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EULER–LAGRANGE EQUATIONS ASSOCIATED WITH EXTREMAL FUNCTIONS OF SEVERAL NONLOCAL INEQUALITIES

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Abstract. This paper is concerned with the extremal functions of several kinds of non-local inequalities, including the Hardy-Littlewood-Sobolev inequality, fractional Gagliardo-Nirenberg inequality, nonlocal Gagliardo-Nirenberg inequality and Coulomb-Sobolev inequality. First, we derive the Euler-Lagrange equations which they satisfy. Second, we investigate the existence of some integrable classical solutions for these equations, where the Pohozaev identity plays a key role.

1. Introduction

This paper is concerned with nonlocal Gagliardo-Nirenberg inequalities. We study the Euler-Lagrange equations which the extremal functions satisfy.

Recall the Hardy-Littlewood-Sobolev (HLS) inequality (cf. [16])

$$\left|\int_{R^n}\int_{R^n}\frac{f(x)g(y)}{|x-y|^{n-\alpha}}dxdy\right| \leqslant C\|f\|_s\|g\|_t,\ \forall\ f\in L^s,\ g\in L^t, \tag{1}$$

where $0<\alpha<n$, $s,t>1$, and $\frac{1}{s}+\frac{1}{t}+\frac{n-\alpha}{n}=2$. Such an inequality comes into play in the study of estimating the Coulomb energy (cf. [3], [4], [12])

$$\int_{R^n}\int_{R^n}\frac{u^p(x)u^p(y)}{|x-y|^{n-\alpha}}dxdy.$$

In order to obtain the upper bound of the Coulomb energy, investigating the best constant of (1) is necessary. In 1983, Lieb [11] employed Schwarz symmetrization to figure successfully out the Euler-Lagrange equation as

$$\begin{cases} u(x)=\displaystyle\int_{R^n}\frac{v^q(y)}{|x-y|^{n-\alpha}}dy,\\ v(x)=\displaystyle\int_{R^n}\frac{u^p(y)}{|x-y|^{n-\alpha}}dy, \end{cases} \tag{2}$$

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and the explicit representation of the extremal functions when $u=v$ and $p=q$ as

$$u\equiv v\equiv c\left(\frac{\delta}{\delta^2+|x-x_0|^2}\right)^{\frac{n-\alpha}{2}}. \tag{3}$$

Afterwards, Chen-Li-Ou [5] and Li [10] proved that all the regular solutions of (2) with $u=v$ and $p=q$ are the form of (3).

By the Hölder inequality and the definition of norm of operator, the classical HLS inequality (1) in R^n with $n\geqslant 2$ is equivalent to the following inequality

$$\|Tg\|_r\leqslant C\|g\|_{\frac{nr}{n+\alpha r}}, \tag{4}$$

where $Tg=\displaystyle\int_{R^n}\frac{g(y)}{|x-y|^{n-\alpha}}dy$ and $r>\frac{n}{n-\alpha}$. Moreover, if u is a rapidly decreasing function, then (4) is equivalent to the inequality below

$$\|u\|_r\leqslant c\|(-\Delta)^{\frac{\alpha}{2}}u\|_{\frac{nr}{n+\alpha r}}, \tag{5}$$

where $r>\frac{n}{n-\alpha}$ and

$$(-\Delta)^{\frac{\alpha}{2}}u:=C_{n,\alpha}P.V.\int_{R^n}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dy=C_{n,\alpha}\lim_{\varepsilon\to 0^+}\int_{|x-y|\geqslant\varepsilon}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dy.$$

Here $C_{n,\alpha}$ is a positive constant. So it is natural that (1) and (5) are equivalent.

In the following special cases: (i) $f=g$, and $s=t$ in (1); (ii) $r=\frac{2n}{n-\alpha}$ in (4) (or $r=\frac{2n}{n-\alpha}$ in (5)); (iii) $u=v$, and $p=q$ in (2), they have the same optimal function (3) (cf. [11], [5]).

Except for these special cases, what the extremal functions are is still an open problem.

Recall the fractional Gagliardo-Nirenberg (GN) inequality (cf. [7] and the references therein)

$$\|u\|_p\leqslant C\|(-\Delta)^{\frac{\alpha}{4}}u\|_2^{\theta}\|u\|_q^{1-\theta},\ \forall\, u\in\mathscr{D}^{\alpha,2}\cap L^q, \tag{6}$$

where $n\geqslant 2$, $1<q<p\leqslant\frac{2n}{n-2\alpha}$ and $\theta=\frac{2n(p-q)}{p[2n-(n-\alpha)q]}$. When $\alpha=2$ and $1<q\leqslant\frac{2n-2}{n-2}$, $p=2q-2$, Del Pino and Dolbeault [6] obtained the best constant. When $\alpha=q=2$, $p=\frac{2n+4}{n}$ and $\theta=\frac{n}{n+2}$, Weinstein [17] proved the extremal functions of (6) satisfy a static Schrödinger equation

$$-\Delta R+R=|R|^{\gamma}R,$$

where $\gamma=\frac{4}{n}$.

For more general exponents, the Euler-Lagrange equation is

$$-\Delta u+|u|^{q-2}u=|u|^{p-2}u. \tag{7}$$

This type of problems can be seen as a prototype of the pattern formation in biology, which is related to the steady-state problem for a chemotactic aggregation model introduced by Keller and Segel. This equation also plays an important role in the study of biological patterning of the activator-inhibitor system, which was proposed by Gierer and Meinhardt. This type of problems, as well as the associated evolutionary equations, describes the phenomenon of super-diffusion. De Gennes presented the models to describe the long van der Waals interaction on the solid surface.

Inserting the HLS inequality into the GN inequality yields

$$\|Tu\|_r \leqslant C\|(-\Delta)^{\frac{\alpha}{4}}u\|_2^{\theta}\|u\|_q^{1-\theta}, \tag{8}$$

where $1 < q < \frac{nr}{n+\alpha r}$ and $\theta = \frac{2[nr-(n+\alpha r)q]}{r[2n-(n-\alpha)q]}$. This is also a fractional GN inequality. Another fractional GN inequality is the following Coulomb-Sobolev inequality (cf. [1], [14])

$$\|u\|_p \leqslant C\|\nabla u\|_2^{\theta}\left(\int\int_{R^n\times R^n}\frac{u^q(x)u^q(y)}{|x-y|^{n-\alpha}}dxdy\right)^{\tau}, \ \forall\, u\in\ X^{1,\alpha}, \tag{9}$$

where

$$\frac{1}{p} = \theta(\frac{1}{2}-\frac{1}{n})+(1-\theta)\frac{n+\alpha}{2nq},\ \theta = \frac{n+\alpha-\frac{2qn}{p}}{(n+\alpha)-q(n-2)},\ \tau = \frac{2n-(n-2)p}{p(n+2+2\alpha)},$$

and

$$X^{1,\alpha} = \left\{u\in\mathscr{D}^{1,2};\int\int_{R^n\times R^n}\frac{u^q(x)u^q(y)}{|x-y|^{n-\alpha}}dxdy<\infty\right\}.$$

Such an inequality plays a key role in estimating the lower bound of the Coulomb energy (cf. [2]). In addition, this inequality is equivalent partly to the Lieb-Thirring type inequality (cf. [13]). The extremal functions belong to the Coulomb-Sobolev space. In 2010, Ruiz used this space to study a Schrödinger-Poisson-Slater equation (cf. [8], [15]).

We will prove the following results.

THEOREM 1. *The extremal functions in (6) satisfy the elliptic equation in the weak sense*

$$(-\Delta)^{\frac{\alpha}{2}}u+u^{q-1} = u^{p-1}. \tag{10}$$

THEOREM 2. *The extremal functions in (8) satisfy the semilinear equation in the weak sense*

$$\theta(-\Delta)^{\frac{\alpha}{2}}v+(1-\theta)v^{q-1} = \sigma^p\int_{R^n}\frac{w^{p-1}(x)}{|x-y|^{n-\alpha}}dx, \tag{11}$$

where σ *is a positive constant associated with the best constant* C, $\theta = \frac{2n(p-q)}{p[2n-(n-\alpha)q]}$, *and* $w(x) := \int_{R^n}\frac{v(y)}{|x-y|^{n-\alpha}}dy$.

THEOREM 3. *The extremal functions in (9) satisfy the elliptic equation in the weak sense*

$$-\Delta u+\frac{2\sigma}{\theta}q\cdot u^{q-1}V(x) = \sigma^p u^{p-1}, \tag{12}$$

where σ is a positive constant associated with the best constant C, $\theta = \frac{n+\alpha-\frac{2qn}{p}}{(n+\alpha)-q(n-2)}$, and $V(x) := \int_{R^n} \frac{u^q(y)}{|x-y|^{n-\alpha}}dy$.

Next, we consider the simplified forms of (10), (11) and (12), i.e.

$$-\Delta u + u^{q-1} = u^{p-1}; \tag{13}$$

$$-\Delta v + v^{q-1} = \int_{R^n} \frac{w^{p-1}(x)}{|x-y|^{n-\alpha}}dx \tag{14}$$

with $w(x) := \int_{R^n} \frac{v(y)}{|x-y|^{n-\alpha}}dy$; and

$$-\Delta u + u^{q-1}V(x) = u^{p-1} \tag{15}$$

with $V(x) := \int_{R^n} \frac{u^q(y)}{|x-y|^{n-\alpha}}dy$.

And we will prove the following results.

THEOREM 4. *If the elliptic equation (13) has positive classical solutions in $\mathscr{D}^{1,2} \cap L^q$, then one of the following holds:*

(i) $q < p < \frac{2n}{n-2}$;

(ii) $q > p > \frac{2n}{n-2}$;

(iii) $q = p = \frac{2n}{n-2}$.

THEOREM 5. *If the elliptic equation (14) has positive classical solutions in $\mathscr{D}^{1,2} \cap L^q$, then one of the following holds:*

(i) $q < \frac{np}{n+\alpha}$ *and* $p < \frac{2(n+\alpha)}{n-2}$;

(ii) $q > \frac{np}{n+\alpha}$ *and* $p > \frac{2(n+\alpha)}{n-2}$;

(iii) $q = \frac{2n}{n-2}$ *and* $p = \frac{2(n+\alpha)}{n-2}$.

THEOREM 6. *If the elliptic equation (15) has positive classical solutions in $X^{1,\alpha}$, then one of the following holds:*

(i) $q < \frac{p(n+\alpha)}{2n}$ *and* $p < \frac{2n}{n-2}$;

(ii) $q > \frac{p(n+\alpha)}{2n}$ *and* $p > \frac{2n}{n-2}$;

(iii) $q = \frac{n+\alpha}{n-2}$ *and* $p = \frac{2n}{n-2}$.

We use variational calculations to derive the Euler-Lagrange equations satisfied by the extremal functions of the inequalities. This method comes from [17]. And the Pohozaev identities play a key role in proving the non-existence of solutions. We use the method in [9] to derive the Pohozaev identities.

2. Euler-Lagrange equations

In this section, we derive the Euler-Lagrange equations satisfied by the extremal functions of three non-local inequalities.

Proof of Theorem 1.
We set

$$J(u)=\frac{\|(-\Delta)^{\frac{\alpha}{4}}u\|_2^{\theta}\|u\|_q^{1-\theta}}{\|u\|_p}, \tag{16}$$

and

$$\sigma=\inf_{u\in\mathscr{D}^{\alpha,2}\cap L^q,u\neq 0}J(u). \tag{17}$$

Now we consider a minimizing sequence $(u_n)_{n\geqslant 0}$. By the GN inequality, we know that $\sigma>0$. We consider v_n defined by $v_n(x)=\mu_n u_n(\lambda_n x)$ with

$$\lambda_n=\frac{\|u_n\|_q^{\theta_1}}{\|(-\Delta)^{\frac{\alpha}{4}}u_n\|_2^{\theta_2}} \text{ and } \mu_n=\frac{\|u_n\|_q^{\theta_3}}{\|(-\Delta)^{\frac{\alpha}{4}}u_n\|_2^{\theta_4}},$$

where,

$$\theta_1=\theta_2=\frac{2q}{2n-(n-\alpha)q},\ \theta_3=\frac{(n-\alpha)q}{2n-(n-\alpha)q} \text{ and } \theta_4=\frac{2n}{2n-(n-\alpha)q}.$$

Thus,

$$\|v_n\|_q=\|(-\Delta)^{\frac{\alpha}{4}}v_n\|_2=1,$$

and

$$\|v_n\|_p^{-1}=J(v_n)=J(u_n)\to\sigma>0, \text{ as } n\to\infty.$$

By symmetrization, we may assume that v_n is spherically symmetric, and hence there exists a subsequence, which we still denote by $(v_n)_{n\geqslant 0}$, and $v\in\mathscr{D}^{1,2}\cap L^q(R^n)$ such that $v_n\to v$ in $\mathscr{D}^{1,2}\cap L^q(R^n)$ weakly and in $L^p(R^n)$ strongly. Since

$$\|v_n\|_p=\lim_{n\to\infty}\|v_n\|_p=\sigma^{-1}>0,$$

it follows that $v\neq 0$. This implies that

$$J(v)=\sigma \text{ and } \|v\|_q=\|(-\Delta)^{\frac{\alpha}{4}}v\|_2=1. \tag{18}$$

The corresponding functional is

$$E(v(x))=\left(\int_{R^n}|(-\Delta)^{\frac{\alpha}{4}}v(x)|^2dx\right)^{\frac{\theta}{2}}\left(\int_{R^n}|v(x)|^qdx\right)^{\frac{1-\theta}{q}}-\frac{\sigma^p}{p}\left(\int_{R^n}|v(x)|^pdx-\frac{1}{\sigma^p}\right). \tag{19}$$

For all $\varphi \in H^1(R^n)$, we get

$$\frac{d}{dt}E(v(x)+t\varphi(x))\big|_{t=0} = \frac{d}{dt}\Big[\left(\int_{R^n}|(-\Delta)^{\frac{\alpha}{4}}(v(x)+t\varphi(x))|^2dx\right)^{\frac{\theta}{2}} \cdot\left(\int_{R^n}|v(x)+t\varphi(x)|^q dx\right)^{\frac{1-\theta}{q}} -\frac{\sigma^p}{p}\left(\int_{R^n}|v(x)+t\varphi(x)|^p dx-\frac{1}{\sigma^p}\right)\Big]_{t=0} = 0.$$

Taking into account (18), we obtain

$$\theta(-\Delta)^{\frac{\alpha}{2}}v+(1-\theta)v^{q-1} = \sigma^p v^{p-1}.$$

Let now u be defined by $v(x) = au(bx)$ with $a = (\frac{1-\theta}{\sigma^p})^{\frac{1}{p-q}}$ and $b = [\frac{1-\theta}{\theta}(\frac{1-\theta}{\sigma^p})^{\frac{q-2}{p-q}}]^{\frac{1}{\alpha}}$, so u is a solution of

$$(-\Delta)^{\frac{\alpha}{2}}u+u^{q-1} = u^{p-1},$$

and

$$J(u) = J(v) = \sigma.$$

Thus we complete the proof of the Theorem 1.

Proof of Theorem 2.
We set

$$J(u) = \frac{\|(-\Delta)^{\frac{\alpha}{4}}u\|_2^{\theta}\|u\|_q^{1-\theta}}{\|Tu\|_p}, \tag{20}$$

and

$$\sigma = \inf_{u\in\mathscr{D}^{\alpha,2}\cap L^q, u\neq 0} J(u). \tag{21}$$

Now we consider a minimizing sequence $(u_n)_{n\geqslant 0}$. By the GN inequality, we know that $\sigma > 0$. We consider v_n defined by $v_n(x) = \mu_n u_n(\lambda_n x)$ with

$$\lambda_n = \frac{\|u_n\|_q^{\theta_1}}{\|(-\Delta)^{\frac{\alpha}{4}}u_n\|_2^{\theta_2}} \text{ and } \mu_n = \frac{\|u_n\|_q^{\theta_3}}{\|(-\Delta)^{\frac{\alpha}{4}}u_n\|_2^{\theta_4}},$$

where,

$$\theta_1 = \theta_2 = \frac{2q}{2n-(n-\alpha)q},\ \theta_3 = \frac{(n-\alpha)q}{2n-(n-\alpha)q} \text{ and } \theta_4 = \frac{2n}{2n-(n-\alpha)q}.$$

Thus,

$$\|v_n\|_q = \|(-\Delta)^{\frac{\alpha}{4}}v_n\|_2 = 1$$

and

$$\|Tv_n\|_p^{-1} = J(v_n) = J(u_n) \to \sigma > 0, \text{ as } n\to\infty.$$

By symmetrization, we may assume that v_n is spherically symmetric, and so there exists a subsequence, which we still denote by $(v_n)_{n\geqslant 0}$, and $v \in \mathscr{D}^{1,2} \cap L^q(R^n)$ such that $v_n \to v$ in $\mathscr{D}^{1,2} \cap L^q(R^n)$ weakly and in $L^p(R^n)$ strongly. Since

$$\|Tv\|_p = \lim_{n\to\infty} \|Tv_n\|_p = \sigma^{-1} > 0,$$

it follows that $v \neq 0$. This implies that

$$J(v) = \sigma \text{ and } \|v\|_q = \|(-\Delta)^{\frac{\alpha}{4}} v\|_2 = 1. \tag{22}$$

The corresponding functional is

$$E(v(x)) = \left(\int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} v(x)|^2 dx\right)^{\frac{\theta}{2}} \left(\int_{R^n} |v(x)|^q dx\right)^{\frac{1-\theta}{q}} - \frac{\sigma^p}{p}\left(\int_{R^n} |Tv(x)|^p dx - \frac{1}{\sigma^p}\right). \tag{23}$$

For all $\varphi \in H^1(R^n)$, by letting

$$\begin{aligned} \frac{d}{dt} E(v(x)+t\varphi(x))|_{t=0} = \frac{d}{dt}\Bigg[& \left(\int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} (v(x)+t\varphi(x))|^2 dx\right)^{\frac{\theta}{2}} \\ & \cdot \left(\int_{R^n} |v(x)+t\varphi(x)|^q dx\right)^{\frac{1-\theta}{q}} \\ & - \frac{\sigma^p}{p}\left(\int_{R^n} |T(v(x)+t\varphi(x))|^p dx - \frac{1}{\sigma^p}\right)\Bigg]_{t=0} = 0, \end{aligned}$$

we have

$$\begin{aligned} & \theta\left(\int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} v(x)|^2 dx\right)^{\frac{\theta}{2}-1} \left(\int_{R^n} (-\Delta)^{\frac{\alpha}{4}} v(x) (-\Delta)^{\frac{\alpha}{4}} \varphi(x) dx\right) \left(\int_{R^n} |v(x)|^q dx\right)^{\frac{1-\theta}{q}} \\ & +(1-\theta)\left(\int_{R^n} |(-\Delta)^{\frac{\alpha}{4}} v(x)|^2 dx\right)^{\frac{\theta}{2}} \left(\int_{R^n} |v(x)|^q dx\right)^{\frac{1-\theta}{q}-1} \int_{R^n} |v(x)|^{q-2} v\varphi dx \\ & -\sigma^p \int_{R^n} \left(\int_{R^n} \frac{v(y)}{|x-y|^{n-\beta}} dy\right)^{p-1} \left(\int_{R^n} \frac{\varphi(y)}{|x-y|^{n-\beta}} dy\right) dx = 0. \end{aligned}$$

Taking into account (22), we obtain

$$\theta(-\Delta)^{\frac{\alpha}{2}} v + (1-\theta) v^{q-1} = \sigma^p \int_{R^n} \frac{w^{p-1}(x)}{|x-y|^{n-\beta}} dx, \tag{24}$$

here $w(x) := \int_{R^n} \frac{v(y)}{|x-y|^{n-\beta}} dy$. Thus we complete the proof of the Theorem 2.

Proof of Theorem 3.

Similar to the proof of the Theorem 2, we consider v_n defined by $v_n(x) = \mu_n u_n(\lambda_n x)$ with

$$\lambda_n = \frac{\left(\int\int_{R^n \times R^n} \frac{u_n^q(x) u_n^q(y)}{|x-y|^{n-\alpha}} dxdy\right)^{\theta_1}}{\|\nabla u_n\|_2^{\theta_2}},$$

$$\mu_n = \frac{\left(\int\int_{R^n\times R^n} \frac{u_n^q(x)u_n^q(y)}{|x-y|^{n-\alpha}} dxdy\right)^{\theta_3}}{\|\nabla u_n\|_2^{\theta_4}}.$$

Where

$$\theta_1 = \frac{2}{n+2+2\alpha},\ \theta_2 = \frac{4q}{n+2+2\alpha},\ \theta_3 = \frac{n-2}{n+2+2\alpha} \text{ and } \theta_4 = \frac{2q(n-2)+n+2+2\alpha}{n+2+2\alpha}.$$

Thus,

$$\int\int_{R^n\times R^n} \frac{v_n^q(x)v_n^q(y)}{|x-y|^{n-\alpha}} dxdy = \|\nabla v_n\|_2 = 1,$$

and

$$\|v_n\|_p^{-1} = J(v_n) = J(u_n) \to \sigma > 0, \text{ as } n \to \infty.$$

Using the similar argument in the proof of the Theorem 2, we get the Euler-Lagrange equation is

$$-\Delta u + \frac{2\sigma}{\theta} q \cdot u^{q-1} V(x) = \sigma^p u^{p-1}, \tag{25}$$

here

$$V(x) := \int_{R^n} \frac{u^q(y)}{|x-y|^{n-\alpha}} dy.$$

Thus we complete the proof of the Theorem 3.

3. Necessary conditions

In this section, we will prove Theorems 4-6.

Proof of Theorem 4.

Assume $u \in \mathscr{D}^{1,2} \cap L^q$ is a positive classical solution of (13). Then we can find $R_j \to \infty$ such that

$$R_j \int_{\partial B_j} (u^{\frac{2n}{n-2}} + |\nabla u|^2) ds \to 0.$$

By means of the Hölder inequality and $u \in \mathscr{D}^{1,2}$, we get

$$\begin{aligned} &\left|\int_{\partial B_j} u \frac{\partial u}{\partial \nu} ds\right| \\ \leqslant &\left(R_j \int_{\partial B_j} u^{\frac{2n}{n-2}} ds\right)^{\frac{n-2}{2n}} \left(\int_{\partial B_j} |\frac{\partial u}{\partial \nu}|^2 ds\right)^{\frac{1}{2}} |\partial B_j|^{\frac{1}{2}-\frac{n-2}{2n}} R_j^{-\frac{n-2}{2n}-\frac{1}{2}} \to 0, \end{aligned} \tag{26}$$

when $R = R_j \to \infty$. Multiplying (13) by u and integrating on B, we have

$$\int_B u^p dx = \int_B |\nabla u|^2 dx + \int_B u^q dx - \int_{\partial B} u \frac{\partial u}{\partial \nu} ds. \tag{27}$$

Letting $R \to \infty$ and using the result above, we have

$$\int_{R^n} |\nabla u|^2 dx + \int_{R^n} u^q dx = \int_{R^n} u^p dx. \tag{28}$$

Hence $u \in L^p(R^n)$. The functional corresponding to the equation (13) is

$$E(u(x)) = \frac{1}{2}\int_{R^n} |\nabla u(x)|^2 dx + \frac{1}{q}\int_{R^n} u^q(x) dx - \frac{1}{p}\int_{R^n} u^p(x) dx. \tag{29}$$

Obviously, the definition of $E(u(x))$ makes sense. Clearly,

$$\begin{aligned} E(u(\lambda x)) &= \frac{1}{2}\int_{R^n} |\nabla u(\lambda x)|^2 dx + \frac{1}{q}\int_{R^n} u^q(\lambda x) dx - \frac{1}{p}\int_{R^n} u^p(\lambda x) dx \\ &= \frac{\lambda^{2-n}}{2}\int_{R^n} |\nabla u|^2 dx + \frac{\lambda^{-n}}{q}\int_{R^n} u^q dx - \frac{\lambda^{-n}}{p}\int_{R^n} u^p dx. \end{aligned}$$

Since the solution of (13) is a critical point of $E(u)$, $\frac{d}{d\lambda}E(u(\lambda x))|_{\lambda=1} = 0$. Therefore, we obtain the Pohozaev identity

$$\frac{2-n}{2}\int_{R^n} |\nabla u|^2 dx + \frac{n}{p}\int_{R^n} u^p dx = \frac{n}{q}\int_{R^n} u^q dx. \tag{30}$$

Combining (28) and (30), it follows that

$$\begin{cases} \left(\frac{n}{p} - \frac{n-2}{2}\right)\int_{R^n} |\nabla u|^2 dx = \left(\frac{n}{q} - \frac{n}{p}\right)\int_{R^n} u^q dx, \\ \left(\frac{n}{q} - \frac{n-2}{2}\right)\int_{R^n} |\nabla u|^2 dx = \left(\frac{n}{q} - \frac{n}{p}\right)\int_{R^n} u^p dx. \end{cases}$$

Then one of the following consequences holds

(i)

$$\begin{cases} \frac{n}{p} - \frac{n-2}{2} > 0, \\ \frac{n}{q} - \frac{n}{p} > 0, \\ \frac{n}{q} - \frac{n-2}{2} > 0, \end{cases}$$

which implies $q < p < \frac{2n}{n-2}$.

(ii)

$$\begin{cases} \frac{n}{p} - \frac{n-2}{2} < 0, \\ \frac{n}{q} - \frac{n}{p} < 0, \\ \frac{n}{q} - \frac{n-2}{2} < 0, \end{cases}$$

which implies $q > p > \frac{2n}{n-2}$.

(iii)

$$\begin{cases} \frac{n}{p} - \frac{n-2}{2} = 0, \\ \frac{n}{q} - \frac{n}{p} = 0, \\ \frac{n}{q} - \frac{n-2}{2} = 0, \end{cases}$$

which implies $q = p = \frac{2n}{n-2}$. Theorem 4 is proved.

Proof of Theorem 5.

Assume $u \in \mathscr{D}^{1,2} \cap L^q$ is a positive classical solution of (14). Similar to the argument in the proof of Theorem 4, we have the same conclusion with (26). Then multiplying (14) by v and integrating on B and letting $R \to \infty$, we have

$$\int_{R^n} |\nabla v|^2 dx + \int_{R^n} v^q dx = \int_{R^n} (Tv)^p dx. \tag{31}$$

The functional corresponding to the equation (14) is

$$E(v(x)) = \frac{1}{2} \int_{R^n} |\nabla v(x)|^2 dx + \frac{1}{q} \int_{R^n} v^q(x) dx - \frac{1}{p} \int_{R^n} \left(\int_{R^n} \frac{v(y)}{|x-y|^{n-\alpha}} dy \right)^p dx. \tag{32}$$

The definition of $E(v(x))$ makes sense, too. Clearly,

$$\begin{aligned} E(v(\lambda x)) &= \frac{1}{2} \int_{R^n} |\nabla v(\lambda x)|^2 dx + \frac{1}{q} \int_{R^n} v^q(\lambda x) dx - \frac{1}{p} \int_{R^n} \left(\int_{R^n} \frac{v(\lambda y)}{|x-y|^{n-\alpha}} dy \right)^p dx \\ &= \frac{\lambda^{2-n}}{2} \int_{R^n} |\nabla v(x)|^2 dx + \frac{\lambda^{-n}}{q} \int_{R^n} v^q(x) dx - \frac{\lambda^{-(n+\alpha)}}{p} \int_{R^n} (Tv(x))^p dx. \end{aligned}$$

Since the solution of (14) is a critical point of $E(v)$, $\frac{d}{d\lambda} E(v(\lambda x))|_{\lambda=1} = 0$. Therefore, we obtain the Pohozaev identity

$$\frac{2-n}{2} \int_{R^n} |\nabla v|^2 dx - \frac{n}{q} \int_{R^n} v^q dx = -\frac{n+\alpha}{p} \int_{R^n} (Tv)^p dx. \tag{33}$$

Inserting this into (34) we get

$$\begin{cases} \left(\frac{n}{q} - \frac{n-2}{2} \right) \int_{R^n} |\nabla v|^2 dx = \left(\frac{n}{q} - \frac{n+\alpha}{p} \right) \int_{R^n} (Tv)^p dx, \\ \left(\frac{n+\alpha}{p} - \frac{n-2}{2} \right) \int_{R^n} |\nabla v|^2 dx = \left(\frac{n}{q} - \frac{n+\alpha}{p} \right) \int_{R^n} (Tv)^p dx. \end{cases}$$

Then one of the following consequences holds

(i)

$$\begin{cases} \frac{n}{q} - \frac{n-2}{2} > 0, \\ \frac{n+\alpha}{p} - \frac{n-2}{2} > 0, \\ \frac{n}{q} - \frac{n+\alpha}{p} > 0, \end{cases}$$

which implies $q < \frac{np}{n+\alpha}$ and $p < \frac{2(n+\alpha)}{n-2}$.

(ii)

$$\begin{cases} \frac{n}{q} - \frac{n-2}{2} < 0, \\ \frac{n+\alpha}{p} - \frac{n-2}{2} < 0, \\ \frac{n}{q} - \frac{n+\alpha}{p} < 0, \end{cases}$$

which implies $q > \frac{np}{n+\alpha}$ and $p > \frac{2(n+\alpha)}{n-2}$.

(iii)

$$\begin{cases} \frac{n}{q} - \frac{n-2}{2} = 0, \\ \frac{n+\alpha}{p} - \frac{n-2}{2} = 0, \\ \frac{n}{q} - \frac{n+\alpha}{p} = 0, \end{cases}$$

which implies $q = \frac{2n}{n-2}$ and $p = \frac{2(n+\alpha)}{n-2}$. Theorem 5 is proved.

Proof of Theorem 6.

Assume $u \in X^{1,\alpha}$ is a positive classical solution of (15). By the same argument in Theorem 5, we have

$$\int_{R^n} |\nabla u|^2 dx + \int_{R^n} u^q V dx = \int_{R^n} u^p dx. \tag{34}$$

Then the functional corresponding to the equation (15) is

$$E(u(x)) = \frac{1}{2}\int_{R^n} |\nabla u(x)|^2 dx + \frac{1}{2q}\int_{R^n} u^q(x)V(x)dx - \frac{1}{p}\int_{R^n} u^p(x)dx. \tag{35}$$

The definition of $E(u(x))$ makes sense, too. Clearly,

$$\begin{aligned} E(u(\lambda x)) &= \frac{1}{2}\int_{R^n} |\nabla u(\lambda x)|^2 dx + \frac{1}{2q}\int_{R^n} u^q(\lambda x)\int_{R^n} \frac{u^q(\lambda y)}{|x-y|^{n-\alpha}} dydx - \frac{1}{p}\int_{R^n} u^p(\lambda x)dx \\ &= \frac{\lambda^{2-n}}{2}\int_{R^n} |\nabla u(x)|^2 dx + \frac{\lambda^{-(n+\alpha)}}{2q}\int_{R^n} u^q(x)V(x)dx - \frac{\lambda^{-n}}{p}\int_{R^n} u^p(x)dx. \end{aligned}$$

Since the solution of (15) is a critical point of $E(u)$, $\frac{d}{d\lambda}E(u(\lambda x))|_{\lambda=1}=0$. Therefore, we obtain the Pohozaev identity

$$\frac{2-n}{2}\int_{R^n}|\nabla u|^2dx-\frac{n+\alpha}{2q}\int_{R^n}u^qV\,dx=-\frac{n}{p}\int_{R^n}u^p\,dx. \tag{36}$$

Combining this result with (34) yields

$$\begin{cases}\left(\frac{n}{p}-\frac{n-2}{2}\right)\int_{R^n}|\nabla u|^2dx=\left(\frac{n+\alpha}{2q}-\frac{n}{p}\right)\int_{R^n}u^qV\,dx,\\ \left(\frac{n+\alpha}{2q}-\frac{n-2}{2}\right)\int_{R^n}|\nabla u|^2dx=\left(\frac{n+\alpha}{2q}-\frac{n}{p}\right)\int_{R^n}u^p\,dx.\end{cases}$$

Then one of the following consequences holds

(i)

$$\begin{cases}\frac{n}{p}-\frac{n-2}{2}>0,\\ \frac{n+\alpha}{2q}-\frac{n}{p}>0,\\ \frac{n+\alpha}{2q}-\frac{n-2}{2}>0,\end{cases}$$

which implies $q<\frac{p(n+\alpha)}{2n}$ and $p<\frac{2n}{n-2}$.

(ii)

$$\begin{cases}\frac{n}{p}-\frac{n-2}{2}<0,\\ \frac{n+\alpha}{2q}-\frac{n}{p}<0,\\ \frac{n+\alpha}{2q}-\frac{n-2}{2}<0,\end{cases}$$

which implies $q>\frac{p(n+\alpha)}{2n}$ and $p>\frac{2n}{n-2}$.

(iii)

$$\begin{cases}\frac{n}{p}-\frac{n-2}{2}=0,\\ \frac{n+\alpha}{2q}-\frac{n}{p}=0,\\ \frac{n+\alpha}{2q}-\frac{n-2}{2}=0,\end{cases}$$

which implies $q=\frac{n+\alpha}{n-2}$ and $p=\frac{2n}{n-2}$. Theorem 6 is proved.

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