2r}} \leqslant I. \tag{2.6}$$

Then $\widetilde{Q}_{\alpha,p,r,t,\theta}(A,B) \leqslant I$ holds obviously.

Next, we prove that Form I of Furuta inequality with negative powers can be derived from (2.1).

Notice that $P_{\alpha,r}(A,B) \leqslant I$ ensures that $\widetilde{Q}_{\alpha,p,r,t,\theta}(A,B) \leqslant I$. Because $P_{\alpha,r}(A,B) \leqslant I$ is equivalent to (2.3) and $\widetilde{Q}_{\alpha,p,r,t,\theta}(A,B) \leqslant I$ is equivalent to (2.5).

Put $\theta = 1$, $B = A_1^{-r}$ and $A = (A_1^{-\frac{1}{2}}B_1^{\frac{1}{\alpha}}A_1^{-\frac{1}{2}})^r$ in (2.3) and (2.5). We can obtain that $A_1 \geqslant B_1$ ensures that

$$A_1^{-t} \geqslant (A_1^{-\frac{t}{2}}B_1^pA_1^{-\frac{t}{2}})^{-\frac{t}{p-t}} \tag{2.7}$$

for $p \leqslant \frac{1}{2}$ and $0 \leqslant p < t \leqslant 1$, which is just Form I of Furuta inequality with negative powers. □

COROLLARY 2.2.. *If* $r > 0$, $\frac{1}{\alpha} < t \leqslant 1$, $0 < \theta \leqslant 1$ *and* $\alpha \geqslant 2$, *then*

$$\{B^{\frac{1}{2r}}(B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{-\frac{1}{2r}})^{\alpha}B^{\frac{1}{2r}}\}^r \succ_{\log} \{B^{-\frac{t\theta}{2r}}(B^{\frac{t-1}{2r}}A^{\frac{1}{r}}B^{\frac{t-1}{2r}})^{-\frac{\alpha t\theta}{1-\alpha t}}B^{-\frac{t\theta}{2r}}\}^{-\frac{r(1-\alpha t)}{t\theta}}$$

holds for every $A,B > O$.

Proof. Put $p = \frac{1}{\alpha}$ in Theorem 2.1. □

COROLLARY 2.3. *If* $r > 0$, $0 < \theta \leqslant 1$ *and* $\alpha \geqslant 2$, *then*

$$\{B^{\frac{1}{2r}}(B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{-\frac{1}{2r}})^{\alpha}B^{\frac{1}{2r}}\}^r \succ_{\log} \{B^{-\frac{\theta}{2r}}A^{-\frac{\alpha\theta}{r(1-\alpha)}}B^{-\frac{\theta}{2r}}\}^{-\frac{r(1-\alpha)}{\theta}}$$

holds for every $A,B > O$.

Proof. Put $t = 1$ in Corollary 2.2. □

THEOREM 2.4. *If* $r>0$, $1 \geqslant t > p \geqslant \frac{1}{2}$, $0<\theta \leqslant 1$ *and* $\alpha>1$, *then*

$$P_{\alpha,r}(A,B) \succ_{\log} \widehat{Q}_{\alpha,p,r,t,\theta}(A,B) \tag{2.8}$$

holds for every $A,B>O$, *where*

$$\widehat{Q}_{\alpha,p,r,t,\theta} = \{B^{\frac{\theta(2p-t-1)}{2r}}[B^{\frac{t}{2r}}(B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{-\frac{1}{2r}})^{p\alpha}B^{\frac{t}{2r}}]^{\frac{\theta(2p-t-1)}{p-t}}B^{\frac{\theta(2p-t-1)}{2r}}\}^{\frac{r(p-t)}{p(2p-t-1)\theta}}. \tag{2.9}$$

Furthermore, (2.8) is equivalent to Form II of Furuta inequality with negative powers.

Proof. First, we prove that (2.8) can be obtained by Form II of Furuta inequality with negative powers.

We only neeed to prove that $P_{\alpha,r}(A,B) \leqslant I$ ensures that $\widehat{Q}_{\alpha,p,r,t,\theta}(A,B) \leqslant I$.

$P_{\alpha,r}(A,B) \leqslant I$ means that

$$(B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{-\frac{1}{2r}})^{\alpha} \leqslant B^{-\frac{1}{r}}. \tag{2.10}$$

Put $B_1 = (B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{-\frac{1}{2r}})^{\alpha}$, $A_1 = B^{-\frac{1}{r}}$. By Form II of Furuta inequality with negative powers,

$$A_1^{(2p-1-t)\theta} \geqslant (A_1^{-\frac{t}{2}}B_1^pA_1^{-\frac{t}{2}})^{\frac{(2p-1-t)\theta}{p-t}} \tag{2.11}$$

holds for $\frac{1}{2} \leqslant p < t \leqslant 1$ and $0 \leqslant \theta \leqslant 1$.

It follows that

$$B^{-\frac{(2p-1-t)\theta}{r}} \geqslant [B^{\frac{t}{2r}}(B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{\frac{-1}{2r}})^{p\alpha}B^{\frac{t}{2r}}]^{\frac{(2p-1-t)\theta}{p-t}}, \tag{2.12}$$

which is equivalent to that

$$B^{\frac{\theta(2p-t-1)}{2r}}[B^{\frac{t}{2r}}(B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{-\frac{1}{2r}})^{p\alpha}B^{\frac{t}{2r}}]^{\frac{\theta(2p-t-1)}{p-t}}B^{\frac{\theta(2p-t-1)}{2r}} \leqslant I. \tag{2.13}$$

Then $\widehat{Q}_{\alpha,p,r,t,\theta}(A,B) \leqslant I$ holds obviously.

Next, we prove that Form II of Furuta inequality with negative powers can be derived from (2.8).

Notice that $P_{\alpha,r}(A,B) \leqslant I$ ensures that $\widehat{Q}_{\alpha,p,r,t,\theta}(A,B) \leqslant I$. Because $P_{\alpha,r}(A,B) \leqslant I$ is equivalent to (2.10) and $\widehat{Q}_{\alpha,p,r,t,\theta}(A,B) \leqslant I$ is equivalent to (2.12).

Put $\theta = 1$, $B = A_1^{-r}$ and $A = (A_1^{-\frac{1}{2}}B_1^{\frac{1}{\alpha}}A_1^{-\frac{1}{2}})^r$ in (2.10) and (2.12). We can obtain that $A_1 \geqslant B_1$ ensures that

$$A_1^{2p-t-1} \geqslant (A_1^{-\frac{t}{2}}B_1^pA_1^{-\frac{t}{2}})^{\frac{2p-t-1}{p-t}} \tag{2.14}$$

for $\frac{1}{2} \leqslant p < t \leqslant 1$, which is just Form II of Furuta inequality with negative powers. □

COROLLARY 2.5. *If* $r>0$, $\frac{1}{\alpha} < t \leqslant 1$, $0<\theta \leqslant 1$ *and* $1<\alpha \leqslant 2$, *then*

$$\{B^{\frac{1}{2r}}(B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{-\frac{1}{2r}})^{\alpha}B^{\frac{1}{2r}}\}^r \succ_{\log} \{B^{\frac{\theta(2-t\alpha-\alpha)}{2r\alpha}}(B^{\frac{t-1}{2r}}A^{\frac{1}{r}}B^{\frac{t-1}{2r}})^{\frac{\theta(2-t\alpha-\alpha)}{1-t\alpha}}B^{\frac{\theta(2-t\alpha-\alpha)}{2r\alpha}}\}^{\frac{r\alpha(1-t\alpha)}{\theta(2-t\alpha-\alpha)}}$$

holds for every $A,B>O$.

Proof. Put $p = \frac{1}{\alpha}$ in Theorem 2.4. □

COROLLARY 2.6. *If* $r>0$, $0<\theta\leqslant 1$ *and* $1<\alpha\leqslant 2$, *then*

$$\{B^{\frac{1}{2r}}(B^{-\frac{1}{2r}}A^{\frac{1}{r}}B^{-\frac{1}{2r}})^{\alpha}B^{\frac{1}{2r}}\}^{r}\succ_{\log}\{B^{\frac{\theta(1-\alpha)}{r\alpha}}A^{\frac{2\theta}{r}}B^{\frac{\theta(1-\alpha)}{r\alpha}}\}^{\frac{r\alpha}{2\theta}}$$

holds for every $A,B>O$.

Proof. Put $t=1$ in Corollary 2.5. □

REMARK. Put $\theta=-\frac{r(1-\alpha)}{z}$ in Corollary 2.3, and put $\theta=\frac{r\alpha}{2z}$ in Corollary 2.6, they are just the first part of Theorem 1.1.

THEOREM 2.7. *If* $z>0$, $p\leqslant\frac{1}{2}$, $0\leqslant p<t\leqslant 1$ *and* $\alpha>1$, *then*

$$Q_{\alpha,z}(A,B)\succ_{\log}\widetilde{P}_{\alpha,p,t,z}(A,B) \tag{2.15}$$

holds for every $A,B>O$, *where*

$$\widetilde{P}_{\alpha,p,t,z}(A,B)=\{B^{\frac{t(\alpha-1)}{2z}}(B^{\frac{t(1-\alpha)}{2z}}A^{\frac{\alpha p}{z}}B^{\frac{t(1-\alpha)}{2z}})^{-\frac{t}{p-t}}B^{\frac{t(\alpha-1)}{2z}}\}^{-\frac{z(p-t)}{pt}}. \tag{2.16}$$

Furthermore, (2.15) is equivalent to Form I of Furuta inequality with negative powers.

Proof. First, we prove that (2.15) can be obtained by Form I of Furuta inequality with negative powers.

We only neeed to prove that $Q_{\alpha,z}(A,B)\leqslant I$ ensures that $\widetilde{P}_{\alpha,p,t,z}(A,B)\leqslant I$.

$Q_{\alpha,z}(A,B)\leqslant I$ means that

$$B^{\frac{1-\alpha}{2z}}A^{\frac{\alpha}{z}}B^{\frac{1-\alpha}{2z}}\leqslant I. \tag{2.17}$$

It follows that

$$A^{\frac{\alpha}{z}}\leqslant B^{\frac{\alpha-1}{z}}. \tag{2.18}$$

Put $B_1=A^{\frac{\alpha}{z}}$, $A_1=B^{\frac{\alpha-1}{z}}$. By Form I of Furuta inequality with negative powers,

$$A_1^{-t}\geqslant(A_1^{-\frac{t}{2}}B_1^{p}A_1^{-\frac{t}{2}})^{-\frac{t}{p-t}} \tag{2.19}$$

holds for $p\leqslant\frac{1}{2}$, $0\leqslant p<t\leqslant 1$.

It follows that

$$B^{-\frac{t(\alpha-1)}{z}}\geqslant(B^{\frac{t(1-\alpha)}{2z}}A^{\frac{p\alpha}{z}}B^{\frac{t(1-\alpha)}{2z}})^{-\frac{t}{p-t}}, \tag{2.20}$$

which is equivalent to that

$$B^{\frac{t(\alpha-1)}{2z}}(B^{\frac{t(1-\alpha)}{2z}}A^{\frac{p\alpha}{z}}B^{\frac{t(1-\alpha)}{2z}})^{-\frac{t}{p-t}}B^{\frac{t(\alpha-1)}{2z}}\leqslant I. \tag{2.21}$$

Then $\widetilde{P}_{\alpha,p,t,z}(A,B)\leqslant I$ holds obviously.

Next, we prove that Form I of Furuta inequality with negative powers can be derived from (2.15).

Notice that $Q_{\alpha,z}(A,B)\leqslant I$ ensures that $\widetilde{P}_{\alpha,p,t,z}(A,B)\leqslant I$. Because $Q_{\alpha,z}(A,B)\leqslant I$ is equivalent to (2.18) and $\widetilde{P}_{\alpha,p,t,z}(A,B)\leqslant I$ is equivalent to (2.20).

Put $B = A_1^{\frac{z}{\alpha-1}}$ and $A = B_1^{\frac{z}{\alpha}}$ in (2.18) and (2.20). We can obtain that $A_1 \geqslant B_1$ ensures that

$$A_1^{-t} \geqslant (A_1^{-\frac{t}{2}} B_1^p A_1^{-\frac{t}{2}})^{-\frac{t}{p-t}} \tag{2.22}$$

for $p \leqslant \frac{1}{2}$ and $0 \leqslant p < t \leqslant 1$, which is just Form I of Furuta inequality with negative powers. □

REMARK. If we put $p = \frac{z}{r} \cdot \frac{1}{\alpha}$ and $t = \frac{z}{r} \cdot \frac{1}{\alpha-1}$, then Theorem 2.7 is just the second part of Theorem 1.1.

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