

## WEAK TYPE ENDPOINT ESTIMATES FOR THE COMMUTATORS OF ROUGH SINGULAR INTEGRAL OPERATORS

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*Abstract.* Let  $\Omega$  be homogeneous of degree zero and have mean value zero on the unit sphere  $S^{n-1}$ ,  $T_\Omega$  be the convolution singular integral operator with kernel  $\frac{\Omega(x)}{|x|^n}$ . For  $b \in \text{BMO}(\mathbb{R}^n)$ , let  $T_{\Omega,b}$  be the commutator of  $T_\Omega$ . In this paper, by establishing suitable sparse dominations, the authors establish some weak type endpoint estimates of  $L \log L$  type for  $T_{\Omega,b}$  when  $\Omega \in L^q(S^{n-1})$  for some  $q \in (1, \infty]$ .

### 1. Introduction

We will work on  $\mathbb{R}^n$ ,  $n \geq 2$ . Let  $\Omega$  be homogeneous of degree zero, integrable and have mean value zero on the unit sphere  $S^{n-1}$ . Define the singular integral operator  $T_\Omega$  by

$$T_\Omega f(x) = \text{p. v.} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y) dy, \quad (1.1)$$

where and in the following,  $y' = y/|y|$  for  $y \in \mathbb{R}^n$ . This operator was introduced by Calderón and Zygmund [2], and then studied by many authors in the last sixty years. Calderón and Zygmund [3] proved that if  $\Omega \in L \log L(S^{n-1})$ , then  $T_\Omega$  is bounded on  $L^p(\mathbb{R}^n)$  for  $p \in (1, \infty)$ . Ricci and Weiss [23] improved the result of Calderón-Zygmund, and showed that  $\Omega \in H^1(S^{n-1})$  guarantees the  $L^p(\mathbb{R}^n)$  boundedness on  $L^p(\mathbb{R}^n)$  for  $p \in (1, \infty)$ . Seeger [25] showed that  $\Omega \in L \log L(S^{n-1})$  is a sufficient condition such that  $T_\Omega$  is bounded from  $L^1(\mathbb{R}^n)$  to  $L^{1,\infty}(\mathbb{R}^n)$ . For other works about the  $L^p(\mathbb{R}^n)$  boundedness and weak type endpoint estimates for  $T_\Omega$ , we refer the papers see [4, 7, 8, 9, 12, 23, 27] and the references therein.

Now let  $T$  be a linear operator from  $\mathcal{S}'(\mathbb{R}^n)$  to  $\mathcal{S}'(\mathbb{R}^n)$  and  $b \in \text{BMO}(\mathbb{R}^n)$ . The commutator of  $T$  with symbol  $b$ , is defined by

$$T_b f(x) = b(x)Tf(x) - T(bf)(x).$$

A celebrated result of Coifman, Rochberg and Weiss [6] states that if  $T$  is a Calderón-Zygmund operator, then  $T_b$  is bounded on  $L^p(\mathbb{R}^n)$  for every  $p \in (1, \infty)$  and also a converse result in terms of the Riesz transforms. Pérez [21] considered the weak type endpoint estimate for the commutator of Calderón-Zygmund operator, and proved the following result.

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**THEOREM 1.1.** *Let  $T$  be a Calderón-Zygmund operator and  $b \in \text{BMO}(\mathbb{R}^n)$ . Then for any  $\lambda > 0$ ,*

$$|\{x \in \mathbb{R}^n : |T_b f(x)| > \lambda\}| \lesssim_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log \left( e + \frac{|f(x)|}{\lambda} \right) dx.$$

By Theorem 1.1, we know that if  $\Omega \in \text{Lip}_\alpha(S^{n-1})$  with  $\alpha \in (0, 1]$ , then for  $b \in \text{BMO}(\mathbb{R}^n)$ ,  $T_{\Omega,b}$ , the commutator of  $T_\Omega$ , satisfies that,

$$|\{x \in \mathbb{R}^n : |T_{\Omega,b} f(x)| > \lambda\}| \lesssim_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log \left( e + \frac{|f(x)|}{\lambda} \right) dx. \tag{1.2}$$

Let  $p \in [1, \infty)$  and  $w$  be a nonnegative, locally integrable function on  $\mathbb{R}^n$ . We say that  $w \in A_p(\mathbb{R}^n)$  if the  $A_p$  constant  $[w]_{A_p}$  is finite, with

$$[w]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w^{1-p'}(x) dx \right)^{p-1}, \quad p \in (1, \infty),$$

the supremum is taken over all cubes in  $\mathbb{R}^n$ ,  $p' = p/(p-1)$  and

$$[w]_{A_1} := \sup_{x \in \mathbb{R}^n} \frac{Mw(x)}{w(x)},$$

see [11] for the properties of  $A_p(\mathbb{R}^n)$ . For a weight  $w \in A_\infty(\mathbb{R}^n) = \cup_{p \geq 1} A_p(\mathbb{R}^n)$ , define  $[w]_{A_\infty}$ , the  $A_\infty$  constant of  $w$ , by

$$[w]_{A_\infty} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{w(Q)} \int_Q M(w\chi_Q)(x) dx,$$

see [28]. By the result of Duandikoetxea and Rubio de Francia [8], and the result in [7], we know that if  $\Omega \in L^q(S^{n-1})$  for some  $q \in (1, \infty]$ , then for  $p \in (q', \infty)$  and  $w \in A_{p/q'}(\mathbb{R}^n)$

$$\|T_\Omega f\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n,p,w} \|f\|_{L^p(\mathbb{R}^n, w)}.$$

This, together with Theorem 2.13 in [1], tells us that if  $\Omega \in L^q(S^{n-1})$  for  $q \in (1, \infty]$ , then for  $b \in \text{BMO}(\mathbb{R}^n)$ ,

$$\|T_{\Omega,b} f\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n,p,w} \|b\|_{\text{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n, w)}, \quad p \in (q', \infty), \quad w \in A_{p/q'}(\mathbb{R}^n).$$

Hu [13] proved that  $\Omega \in L(\log L)^2(S^{n-1})$  is a sufficient condition such that  $T_{\Omega,b}$  is bounded on  $L^p(\mathbb{R}^n)$  for all  $p \in (1, \infty)$  and  $b \in \text{BMO}(\mathbb{R}^n)$ . However, as far as we know, there is no result concerning the weak type endpoint estimate for  $T_{\Omega,b}$  when  $\Omega$  only satisfies size condition. In this paper, we consider this question. Our first result can be stated as follows.

**THEOREM 1.2.** *Let  $\Omega$  be homogeneous of degree zero and have mean value zero on  $S^{n-1}$ ,  $b \in \text{BMO}(\mathbb{R}^n)$ . Suppose that  $\Omega \in L^q(S^{n-1})$  for some  $q \in (1, \infty)$ , then for any  $\lambda > 0$  and weight  $w$  such that  $w^{q'} \in A_1(\mathbb{R}^n)$ ,*

$$w(\{x \in \mathbb{R}^n : |T_{\Omega,b} f(x)| > \lambda\}) \lesssim_{n,w} \int_{\mathbb{R}^n} \frac{D|f(x)|}{\lambda} \log \left( e + \frac{D|f(x)|}{\lambda} \right) w(x) dx,$$

with  $D = \|\Omega\|_{L^q(S^{n-1})} \|b\|_{\text{BMO}(\mathbb{R}^n)}$ .

In the last several years, considerable attention has been paid to the quantitative weighted bounds for  $T_\Omega$  when  $\Omega \in L^\infty(S^{n-1})$ . The first result in this area was established by Hytönen, Roncal and Tapiola [16], who proved that for  $p \in (1, \infty)$  and  $w \in A_p(\mathbb{R}^n)$ ,

$$\|T_\Omega f\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n,p} \|\Omega\|_{L^\infty(S^{n-1})} [w]^{2\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(\mathbb{R}^n, w)}. \tag{1.3}$$

Li, Pérez, Rivera-Rios and Roncal [19] improved (1.3) and showed that for  $p \in (1, \infty)$  and  $w \in A_p(\mathbb{R}^n)$

$$\|T_\Omega f\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n,p} [w]_{A_p}^{\frac{1}{p}} ([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}) \min\{[\sigma]_{A_\infty}, [w]_{A_\infty}\} \|f\|_{L^p(\mathbb{R}^n, w)}, \tag{1.4}$$

where and in the following, for  $w \in A_p(\mathbb{R}^n)$ ,  $\sigma = w^{1-p'}$ . The estimate (1.4), via the method in [5], implies the following quantitative weighted estimate

$$\begin{aligned} \|T_{\Omega, b} f\|_{L^p(\mathbb{R}^n, w)} &\lesssim_{n,p} [w]_{A_p}^{\frac{1}{p}} ([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}) \min\{[\sigma]_{A_\infty}, [w]_{A_\infty}\} \\ &\quad \times ([w]_{A_\infty} + [\sigma]_{A_\infty}) \|f\|_{L^p(\mathbb{R}^n, w)}. \end{aligned}$$

Rivera-Ríos [24] established the sparse domination for  $T_{\Omega, b}$  when  $\Omega \in L^\infty(S^{n-1})$ , and proved that for  $p \in (1, \infty)$  and  $w \in A_1(\mathbb{R}^n)$ ,

$$\|T_{\Omega, b} f\|_{L^p(\mathbb{R}^n, w)} \lesssim_{n,p} \|\Omega\|_{L^\infty(S^{n-1})} p'^3 p^2 [w]_{A_1}^{\frac{1}{p}} [w]_{A_\infty}^{1+\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}^n, w)}.$$

Our second result is the following quantitative weighted weak type estimate for  $T_{\Omega, b}$ .

**THEOREM 1.3.** *Let  $\Omega$  be homogeneous of degree zero and have mean value zero on  $S^{n-1}$ ,  $b \in \text{BMO}(\mathbb{R}^n)$ . Suppose that  $\Omega \in L^\infty(S^{n-1})$  and  $w \in A_1(\mathbb{R}^n)$ , then for any  $\lambda > 0$ ,*

$$\begin{aligned} &w(\{x \in \mathbb{R}^n : |T_{\Omega, b} f(x)| > \lambda\}) \\ &\lesssim_n [w]_{A_1} [w]_{A_\infty}^2 \log(e + [w]_{A_\infty}) \int_{\mathbb{R}^n} \frac{D_\infty |f(x)|}{\lambda} \log\left(e + \frac{D_\infty |f(x)|}{\lambda}\right) w(x) dx, \end{aligned}$$

with  $D_\infty = \|\Omega\|_{L^\infty(S^{n-1})} \|b\|_{\text{BMO}(\mathbb{R}^n)}$ .

**REMARK 1.4.** Proofs of Theorem 1.2 and Theorem 1.3 depend essentially on the weak type endpoint estimates for the maximal operator defined by

$$\mathcal{M}_{r, T_\Omega} f(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |T_\Omega(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi)|^r d\xi \right)^{1/r}, \tag{1.5}$$

where the supremum is taken over all cubes  $Q \subset \mathbb{R}^n$  containing  $x$ . This operator was introduced by Lerner [18], who proved that for any  $r \in (1, \infty)$ ,

$$\|\mathcal{M}_{r, T_\Omega} f\|_{L^{1, \infty}(\mathbb{R}^n)} \lesssim r \|\Omega\|_{L^\infty(S^{n-1})} \|f\|_{L^1(\mathbb{R}^n)}, \tag{1.6}$$

see [18, Lemma 3.3]. Although we can show that

$$\|M_{r, T_\Omega} f\|_{L^{1, \infty}(\mathbb{R}^n)} \lesssim r \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^1(\mathbb{R}^n)},$$

we do not know if there exists a  $\alpha \in (0, \infty)$  such that the estimate

$$\|M_{r, T_\Omega} f\|_{L^{1, \infty}(\mathbb{R}^n)} \lesssim r^\alpha \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^1(\mathbb{R}^n)}$$

holds true when  $\Omega \in L^q(S^{n-1})$  for some  $q \in (1, \infty)$ . This is the main difficult which prevent us obtaining a desired quantitative weighted weak type endpoint estimates for  $T_{\Omega, b}$  when  $\Omega \in L^q(S^{n-1})$  for  $q \in (1, \infty)$ .

In what follows,  $C$  always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol  $A \lesssim B$  to denote that there exists a positive constant  $C$  such that  $A \leq CB$ . Specially, we use  $A \lesssim_{n, p} B$  to denote that there exists a positive constant  $C$  depending only on  $n, p$  such that  $A \leq CB$ . Constant with subscript such as  $c_1$ , does not change in different occurrences. For any set  $E \subset \mathbb{R}^n$ ,  $\chi_E$  denotes its characteristic function. For a cube  $Q \subset \mathbb{R}^n$  and  $\lambda \in (0, \infty)$ , we use  $\lambda Q$  to denote the cube with the same center as  $Q$  and whose side length is  $\lambda$  times that of  $Q$ . For a fixed cube  $Q$ , denote by  $\mathcal{D}(Q)$  the set of dyadic cubes with respect to  $Q$ , that is, the cubes from  $\mathcal{D}(Q)$  are formed by repeating subdivision of  $Q$  and each of descendants into  $2^n$  congruent subcubes. For a function  $f$  and cube  $Q$ ,  $\langle f \rangle_Q$  denotes the mean value of  $f$  on  $Q$ , and  $\langle |f| \rangle_{Q, r} = (\langle |f|^r \rangle_Q)^{1/r}$  for  $r \in (0, \infty)$ .

For a cube  $Q$ ,  $\beta \in (0, \infty)$  and suitable function  $f$ , define  $\|f\|_{L(\log L)^\beta, Q}$  by

$$\|f\|_{L(\log L)^\beta, Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \frac{|f(y)|}{\lambda} \log^\beta \left( e + \frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}.$$

Also, we define  $\|h\|_{\exp L, Q}$  as

$$\|h\|_{\exp L, Q} = \inf \left\{ t > 0 : \frac{1}{|Q|} \int_Q \exp \left( \frac{|h(y)|}{t} \right) dy \leq 2 \right\}.$$

By the generalization of Hölder’s inequality (see [22, p. 64]), we know that for any cube  $Q$  and suitable functions  $f$  and  $h$ ,

$$\int_Q |f(x)h(x)| dx \lesssim \|f\|_{L \log L, Q} \|h\|_{\exp L, Q} |Q|. \tag{1.7}$$

### 2. Proof of theorems

Given an operator  $T$ , define the maximal operator  $M_{\lambda, T}$  by

$$M_{\lambda, T} f(x) = \sup_{Q \ni x} \left( T(f \chi_{\mathbb{R}^n \setminus 3Q}) \chi_Q \right)^* (\lambda |Q|), \quad (0 < \lambda < 1),$$

where the supremum is taken over all cubes  $Q \subset \mathbb{R}^n$  containing  $x$ , and  $h^*$  denotes the non-increasing rearrangement of  $h$ . This operator was introduced by Lerner [18] and is useful in the study of weighted bounds for rough operators, see [18, 24].

LEMMA 2.1. Let  $\Omega$  be homogeneous of degree zero, have mean value zero and  $\Omega \in L^\infty(S^{n-1})$ . Then for any  $\lambda \in (0, 1)$ ,

$$\|M_{\lambda, T_\Omega} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim n \|\Omega\|_{L^\infty(S^{n-1})} \left(1 + \log\left(\frac{1}{\lambda}\right)\right) \|f\|_{L^1(\mathbb{R}^n)}.$$

Lemma 2.1 is Theorem 1.1 in [18].

For a function  $\Omega$  on  $S^{n-1}$ , define  $\|\Omega\|_{L \log L(S^{n-1})}^*$  by

$$\|\Omega\|_{L \log L(S^{n-1})}^* = \inf \left\{ \lambda > 0 : \int_{S^{n-1}} \frac{|\Omega(\theta)|}{\lambda} \log \left( e + \frac{|\Omega(\theta)|}{\lambda} \right) d\theta \leq 1 \right\}.$$

LEMMA 2.2. Let  $\Omega$  be homogeneous of degree zero, have mean value zero and  $\|\Omega\|_{L \log L(S^{n-1})}^* < \infty$ , then

$$\|T_\Omega f\|_{L^{1,\infty}(S^{n-1})} \lesssim \|\Omega\|_{L \log L(S^{n-1})}^* \|f\|_{L^1(\mathbb{R}^n)}.$$

*Proof.* This lemma is essentially a corollary of estimate (3.1) in [25]. At first, we claim that

$$\int_{S^{n-1}} |\Omega(\theta)| \log \left( e + \frac{|\Omega(\theta)|}{\|\Omega\|_{L^1(S^{n-1})}} \right) d\theta \lesssim \|\Omega\|_{L \log L(S^{n-1})}^*. \tag{2.1}$$

In fact, by homogeneity, it suffices to prove (2.1) for the case  $\|\Omega\|_{L^1(S^{n-1})} = 1$ . Let

$$\lambda_0 = \int_{S^{n-1}} |\Omega(\theta)| \log(e + |\Omega(\theta)|) d\theta.$$

We consider the following two cases.

Case I.  $\lambda_0 > e^{10}$ . Let  $S_0 = \{\theta \in S^{n-1} : |\Omega(\theta)| \leq 2\}$ , and

$$S_k = \left\{ \theta \in S^{n-1} : 2^k < |\Omega(\theta)| \leq 2^{k+1} \right\}, k \in \mathbb{N}.$$

Set  $k_0 \in \mathbb{N}$  such that  $2^{k_0-1} < \lambda_0 \leq 2^{k_0}$ . Then  $k_0 \leq \lambda_0/8$

$$\begin{aligned} \int_{S^{n-1}} \frac{|\Omega(\theta)|}{\lambda_0} \log \left( e + \frac{|\Omega(\theta)|}{\lambda_0} \right) d\theta &> \lambda_0^{-1} \sum_{k=k_0+1}^\infty |S_k| 2^k (k - k_0) + \lambda_0^{-1} \sum_{k \leq k_0} |S_k| 2^k \\ &> \lambda_0^{-1} \left( \sum_{k=1}^\infty 2^k k |S_k| + |S_0| \right) \\ &\quad - \lambda_0^{-1} \left( k_0 \sum_{k \geq k_0+1} 2^k |S_k| + \sum_{1 \leq k \leq k_0} k 2^k |S_k| \right). \end{aligned}$$

Obviously,

$$\sum_{k=1}^\infty 2^k k |S_k| + |S_0| \geq \frac{1}{4} \int_{S^{n-1}} |\Omega(\theta)| \log(e + |\Omega(\theta)|) d\theta = \frac{\lambda_0}{4},$$

and

$$k_0 \sum_{k \geq k_0+1} 2^k |S_k| + \sum_{1 \leq k \leq k_0} k 2^k |S_k| \leq k_0 \sum_{k \geq 1} 2^k |S_k| \leq k_0 \|\Omega\|_{L^1(S^{n-1})}.$$

Recall that  $\|\Omega\|_{L^1(S^{n-1})} = 1$ . It then follows that

$$\int_{S^{n-1}} \frac{|\Omega(\theta)|}{\lambda_0} \log \left( e + \frac{|\Omega(\theta)|}{\lambda_0} \right) d\theta > \frac{1}{8}.$$

This in turn leads to that

$$\|\Omega\|_{L \log L(S^{n-1})}^* > \lambda_0/8.$$

*Case II.*  $\lambda_0 \leq e^{10}$ . Let  $\lambda > 0$  satisfies that

$$\int_{S^{n-1}} \frac{|\Omega(\theta)|}{\lambda} \log \left( e + \frac{|\Omega(\theta)|}{\lambda} \right) d\theta \leq 1. \tag{2.2}$$

If  $10e^{10}\lambda < \lambda_0$ , we then have that

$$\int_{S^{n-1}} \frac{|\Omega(\theta)|}{\lambda_0} \log \left( e + \frac{|\Omega(\theta)|}{\lambda_0} \right) d\theta \leq \int_Q \frac{|\Omega(\theta)|}{10e^{10}\lambda} \log \left( e + \frac{|\Omega(\theta)|}{10e^{10}\lambda} \right) d\theta \leq (10e^{10})^{-1}.$$

On the other hand, a trivial computation gives us that

$$\begin{aligned} \int_{S^{n-1}} \frac{|\Omega(\theta)|}{\lambda_0} \log \left( e + \frac{|\Omega(\theta)|}{\lambda_0} \right) d\theta &> \int_{S^{n-1}} \frac{|\Omega(\theta)|}{e^{10}} \log \left( e + \frac{|\Omega(\theta)|}{e^{10}} \right) d\theta \\ &> \int_{S^{n-1}} |\Omega(\theta)| \log(e + |\Omega(\theta)|) d\theta (10e^{10})^{-1} \\ &> (10e^{10})^{-1}, \end{aligned}$$

where the last inequality follows from the fact that  $\lambda_0 \geq \|\Omega\|_{L^1(S^{n-1})} = 1$  (recall that  $\|\Omega\|_{L^1(S^{n-1})} = 1$ ). This is a contradiction. Thus, the positive numbers  $\lambda$  in (2.2) satisfy  $\lambda \geq (10e^{10})^{-1}\lambda_0$ . Inequality (2.1) holds true in this case.

We now conclude the proof of Lemma 2.2. By the result of Seeger (see inequality (3.1) in [25]), we know that if  $\Omega \in L \log L(S^{n-1})$ , then

$$\begin{aligned} \|T_\Omega f\|_{L^{1,\infty}(\mathbb{R}^n)} &\lesssim n \left[ \|T_\Omega\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} + \|\Omega\|_{L^1(S^{n-1})} \right. \\ &\quad \left. + \int_{S^{n-1}} |\Omega(\theta)| \left( 1 + \log^+ \left( |\Omega(\theta)| / \|\Omega\|_{L^1(S^{n-1})} \right) \right) d\theta \right] \|f\|_{L^1(\mathbb{R}^n)}, \end{aligned}$$

where  $\log^+ s = \log s$  if  $s > 1$  and  $\log^+ s = 0$  if  $s \in (0, 1]$ . Thus by (2.1),

$$\|T_\Omega f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim n \left[ \|T_\Omega\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} + \|\Omega\|_{L^1(S^{n-1})} + \|\Omega\|_{L \log L(S^{n-1})}^* \right] \|f\|_{L^1(\mathbb{R}^n)}.$$

On the other hand, we know that

$$\|T_{\Omega}f\|_{L^2(\mathbb{R}^n)} \lesssim \left[1 + \|\Omega\|_{L\log L(S^{n-1})}\right] \|f\|_{L^2(\mathbb{R}^n)},$$

with

$$\|\Omega\|_{L\log L(S^{n-1})} = \int_{S^{n-1}} |\Omega(\theta)|(1 + \log^+ |\Omega(\theta)|) d\theta.$$

see [10, Theorem 4.2.10]. The last two inequality, along with homogeneity, yields

$$\|T_{\Omega}f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_n \|\Omega\|_{L\log L(S^{n-1})}^* \|f\|_{L^1(\mathbb{R}^n)},$$

and completes the proof of Lemma 2.2.  $\square$

LEMMA 2.3. *Let  $\Omega$  be homogeneous of degree zero, have mean value zero and  $\Omega \in L^q(S^{n-1})$  for some  $q \in (1, \infty)$ . Then for any  $\lambda \in (0, 1)$  and  $\varepsilon \in (0, \min\{1, q - 1\})$ ,*

$$\|M_{\lambda, T_{\Omega}}f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_{q, \varepsilon} \|\Omega\|_{L^q(S^{n-1})} \left(\frac{1}{\lambda}\right)^{\frac{1+2\varepsilon}{q}} \|f\|_{L^1(\mathbb{R}^n)}.$$

*Proof.* For  $\lambda \in (0, 1)$ , let  $M_{0, \lambda}$  be the operator

$$M_{0, \lambda}h(x) = \sup_{Q \ni x} (h\chi_Q)^*(\lambda|Q|),$$

see [17, 26]. It is well known that for  $\alpha > 0$ ,

$$|\{x \in \mathbb{R}^n : M_{0, \lambda}f(x) > \alpha\}| \lesssim \lambda^{-1} |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}|.$$

Let  $S$  be a linear operator which is bounded from  $L^1(\mathbb{R}^n)$  to  $L^{1,\infty}(\mathbb{R}^n)$  with bound 1. We claim that the operator  $S_{\lambda}^*$  defined by

$$S_{\lambda}^*f(x) = \sup_{Q \ni x} (S(f\chi_Q))^*(\lambda|Q|)$$

is bounded from  $L^1(\mathbb{R}^n)$  to  $L^{1,\infty}(\mathbb{R}^n)$  with bound  $C_n\lambda^{-1}$ . To prove this, let

$$E_{\alpha} = \{x \in \mathbb{R}^n : S_{\lambda}^*f(x) > \alpha\}.$$

For each  $x \in E_{\alpha}$ , we can choose a cube  $Q$  such that  $Q \ni x$  and

$$|\{y \in Q : |S(f\chi_Q)(y)| > \alpha\}| > \lambda|Q|.$$

This, via the weak type (1, 1) boundedness of  $S$ , tells us that

$$|Q| \leq \frac{1}{\alpha\lambda} \int_Q |f(y)| dy,$$

and so  $Mf(x) \geq \alpha\lambda$ . Therefore,

$$|E_{\alpha}| \leq |\{x \in \mathbb{R}^n : Mf(x) > \alpha\lambda\}| \lesssim \frac{1}{\lambda\alpha} \|f\|_{L^1(\mathbb{R}^n)}.$$

This verifies our claim.

We now conclude the proof of Lemma 2.3. Using the estimate  $\log t \leq t^\varepsilon/\varepsilon$  when  $t > 1$  and  $\varepsilon > 0$ , we can verify by homogeneity that

$$\|\Omega\|_{L^{\log L}(S^{n-1})}^* \lesssim_\varepsilon \|\Omega\|_{L^{1+\varepsilon}(S^{n-1})}.$$

This, along with Lemma 2.2, tells us that for  $\varepsilon > 0$ ,

$$\|T_\Omega f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_{n,\varepsilon} \|\Omega\|_{L^{1+\varepsilon}(S^{n-1})} \|f\|_{L^1(\mathbb{R}^n)}.$$

Observe that

$$M_{\lambda, T_\Omega} f(x) \leq M_{0, \frac{\lambda}{2}} T_\Omega f(x) + \sup_{Q \ni x} (T_\Omega(f\chi_{3Q})\chi_Q)^* \left( \frac{\lambda}{2} |Q| \right),$$

and

$$\sup_{Q \ni x} (T_\Omega(f\chi_{3Q})\chi_Q)^* \left( \frac{\lambda}{2} |Q| \right) \leq \sup_{Q \ni x} (T_\Omega(f\chi_Q)\chi_Q)^* \left( \frac{1}{3^n} \frac{\lambda}{2} |Q| \right).$$

Our claim states that

$$\|M_{\lambda, T_\Omega} f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim_\varepsilon \frac{1}{\lambda} \|\Omega\|_{L^{1+\varepsilon}(S^{n-1})} \|f\|_{L^1(\mathbb{R}^n)}. \tag{2.3}$$

Now let  $\Omega \in L^q(S^{n-1})$ , have mean value zero on  $S^{n-1}$ . Without loss of generality, we assume that  $\|\Omega\|_{L^q(S^{n-1})} = 1$ . Set

$$t_0 = \left( \frac{1}{\lambda} \right)^{\frac{1+\varepsilon}{q}} \left[ 1 + \log \left( \frac{1}{\lambda} \right) \right]^{-\frac{1+\varepsilon}{q}}.$$

Let

$$\Omega^{t_0}(\theta) = \Omega(\theta)\chi_{\{|\Omega(\theta)| > t_0\}}(\theta), \quad \Omega_{t_0}(\theta) = \Omega(\theta)\chi_{\{|\Omega(\theta)| \leq t_0\}}(\theta),$$

and

$$\tilde{\Omega}^{t_0}(\theta) = \Omega^{t_0}(\theta) - A^{t_0}, \quad \tilde{\Omega}_{t_0}(\theta) = \Omega_{t_0}(\theta) - A_{t_0},$$

where

$$A^{t_0} = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \Omega^{t_0}(\theta) d\theta, \quad A_{t_0} = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \Omega_{t_0}(\theta) d\theta.$$

Both of  $\tilde{\Omega}^{t_0}$  and  $\tilde{\Omega}_{t_0}$  have mean value zero. Moreover,

$$\|\tilde{\Omega}^{t_0}\|_{L^{1+\varepsilon}(S^{n-1})} \lesssim t_0^{1-\frac{q}{1+\varepsilon}}, \quad \|\tilde{\Omega}_{t_0}\|_{L^\infty(S^{n-1})} \lesssim t_0,$$

and  $\Omega(\theta) = \tilde{\Omega}^{t_0}(\theta) + \tilde{\Omega}_{t_0}(\theta)$ . Applying Lemma 2.1 and (2.3), we deduce that

$$\begin{aligned} \|M_{\lambda, T_\Omega} f\|_{L^{1,\infty}(\mathbb{R}^n)} &\lesssim \|M_{\lambda, T_{\tilde{\Omega}^{t_0}}} f\|_{L^{1,\infty}(\mathbb{R}^n)} + \|M_{\lambda, T_{\tilde{\Omega}_{t_0}}} f\|_{L^{1,\infty}(\mathbb{R}^n)} \\ &\lesssim_\varepsilon \frac{1}{\lambda} \|\tilde{\Omega}^{t_0}\|_{L^{1+\varepsilon}(S^{n-1})} \|f\|_{L^1(\mathbb{R}^n)} \end{aligned}$$



$$\begin{aligned}
 &+ \left[ 1 + \log \left( \frac{1}{\lambda} \right) \right] \|\tilde{\Omega}_{t_0}\|_{L^\infty(S^{n-1})} \|f\|_{L^1(\mathbb{R}^n)} \\
 &\lesssim_{q,\varepsilon} \left( \frac{1}{\lambda} \right)^{\frac{1+\varepsilon}{q}} \left[ 1 + \log \left( \frac{1}{\lambda} \right) \right]^{1-\frac{1+\varepsilon}{q}} \|f\|_{L^1(\mathbb{R}^n)} \\
 &\lesssim_{q,\varepsilon} \left( \frac{1}{\lambda} \right)^{\frac{1+2\varepsilon}{q}} \|f\|_{L^1(\mathbb{R}^n)},
 \end{aligned}$$

where in the last inequality, we again invoked the fact that  $\log t \leq t^\alpha/\alpha$  for all  $t > 1$  and  $\alpha > 0$ . This completes the proof of Lemma 2.3.  $\square$

LEMMA 2.4. *Let  $r \in (1, \infty)$  and  $w$  be a weight. The following two statements are equivalent.*

- (i)  $w \in A_1(\mathbb{R}^n)$  and  $w^{1-p'} \in A_{p'/r}(\mathbb{R}^n)$  for some  $p \in (1, r')$ ;
- (ii)  $w^r \in A_1(\mathbb{R}^n)$ .

*Proof.* Let  $w \in A_1(\mathbb{R}^n)$  and  $w^{1-p'} \in A_{p'/r}(\mathbb{R}^n)$  for some  $p \in (1, r')$ , then for any cube  $Q \subset \mathbb{R}^n$ ,

$$\left( \frac{1}{|Q|} \int_Q w^{1-p'}(x) dx \right) \left( \frac{1}{|Q|} \int_Q w^{r\frac{p'-1}{p'-r}}(x) dx \right)^{\frac{p'}{r}-1} \leq [w^{1-p'}]_{A_{p'/r}},$$

and so

$$\begin{aligned}
 \frac{1}{|Q|} \int_Q w^{r\frac{p'-1}{p'-r}}(x) dx &\leq [w^{1-p'}]_{A_{p'/r}}^{\frac{p'}{r}-1} \left( \frac{1}{|Q|} \int_Q w^{1-p'}(x) dx \right)^{-\frac{1}{\frac{p'}{r}-1}} \\
 &\leq [w^{1-p'}]_{A_{p'/r}}^{\frac{1}{\frac{p'}{r}-1}} [w]_{A_1}^{\frac{1}{\frac{p'}{r}-1} \frac{1}{p-1}} \left( \frac{1}{|Q|} \int_Q w(x) dx \right)^{\frac{1}{\frac{p'}{r}-1} \frac{1}{p-1}} \\
 &\leq [w^{1-p'}]_{A_{p'/r}}^{\frac{1}{\frac{p'}{r}-1}} [w]_{A_1}^{\frac{1}{\frac{p'}{r}-1} \frac{1}{p-1}} (\text{essinf}_{y \in Q} w(y))^{\frac{p'-1}{\frac{p'}{r}-1}},
 \end{aligned}$$

where the second inequality follows from the fact that

$$\left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w^{1-p'}(x) dx \right)^{p-1} \geq 1.$$

We thus deduce that  $w^r \in A_1(\mathbb{R}^n)$ , with  $[w^r]_{A_1} \leq [w^{1-p'}]_{A_{p'/r}}^{\frac{1}{p-1}} [w]_{A_1}^r$ .

Let  $w^r \in A_1(\mathbb{R}^n)$ . By the reverse Hölder inequality, we know that  $w^{r\frac{p'-1}{p'-r}} \in A_1(\mathbb{R}^n)$  for some  $p \in (1, r')$ , and  $[w]_{A_1} \leq [w^r]_{A_1}$ ,  $[w^{r\frac{p'-1}{p'-r}}]_{A_1} \leq [w^r]_{A_1}^{(p'-1)/(p'-r)}$ . Thus for any cube  $Q \subset \mathbb{R}^n$ ,

$$\left( \frac{1}{|Q|} \int_Q w^{1-p'}(x) dx \right) \left( \frac{1}{|Q|} \int_Q w^{r\frac{p'-1}{p'-r}}(x) dx \right)^{\frac{p'}{r}-1}$$

$$\leq [\operatorname{ess\,inf}_{y \in Q} w(y)]^{1-p'} \left[ w^r \Big|_{A_1}^{r \frac{p'-1}{p'-r}} \right]^{\frac{p'}{r}-1} [\operatorname{ess\,inf}_{y \in Q} w(y)]^{p'-1} \leq [w^r]_{A_1}^{\frac{p'-1}{r}}.$$

This shows that  $w^{1-p'} \in A_{p'/r}(\mathbb{R}^n)$ .  $\square$

LEMMA 2.5. *Let  $T$  be a sublinear operator. Suppose that there exists a constant  $\tau \in (0, 1)$ , such that for all  $\lambda \in (0, 1/2)$ ,*

$$\|M_{\lambda, T} f\|_{L^{1, \infty}(\mathbb{R}^n)} \leq \lambda^{-\tau} \|f\|_{L^1(\mathbb{R}^n)}.$$

Then for  $p_0 \in (1, 1/\tau)$ ,

$$\|\mathcal{M}_{p_0, T} f\|_{L^{1, \infty}(\mathbb{R}^n)} \leq 2^{2+\frac{4}{1-\tau p_0}} \|f\|_{L^1(\mathbb{R}^n)},$$

where  $\mathcal{M}_{p_0, T}$  is the maximal operator defined as (1.5).

*Proof.* We employ the argument used in the proof of Lemma 3.3 in [18]. As it was proved in [18],

$$\mathcal{M}_{p_0, T} f(x) \leq \left( \int_0^1 (M_{\lambda, T} f(x))^{p_0} d\lambda \right)^{\frac{1}{p_0}}.$$

For  $N > 0$ , denote

$$G_{p_0, T, N} f(x) = \left( \int_0^1 (\min\{M_{\lambda, T} f(x), N\})^{p_0} d\lambda \right)^{\frac{1}{p_0}},$$

and

$$\mu_f(\alpha, R) = |\{x \in \mathbb{R}^n : |x| \leq R, |f(x)| > \alpha\}|, \quad \alpha, R > 0.$$

Let  $p_0 \in (1, \infty)$  such that  $\tau p_0 \in (0, 1)$ ,  $k = \lfloor \frac{4}{1-\tau p_0} \rfloor + 1$ , where and in the following, for  $a \in \mathbb{R}$ ,  $[a]$  denotes the integer part of  $a$ . By Hölder’s inequality,

$$\begin{aligned} G_{p_0, T, N} f(x) &\leq \left( \int \frac{1}{2^{kp_0}} (\min\{M_{\lambda, T} f(x), N\})^{p_0} d\lambda \right)^{\frac{1}{p_0}} + M_{1/2^{kp_0}, T} f(x) \\ &\leq \frac{1}{2^{k-1}} G_{kp_0, T, N} f(x) + M_{1/2^{kp_0}, T} f(x). \end{aligned}$$

Therefore,

$$\begin{aligned} \mu_{G_{p_0, T, N} f}(\alpha, R) &\leq \mu_{G_{kp_0, T, N} f}(2^{k-2}\alpha, R) + \mu_{M_{1/2^{kp_0}, T} f}(\alpha/2, R) \\ &\leq \mu_{G_{kp_0, T, N} f}(2^{k-2}\alpha, R) + \frac{1}{\alpha} 2^{\tau kp_0+1} \|f\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Repeating the last inequality  $j$  times, we have that

$$\mu_{G_{p_0, T, N} f}(\alpha, R) \leq \mu_{G_{kj p_0, T, N} f}(2^{j(k-2)}\alpha, R) + \frac{2^{k-2}}{\alpha} \sum_{l=1}^j \left( \frac{2^{\tau kp_0+1}}{2^{k-2}} \right)^l \|f\|_{L^1(\mathbb{R}^n)}.$$

Since  $G_{p_0, T, N} f$  is uniformly bounded in  $p_0$ , we obtain that  $\mu_{G_{k_j p_0, T, N} f}(\alpha, R) \rightarrow 0$  as  $j \rightarrow \infty$ . We finally deduce that

$$\mu_{G_{p_0, T, N} f}(\alpha, R) \leq 2^{2+\frac{4}{1-\tau p_0}} \frac{1}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}.$$

This completes the proof of Lemma 2.5.  $\square$

Let  $\eta \in (0, 1)$  and  $\mathcal{S} = \{Q_j\}$  be a family of cubes. We say that  $\mathcal{S}$  is  $\eta$ -sparse, if for each fixed  $Q \in \mathcal{S}$ , there exists a measurable subset  $E_Q \subset Q$ , such that  $|E_Q| \geq \eta|Q|$  and  $E_Q$ 's are pairwise disjoint. For sparse family  $\mathcal{S}$  and constants  $\beta, r \in [0, \infty)$ , we define the bilinear sparse operator  $\mathcal{A}_{\mathcal{S}; L(\log L)^\beta, L^r}$  by

$$\mathcal{A}_{\mathcal{S}; L(\log L)^\beta, L^r}(f, g) = \sum_{Q \in \mathcal{S}} |Q| \|f\|_{L(\log L)^\beta, Q} \langle |g| \rangle_{Q, r}.$$

We denote  $\mathcal{A}_{\mathcal{S}; L(\log L)^1, L^r}$  by  $\mathcal{A}_{\mathcal{S}; L \log L, L^r}$  for simplicity, and  $\mathcal{A}_{\mathcal{S}; L(\log L)^0, L^r}$  by  $\mathcal{A}_{\mathcal{S}; L, L^r}$ .

LEMMA 2.6. *Let  $\alpha, \beta \in \mathbb{N} \cup \{0\}$  and  $U$  be an operator. Suppose that for any  $r \in (1, 3/2)$ , and bounded function  $f$  with compact support, there exists a sparse family of cubes  $\mathcal{S}$ , such that for any function  $g \in L^1(\mathbb{R}^n)$ ,*

$$\left| \int_{\mathbb{R}^n} Uf(x)g(x)dx \right| \leq r'^\alpha \mathcal{A}_{\mathcal{S}; L(\log L)^\beta, L^r}(f, g). \tag{2.4}$$

Then for any  $u \in A_1(\mathbb{R}^n)$  and bounded function  $f$  with compact support,

$$w(\{x \in \mathbb{R}^n : |Uf(x)| > \lambda\}) \lesssim_{n, \alpha, \beta} [w]_{A_\infty}^\alpha \log^{1+\beta} \left( e + [w]_{A_\infty} \right) [w]_{A_1} \int_{\mathbb{R}^d} \frac{|f(x)|}{\lambda} \log^\beta \left( e + \frac{|f(x)|}{\lambda} \right) w(x) dx.$$

Lemma 2.6 is Corollary 3.6 in [14].

THEOREM 2.7. *Let  $p_0 \in (1, \infty)$ ,  $r \in (1, \infty)$ ,  $b \in \text{BMO}(\mathbb{R}^n)$ ,  $T$  be a linear operator and  $T_b$  be the commutator of  $T$ . Suppose that both of operators  $T$  and  $\mathcal{M}_{p_0, T}$  are bounded from  $L^1(\mathbb{R}^n)$  to  $L^{1, \infty}(\mathbb{R}^n)$  with bound 1. Then for bounded functions  $f$  with compact supports, there exists a  $\frac{1}{2} \frac{1}{3^n}$ -sparse family  $\mathcal{S}$  and functions  $H_1 f, H_2 f$ , such that for each function  $g \in L'_{\text{loc}}{}^{p'_0}(\mathbb{R}^n)$ ,*

$$\left| \int_{\mathbb{R}^n} H_1 f(x)g(x)dx \right| \lesssim_n \|b\|_{\text{BMO}(\mathbb{R}^n)} r' p'_0 \mathcal{A}_{\mathcal{S}; L^1, L'^{p'_0}}(f, g), \tag{2.5}$$

$$\left| \int_{\mathbb{R}^n} H_2 f(x)g(x)dx \right| \lesssim_n \|b\|_{\text{BMO}(\mathbb{R}^n)} \mathcal{A}_{\mathcal{S}; L \log L, L'^{p'_0}}(f, g), \tag{2.6}$$

and for a. e.  $x \in \mathbb{R}^n$ ,

$$T_b f(x) = H_1 f(x) + H_2 f(x).$$

*Proof.* We will employ the ideas in [18], see also the proof of Theorem 3.2 in [14]. Without loss of generality, we may assume that  $\|b\|_{\text{BMO}(\mathbb{R}^n)} = 1$ . For a fixed cube  $Q_0$ , define the local analogy of  $\mathcal{M}_{p_0, T}$  by

$$\mathcal{M}_{p_0, T; Q_0} f(x) = \sup_{Q \ni x, Q \subset Q_0} \left( \frac{1}{|Q|} \int_Q |T(f\chi_{3Q_0 \setminus 3Q})(y)|^{p_0} dy \right)^{\frac{1}{p_0}}.$$

Let  $E = \cup_{j=1}^4 E_j$  with

$$\begin{aligned} E_1 &= \{x \in Q_0 : |T(f\chi_{3Q_0})(x)| > D\langle |f| \rangle_{3Q_0}\}, \\ E_2 &= \{x \in Q_0 : |T((b - \langle b \rangle_{Q_0})f\chi_{3Q_0})(x)| > D\langle |(b - \langle b \rangle_{Q_0})f| \rangle_{3Q_0}\}, \\ E_3 &= \{x \in Q_0 : \mathcal{M}_{p_0, T; Q_0} f(x) > D\langle |f| \rangle_{3Q_0}\}, \end{aligned}$$

and

$$E_4 = \{x \in Q_0 : \mathcal{M}_{p_0, T_\Omega; Q_0}((b - \langle b \rangle_{Q_0})f)(x) > D\langle |b - \langle b \rangle_{Q_0}| |f| \rangle_{Q_0}\},$$

where  $D$  is a positive constant. If we choose  $D$  large enough, it then follows from the weak type  $(1, 1)$  boundedness of  $T$  and  $\mathcal{M}_{p_0, T}$  that

$$|E| \leq \frac{1}{2^{n+2}} |Q_0|.$$

Now on the cube  $Q_0$ , we apply the Calderón-Zygmund decomposition to  $\chi_E$  at level  $\frac{1}{2^{n+1}}$ , and obtain pairwise disjoint cubes  $\{P_j\} \subset \mathcal{D}(Q_0)$ , such that

$$\frac{1}{2^{n+1}} |P_j| \leq |P_j \cap E| \leq \frac{1}{2} |P_j|$$

and  $|E \setminus \cup_j P_j| = 0$ . Observe that  $\sum_j |P_j| \leq \frac{1}{2} |Q_0|$ . Let

$$\begin{aligned} G_{Q_0}^1(x) &= (b(x) - \langle b \rangle_{Q_0})T(f\chi_{3Q_0})\chi_{Q_0 \setminus \cup_j P_j}(x) + \sum_I (b(x) - \langle b \rangle_{Q_0})T(f\chi_{3Q_0 \setminus 3P_I})\chi_{P_I}(x), \\ G_{Q_0}^2(x) &= T((b - \langle b \rangle_{Q_0})f\chi_{3Q_0})\chi_{Q_0 \setminus \cup_j P_j}(x) + \sum_I T((b - \langle b \rangle_{Q_0})f\chi_{3Q_0 \setminus 3P_I})\chi_{P_I}(x). \end{aligned}$$

It then follows that

$$T_b(f\chi_{3Q_0})(x)\chi_{Q_0}(x) = G_{Q_0}^1(x) + G_{Q_0}^2(x) + \sum_I T_b(f\chi_{3P_I})(x)\chi_{P_I}(x).$$

We now estimate  $G_{Q_0}^1$  and  $G_{Q_0}^2$ . By (1.7) and the John-Nirenberg inequality (see [11, p.128]), we know that

$$\begin{aligned} \int_{Q_0} |b(x) - \langle b \rangle_{Q_0}| |h(x)| dx &\lesssim |Q_0| \|b - \langle b \rangle_{Q_0}\|_{\text{exp}L, Q} \|h\|_{L \log L, Q_0} \\ &\lesssim |Q_0| \|b\|_{\text{BMO}(\mathbb{R}^n)} \|h\|_{L \log L, Q_0}. \end{aligned}$$

This, along with the fact that  $|E \setminus \cup_j P_j| = 0$ , implies that

$$\left| \int_{Q_0 \setminus \cup_j P_j} (b(x) - \langle b \rangle_{Q_0}) T(f\chi_{3Q_0})(x)g(x)dx \right| \lesssim \langle |f| \rangle_{3Q_0} \|g\|_{L \log L, Q_0} |Q_0|,$$

and

$$\left| \int_{Q_0 \setminus \cup_j P_j} T((b - \langle b \rangle_{Q_0})f\chi_{3Q_0})(x)g(x)dx \right| \lesssim \langle |f| \rangle_{L \log L, 3Q_0} \langle |g| \rangle_{Q_0} |Q_0|.$$

On the other hand, the fact that  $P_j \cap E^c \neq \emptyset$  tells us that

$$\begin{aligned} & \sum_l \left| \int_{P_l} (b(x) - \langle b \rangle_{Q_0}) T(f\chi_{3Q_0 \setminus 3P_l})(x)g(x)dx \right| \\ & \lesssim \sum_l \left( \int_{P_l} |b(x) - \langle b \rangle_{Q_0}|^{p'_0} |g(x)|^{p'_0} dx \right)^{\frac{1}{p'_0}} \left( \int_{P_l} |T(f\chi_{3Q_0 \setminus 3P_l})(x)|^{p_0} dx \right)^{p_0} \\ & \lesssim \sum_l \left( \int_{P_l} |b(x) - \langle b \rangle_{Q_0}|^{p'_0} \right)^{\frac{1}{p'_0 r'}} |P_l|^{\frac{1}{p'_0 r'} + \frac{1}{p_0}} \langle |g| \rangle_{P_l, p'_0 r'} \inf_{y \in P_l} \mathcal{M}_{T, p_0, Q_0} f(y) \\ & \lesssim r' p'_0 \langle |f| \rangle_{3Q_0} \sum_l |P_l| \langle |g| \rangle_{P_l, r p'_0} \lesssim r' p'_0 \langle |f| \rangle_{3Q_0} \langle |g| \rangle_{Q_0, r p'_0} |Q_0|, \end{aligned}$$

here we have invoked the following estimate

$$\left( \int_{Q_0} |b(x) - \langle b \rangle_{Q_0}|^{p'_0 r'} dx \right)^{\frac{1}{p'_0 r'}} \lesssim r' p'_0 |Q_0|^{\frac{1}{p'_0 r'}},$$

see [11, p. 128]. Similarly, we can deduce that

$$\begin{aligned} & \sum_l \left| \int_{P_l} T((b - \langle b \rangle_{Q_0})f\chi_{3Q_0 \setminus 3P_l})(x)g(x)dx \right| \\ & \lesssim \sum_l |P_l| \langle |g| \rangle_{P_l, p'_0} \inf_{y \in P_l} \mathcal{M}_{p_0, T; Q_0} (b - \langle b \rangle_{Q_0})f(y) \\ & \lesssim \langle |f| \rangle_{3Q_0} \sum_l |P_l| \langle |g| \rangle_{P_l, p'_0} \lesssim \langle |f| \rangle_{3Q_0} \langle |g| \rangle_{Q_0, p'_0} |Q_0|. \end{aligned}$$

Therefore, for function  $g \in L^r_{loc}(\mathbb{R}^n)$ ,

$$\left| \int_{\mathbb{R}^n} G^1_{Q_0}(x)g(x)dx \right| \lesssim r' p'_0 \langle |f| \rangle_{3Q_0} \langle |g| \rangle_{Q_0, r p'_0} |Q_0|. \tag{2.7}$$

and

$$\left| \int_{\mathbb{R}^n} G^2_{Q_0}(x)g(x)dx \right| \lesssim \|f\|_{L \log L, 3Q_0} \langle |g| \rangle_{Q_0, p'_0} |Q_0|. \tag{2.8}$$

We repeat argument above with  $T(f\chi_{3Q_0})(x)\chi_{Q_0}$  replaced by  $T(\chi_{3P_l})(x)\chi_{P_l}(x)$ , and so on. Let  $Q_0^{j_0} = Q_0$ ,  $Q_0^{j_1} = P_j$ , and for fixed  $j_1, \dots, j_{m-1}$ ,  $\{Q_0^{j_1 \dots j_{m-1} j_m}\}_{j_m}$  be the

cubes obtained at the  $m$ -th stage of the decomposition process to the cube  $Q_0^{j_1 \dots j_{m-1}}$ . Set  $\mathcal{F} = \{Q_0\} \cup_{m=1}^\infty \cup_{j_1, \dots, j_m} \{Q_0^{j_1 \dots j_m}\}$ . Then  $\mathcal{F} \subset \mathcal{D}(Q_0)$  is a  $\frac{1}{2}$ -sparse family. We define the functions  $H_{1, Q_0}$  and  $H_{2, Q_0}$  by

$$H_{1, Q_0}(x) = \sum_{m=1}^\infty \sum_{j_1 \dots j_{m-1}} (b(x) - \langle b \rangle_{Q_0^{j_1 \dots j_{m-1}}}) \times T(f\chi_{3Q_0^{j_1 \dots j_{m-1}}})(x) \chi_{Q_0^{j_1 \dots j_{m-1}} \setminus \cup_{j_m} Q_0^{j_1 \dots j_m}}(x) \\ + \sum_{m=1}^\infty \sum_{j_1 \dots j_m} (b(x) - \langle b \rangle_{Q_0^{j_1 \dots j_{m-1}}}) \times T(f\chi_{3Q_0^{j_1 \dots j_{m-1}} \setminus \cup_{j_m} 3Q_0^{j_1 \dots j_m}})(x) \chi_{Q_0^{j_1 \dots j_m}}(x),$$

and

$$H_{2, Q_0}(x) = \sum_{m=1}^\infty \sum_{j_1 \dots j_{m-1}} T((b(x) - \langle b \rangle_{Q_0^{j_1 \dots j_{m-1}}}) f\chi_{3Q_0^{j_1 \dots j_{m-1}}})(x) \\ \times \chi_{Q_0^{j_1 \dots j_{m-1}} \setminus \cup_{j_m} Q_0^{j_1 \dots j_m}}(x) \\ + \sum_{m=1}^\infty \sum_{j_1 \dots j_m} T((b(x) - \langle b \rangle_{Q_0^{j_1 \dots j_{m-1}}}) f\chi_{3Q_0^{j_1 \dots j_m} \setminus \cup_{j_{m+1}} 3Q_0^{j_1 \dots j_{m+1}}})(x) \\ \times \chi_{Q_0^{j_1 \dots j_{m-1}}}(x).$$

Then for a. e.  $x \in Q_0$ ,

$$T_b(f\chi_{3Q_0})(x) = H_{1, Q_0}(x) + H_{2, Q_0}(x).$$

Moreover, as in inequalities (2.7)-(2.8), the process of producing  $\{Q_0^{j_1 \dots j_m}\}$  leads to that

$$\left| \int_{Q_0} g(x) H_{1, Q_0}(x) dx \right| \lesssim r' p'_0 \sum_{Q \in \mathcal{F}} |Q| \langle |f| \rangle_{3Q} \langle |g| \rangle_{Q, r' p'_0},$$

and

$$\left| \int_{Q_0} g(x) H_{2, Q_0}(x) dx \right| \lesssim \sum_{Q \in \mathcal{F}} |Q| \|f\|_{L \log L, 3Q} \langle |g| \rangle_{Q, p'_0}.$$

We can now conclude the proof of Theorem 2.7. In fact, as in [18], we decompose  $\mathbb{R}^n$  by cubes  $\{R_l\}$ , such that  $\text{supp} f \subset 3R_l$  for each  $l$ , and  $R_l$ 's have disjoint interiors. Then for a. e.  $x \in \mathbb{R}^n$ ,

$$T_b f(x) = \sum_l H_{1, R_l} f(x) + \sum_l H_{2, R_l} f(x) =: H_1 f(x) + H_2 f(x).$$

Obviously,  $H_1, H_2$  satisfies (2.5) and (2.6). Our desired conclusion then follows directly.  $\square$

LEMMA 2.8. *Let  $\gamma \in \mathbb{N} \cup \{0\}$ ,  $r \in [1, \infty)$ , and  $U$  be an operator. Suppose that for any bounded function  $f$  with compact support, there exists a sparse family of cubes  $\mathcal{S}$ , such that for any function  $g \in L^r_{\text{loc}}(\mathbb{R}^n)$ ,*

$$\left| \int_{\mathbb{R}^n} Uf(x)g(x)dx \right| \leq \mathcal{A}_{\mathcal{S}, L(\log L)^\gamma, L^r}(f, g). \tag{2.9}$$

Then for any  $w$  with  $w^r \in A_1(\mathbb{R}^n)$ ,  $\alpha > 0$  and bounded function  $f$  with compact support,

$$w(\{x \in \mathbb{R}^n : |Uf(x)| > \alpha\}) \lesssim_{n,\gamma,w} \int_{\mathbb{R}^d} \frac{|f(x)|}{\alpha} \log^\gamma \left( e + \frac{|f(x)|}{\alpha} \right) w(x) dx.$$

*Proof.* By Theorem 3.2 in [14], we know that  $U$  satisfies the following estimate:

$$w(\{x \in \mathbb{R}^d : |Uf(x)| > 1\}) \lesssim \left( 1 + \left\{ p_1'^{1+\gamma} \left( \frac{p_1'}{r} \right)' \left( t \frac{p_1'/r-1}{p_1'-1} \right)' \frac{1}{p_1'} \right\}^{p_1'} \right) \times \int_{\mathbb{R}^n} |f(y)| \log^\gamma(e + |f(y)|) M_t w(y) dy, \tag{2.10}$$

where  $t \in [1, \infty)$ ,  $p_1 \in (1, r')$  such that  $t \frac{p_1'/r-1}{p_1'-1} > 1$ , and  $M_t$  is defined by

$$M_r f(x) = [M(|f|^r)(x)]^{1/r}.$$

Let  $w^r \in A_1(\mathbb{R}^n)$ . We choose  $\varepsilon > 0$  such that  $w^{r(1+\varepsilon)} \in A_1(\mathbb{R}^n)$ . Set  $t = r(1 + \varepsilon)$  and  $p_1' = 2(r-1) \frac{1+\varepsilon}{\varepsilon} + 1$ . Then  $t \frac{p_1'/r-1}{p_1'-1} = 1 + \frac{\varepsilon}{2}$ . We obtain from (2.10) that

$$w(\{x \in \mathbb{R}^d : |Uf(x)| > 1\}) \lesssim_{n,\gamma,w} \int_{\mathbb{R}^n} |f(y)| \log^\gamma(e + |f(y)|) w(y) dy.$$

This, via homogeneity, leads to our desired conclusion.  $\square$

*Proof of Theorem 1.2.* By homogeneity, we may assume that  $\|\Omega\|_{L^q(S^{n-1})} = 1 = \|b\|_{\text{BMO}(\mathbb{R}^n)}$ . Let  $w^{q'} \in A_1(\mathbb{R}^n)$ . We choose  $\varepsilon > 0$  such that  $\varepsilon \in (0, \min\{1, (q-1)/3\})$  and  $w^{q'(1+\varepsilon)} \in A_1(\mathbb{R}^n)$ . On the other hand, by Lemma 2.3 and Lemma 2.5, we know that for any  $p_0 \in (0, q/(1+2\varepsilon))$ ,

$$\|\mathcal{M}_{p_0, T_\Omega} f\|_{L^1(\mathbb{R}^n)} \lesssim 2^{\frac{4}{1-p_0} \frac{1+2\varepsilon}{q}} \|f\|_{L^1(\mathbb{R}^n)}.$$

Take  $p_0 = q/(1+3\varepsilon)$  and  $r = \frac{q-(1+3\varepsilon)}{q-1}(1+\varepsilon)$ , then  $rp_0' = (1+\varepsilon)q'$ . Applying Theorem 2.7 with such indices  $p_0$  and  $r$ , we see that for any bounded function  $f$  with compact support, there exists a sparse family of cubes  $\mathcal{S}$ , such that for any  $g \in L_{\text{loc}}^{q'(1+\varepsilon)}(\mathbb{R}^n)$ ,

$$\left| \int_{\mathbb{R}^n} T_b f(x) g(x) dx \right| \lesssim p_0' r' 2^{4 \frac{1+3\varepsilon}{\varepsilon}} \mathcal{A}_{\mathcal{S}; L \log L, L^{q'(1+\varepsilon)}}(f, g).$$

Theorem 1.2 now follows from Lemma 2.8 immediately.  $\square$

*Proof of Theorem 1.3.* Again we assume that  $\|\Omega\|_{L^\infty(S^{n-1})} = 1 = \|b\|_{\text{BMO}(\mathbb{R}^n)}$ . Let  $s \in (1, \infty)$ . Applying (1.6) and Theorem 2.7 (with  $p_0 = (\sqrt{s})'$  and  $r = \sqrt{s}$ ), we know

that for bounded function  $f$  with compact support, there exists a  $\frac{1}{2} \frac{1}{3^n}$ -sparse family of cubes  $\mathcal{S} = \{Q\}$ , and functions  $H_1 f, H_2 f$ , such that for each function  $g \in L^s_{\text{loc}}(\mathbb{R}^n)$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} H_1 f(x) g(x) dx \right| &\lesssim (\sqrt{s})^2 \mathcal{A}_{\mathcal{S}; L^1, L^s}(f, g) \lesssim s^2 \mathcal{A}_{\mathcal{S}; L^1, L^s}(f, g), \\ \left| \int_{\mathbb{R}^n} H_2 f(x) g(x) dx \right| &\lesssim (\sqrt{s})^l \mathcal{A}_{\mathcal{S}; L \log L, L \sqrt{s}}(f, g) \lesssim s^l \mathcal{A}_{\mathcal{S}; L \log L, L^s}(f, g), \end{aligned}$$

and for a. e.  $x \in \mathbb{R}^n$ ,

$$T_{\Omega, b} f(x) = H_1 f(x) + H_2 f(x).$$

Let  $w \in A_1(\mathbb{R}^n)$ ,  $\lambda > 0$ ,  $f$  be a bounded function with compact support. It follows from Lemma 2.6 that

$$\begin{aligned} &w(\{x \in \mathbb{R}^n : |T_{\Omega, b} f(x)| > \lambda\}) \\ &\leq w(\{x \in \mathbb{R}^n : |H_1 f(x)| > \lambda/2\}) + w(\{x \in \mathbb{R}^n : |H_2 f(x)| > \lambda/2\}) \\ &\lesssim [w]_{A_1} [w]_{A_\infty}^2 \log(e + [w]_{A_\infty}) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} w(x) dx \\ &\quad + [w]_{A_1} [w]_{A_\infty} \log^2(e + [w]_{A_\infty}) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\left(e + \frac{|f(x)|}{\lambda}\right) w(x) dx \\ &\lesssim [w]_{A_1} [w]_{A_\infty}^2 \log(e + [w]_{A_\infty}) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log\left(e + \frac{|f(x)|}{\lambda}\right) w(x) dx. \end{aligned}$$

This completes the proof of Theorem 1.3.  $\square$

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