Mathematical
Inequalities
& Applications
Volume 25, Number 3 (2022), 631–646 doi:10.7153/mia-2022-25-39

TRIANGULAR CESÀRO SUMMABILITY AND LEBESGUE POINTS OF TWO–DIMENSIONAL FOURIER SERIES

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(*Communicated by I. Perić*)

Abstract. We prove that the triangular Cesàro means of two-dimensional functions $f \in L_1(\mathbb{T}^2)$ converge to f at each strong $(1,\omega)$-Lebesgue point. Moreover, if $f \in L_p(\mathbb{T}^2)$ with $1 < p < \infty$, then the Cesàro means converge to f at each (p,ω)-Lebesgue point. This generalizes the well known classical Lebesgue's theorem.

1. Introduction

It is known that the Fejér means

$$\sigma_n f(x) := \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) \widehat{f}(k) e^{\imath kx}$$

of a one-dimensional integrable function f converge to $f(x)$ at each Lebesgue point (see Lebesgue [11]), where $\widehat{f}(k)$ denotes the kth Fourier coefficient. The generalization of this theorem to Cesàro means is due to M. Riesz [15]. These means were considered in a great number of papers (see e.g. Gát [4, 5, 6], Goginava [7, 8, 9], Simon [16, 17], Nagy, Persson, Tephnadze and Wall [13, 14], Weisz [19, 20] and Zygmund [26]).

Here we investigate the triangular Cesàro means

$$\sigma_n^\alpha f(x) := \frac{1}{A_{n-1}^\alpha} \sum_{|k_1|+|k_2| \leqslant n} A_{n-1-|k_1|-|k_2|}^\alpha \widehat{f}(k) e^{\imath(k_1x_1+k_2x_2)}$$

of two-dimensional functions. These means were also sudied e.g. in Berens, Li and Xu [1, 2, 12, 25], Szili and Vértesi [18]. In [19], we proved that $\sigma_n^\alpha f \to f$ almost everywhere if $f \in L_1(\mathbb{T}^2)$.

In this paper, we partly characterize the set of this convergence, which is a generalization of the result of Lebesgue and Riesz just mentioned. We introduce two new types of Lebesgue points, the (p,ω)- and strong (p,ω)-Lebesgue points. The author proved in [21] that almost every point is a (strong) (p,ω)-Lebesgue point of $f \in L_p(\mathbb{T}^2)$ with

Mathematics subject classification (2020): 42B08, 42A38, 42A24, 42B25.
Keywords and phrases: Cesàro summability, Fejér summability, triangular summability, Hardy-Littlewood maximal function, Lebesgue points.
This research was supported by the Hungarian Scientific Research Funds (OTKA) No KH130426.

$1 \leqslant p < \infty$. We show that if the maximal function $\mathscr{M}_1^{\omega} f(x)$ is finite and x is a strong $(1,\omega)$-Lebesgue point of $f \in L_1(\mathbb{T}^2)$, then

$$\lim_{n\to\infty} \sigma_n^{\alpha} f(x) = f(x).$$

For $1 < p < \infty$, we get a simpler result: the convergence holds for all (p,ω)-Lebesgue points if $f \in L_p(\mathbb{T}^2)$. A more complicated version of the first result can be shown for higher dimensional functions with another method (see [23]). In [22], we verified similar results for triangular θ-means of Fourier transforms. However, the θ-summability does not contain the Cesàro summability and the proof for Fourier series is more complicated than for Fourier transforms, so new ideas are needed here. An analogous result for another type of summability, for the so called rectangular summability was proved in [24].

2. Maximal functions and Lebesgue points

We briefly write $L_p(\mathbb{T}^2)$ instead of the $L_p(\mathbb{T}^2,\lambda)$ space equipped with the norm

$$\|f\|_p := \left(\int_{\mathbb{T}^2} |f|^p \, d\lambda \right)^{1/p} \qquad (1 \leqslant p < \infty),$$

where $\mathbb{T} = [-\pi,\pi]$ is the torus and λ the Lebesgue measure.

For some $\omega > 0$, $1 \leqslant p < \infty$, $x = (x_1,x_2)$ and $f \in L_1(\mathbb{T}^2)$, we introduce the next Hardy-Littlewood maximal functions:

$$\mathscr{M}_p^{\omega} f(x) = \sum_{j=1}^{5} \mathscr{M}_p^{\omega,j} f(x),$$

where

$$\mathscr{M}_p^{\omega,1} f(x) = \sup_{i_1,i_2\in\mathbb{N},h>0} 2^{-\omega(i_1+i_2)} \left(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{-2^{i_2}h}^{2^{i_2}h} |f(x-t)|^p \, dt \right)^{1/p},$$

$$\mathscr{M}_p^{\omega,2} f(x) := \sup_{i_1,i_2\in\mathbb{N},h>0} 2^{-\omega(i_1+i_2)} \left(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{t_1-2^{i_2}h}^{t_1+2^{i_2}h} |f(x-t)|^p \, dt_2 \, dt_1 \right)^{1/p},$$

$$\mathscr{M}_p^{\omega,3} f(x) := \sup_{i_1,i_2\in\mathbb{N},h>0} 2^{-\omega(i_1+i_2)} \left(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{-t_1-2^{i_2}h}^{-t_1+2^{i_2}h} |f(x-t)|^p \, dt_2 \, dt_1 \right)^{1/p},$$

$$\mathscr{M}_p^{\omega,4} f(x) := \sup_{i_1,i_2\in\mathbb{N},h>0} 2^{-\omega(i_1+i_2)} \left(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_2}h}^{2^{i_2}h} \int_{t_2-2^{i_1}h}^{t_2+2^{i_1}h} |f(x-t)|^p \, dt_1 \, dt_2 \right)^{1/p}$$

and

$$\mathscr{M}_p^{\omega,5} f(x) := \sup_{i_1,i_2\in\mathbb{N},h>0} 2^{-\omega(i_1+i_2)} \left(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_2}h}^{2^{i_2}h} \int_{-t_2-2^{i_1}h}^{-t_2+2^{i_1}h} |f(x-t)|^p \, dt_1 \, dt_2 \right)^{1/p}.$$

Note that in $\mathscr{M}_p^{\omega,1} f$, we take the supremum over rectangles with sides parallel to the axes and in $\mathscr{M}_p^{\omega,j} f$ $(j=2,3,4,5)$, over parallelograms with one side parallel to one of the axes and with the other side parallel to one of the diagonals of the square $[0,\pi]^2$. If $\omega=0$, then $\mathscr{M}_p^{\omega,1} f$ is exactly the strong Hardy-Littlewood maximal function, if $\omega=0$ and $i_1=i_2$, then $\mathscr{M}_p^{\omega,1} f$ is equal to the usual Hardy-Littlewood maximal function (see e.g. Feichtinger and Weisz [3]). The next two inequalities were proved in [21]. If $\omega>0$ and $1\leqslant p<\infty$, then

$$\sup_{\rho>0} \rho \lambda (\mathscr{M}_p^{\omega} f > \rho)^{1/p} \leqslant C\|f\|_p \qquad (f\in L_p(\mathbb{T}^2)). \tag{1}$$

Moreover, if $p<r\leqslant\infty$, then

$$\left\|\mathscr{M}_p^{\omega} f\right\|_r \leqslant C_r \|f\|_r \qquad (f\in L_r(\mathbb{T}^d)).$$

In this paper the constants C and C_p may vary from line to line.

Using the above maximal functions, we define the next operators:

$$U_{r,p}^{\omega,1} f(x) := \sup_{i_1,i_2\in\mathbb{N},h>0,2^{i_k}h<r,k=1,2} 2^{-\omega(i_1+i_2)} \left(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{-2^{i_2}h}^{2^{i_2}h} |f(x-t)-f(x)|^p \, dt \right)^{1/p},$$

$$U_{r,p}^{\omega,2} f(x) := \sup_{i_1,i_2\in\mathbb{N},h>0,2^{i_k}h<r,k=1,2} 2^{-\omega(i_1+i_2)} \left(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{t_1-2^{i_2}h}^{t_1+2^{i_2}h} |f(x-t)-f(x)|^p \, dt_2 \, dt_1 \right)^{1/p},$$

$$U_{r,p}^{\omega,3} f(x) := \sup_{i_1,i_2\in\mathbb{N},h>0,2^{i_k}h<r,k=1,2} 2^{-\omega(i_1+i_2)} \left(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_1}h}^{2^{i_1}h} \int_{-t_1-2^{i_2}h}^{-t_1+2^{i_2}h} |f(x-t)-f(x)|^p \, dt_2 \, dt_1 \right)^{1/p},$$

$$U_{r,p}^{\omega,4} f(x) := \sup_{i_1,i_2\in\mathbb{N},h>0,2^{i_k}h<r,k=1,2} 2^{-\omega(i_1+i_2)} \left(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_2}h}^{2^{i_2}h} \int_{t_2-2^{i_1}h}^{t_2+2^{i_1}h} |f(x-t)-f(x)|^p \, dt_1 \, dt_2 \right)^{1/p},$$

$$U_{r,p}^{\omega,5} f(x) := \sup_{i_1,i_2\in\mathbb{N},h>0,2^{i_k}h<r,k=1,2} 2^{-\omega(i_1+i_2)} \Big(\frac{1}{4\cdot 2^{i_1+i_2}h^2} \int_{-2^{i_2}h}^{2^{i_2}h} \int_{-t_2-2^{i_1}h}^{-t_2+2^{i_1}h} |f(x-t)-f(x)|^p \, dt_1\, dt_2 \Big)^{1/p},$$

and

$$U_{r,p}^{\omega} f(x) = \sum_{j=1}^{5} U_{r,p}^{\omega,j} f(x).$$

If $p=1$, then we omit the index p and simply write $\mathscr{M}^{\omega}$, U_r^{ω},...

For $1 \leqslant p < \infty$ and $\omega > 0$, a point $x \in \mathbb{T}^2$ is called a (p,ω)-Lebesgue point of $f \in L_p(\mathbb{T}^d)$ if

$$\lim_{r\to 0} U_{r,p}^{\omega,1} f(x) = 0.$$

If

$$\lim_{r\to 0} U_{p,r}^{\omega} f(x) = 0,$$

then x is called a strong (p,ω)-Lebesgue point. If $\omega = 0$, then the (p,ω)-Lebesgue points are the same as the well known strong Lebesgue points, if $\omega = 0$ and $i_1 = i_2$, then the (p,ω)-Lebesgue points give back the usual Lebesgue points, i.e.,

$$\lim_{h\to 0} \frac{1}{4h_1h_2} \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} |f(x-t)-f(x)| \, dt = 0$$

and

$$\lim_{h\to 0} \frac{1}{(2h)^2} \int_{-h}^{h} \int_{-h}^{h} |f(x-t)-f(x)| \, dt = 0$$

(see Feichtinger and Weisz [3]). Since $U_{p,r}^{\omega_1} f \leqslant U_{p,r}^{\omega_2} f$, every (strong) (p,ω_2)-Lebesgue point is a (strong) (p,ω_1)-Lebesgue point $(0<\omega_2<\omega_1<\infty)$. If f is continuous at x, then x is a (strong) (p,ω)-Lebesgue point of f. The next theorem was proved in [21].

THEOREM 1. *For $1 \leqslant p < \infty$ and $\omega > 0$, almost every point $x \in \mathbb{T}^2$ is a (strong) (p,ω)-Lebesgue point of $f \in L_p(\mathbb{T}^2)$.*

3. The kernel functions

The kth Fourier coefficient of a two-dimensional integrable function $f \in L_1(\mathbb{T}^2)$ is defined by

$$\widehat{f}(k) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(x) e^{-\imath(k_1x_1+k_2x_2)} \, dx \qquad (k \in \mathbb{Z}^2).$$

For $f \in L_1(\mathbb{T}^2)$ and $n \in \mathbb{N}$, we define the nth triangular partial sum $s_n f$ of the Fourier series of f by

$$s_n f(x) := \sum_{|k_1|+|k_2|\leqslant n} \widehat{f}(k) e^{\imath(k_1x_1+k_2x_2)}$$

It is known (see e.g. Grafakos [10] or Weisz [20]) that for $f \in L_p(\mathbb{T}^2)$, $1 < p < \infty$,

$$\lim_{n\to\infty} s_n f = f \qquad \text{in the } L_p(\mathbb{T}^2)\text{-norm and a.e.}$$

To extend this convergence to $p = 1$, we need to consider the Cesàro summation.

For $\alpha \neq -1, -2, \ldots$ and $n \in \mathbb{N}$, let

$$A_n^\alpha := \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}.$$

Then $A_0^\alpha = 1$, $A_n^0 = 1$ and $A_n^1 = n+1$ $(n \in \mathbb{N})$. Let $f \in L_1(\mathbb{T}^d)$, $n \in \mathbb{N}$ and $\alpha \geqslant 0$. In [19], we investigated the triangular Cesàro means $\sigma_n^\alpha f$ given by

$$\sigma_n^\alpha f(x) := \frac{1}{A_{n-1}^\alpha} \sum_{|k_1|+|k_2| \leqslant n} A_{n-1-|k_1|-|k_2|}^\alpha \widehat{f}(k) e^{\imath(k_1x_1+k_2x_2)}.$$

If $\alpha = 0$, we get back $s_n f$, if $\alpha = 1$, then the triangular Fejér means

$$\sigma_n^1 f(x) := \sum_{|k_1|+|k_2| \leqslant n} \left(1 - \frac{|k_1|+|k_2|}{n}\right) \widehat{f}(k) e^{\imath(k_1x_1+k_2x_2)}.$$

It is easy to see that

$$\sigma_n^\alpha f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^2} f(x-t) K_n^\alpha(t)\, dt,$$

where

$$K_n^\alpha(t) := \frac{1}{A_{n-1}^\alpha} \sum_{|k_1|+|k_2| \leqslant n} A_{n-1-|k_1|-|k_2|}^\alpha e^{\imath(k_1x_1+k_2x_2)}$$

are the triangular Cesàro kernels. Zygmund [26] verified the next lemma.

LEMMA 1. *For $\alpha > -1$ and $h > 0$, we have*

$$\sigma_n^{\alpha+h} f = \frac{1}{A_{n-1}^{\alpha+h}} \sum_{k=1}^{n} A_{n-k}^{h-1} A_{k-1}^\alpha \sigma_k^\alpha f.$$

The next two lemmas were proved by the author in [19].

LEMMA 2. *Suppose that $0 < \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$ and $\pi > x_1 > x_2 > 0$. Then*

$$|K_n^\alpha(x_1,x_2)| \leqslant Cn^2, \tag{2}$$

$$\begin{aligned} |K_n^\alpha(x_1,x_2)| \leqslant{}& C(x_1-x_2)^{-1}(x_1+x_2)^{-1} 1_{\{x_2 \leqslant \pi/2\}} \\ &+ C(x_1-x_2)^{-1}(\pi-x_2)^{-1} 1_{\{x_2 > \pi/2\}}. \end{aligned} \tag{3}$$

If $x_2 > 1/n$, *then*

$$1_{\{x_2 \leqslant \pi/2\}} |K_n^\alpha(x_1,x_2)| \leqslant Cn^{-\alpha}(x_1-x_2)^{-1-\beta} x_2^{\beta-\alpha-1} 1_{\{x_2 \leqslant \pi/2\}} \tag{4}$$

and

$$1_{\{x_2 \leqslant \pi/2\}} |K_n^\alpha(x_1,x_2)| \leqslant Cn^{1-\alpha} x_2^{-\alpha-1} 1_{\{x_2 \leqslant \pi/2\}}. \tag{5}$$

If $\pi - x_2 > 1/n$, *then*

$$1_{\{x_2 > \pi/2\}} |K_n^\alpha(x_1,x_2)| \leqslant Cn^{-\alpha}(x_1-x_2)^{-1-\beta} (\pi - x_2)^{\beta-\alpha-1} 1_{\{x_2 > \pi/2\}} \tag{6}$$

and

$$1_{\{x_2 > \pi/2\}} |K_n^\alpha(x_1,x_2)| \leqslant Cn^{1-\alpha} (\pi - x_2)^{-\alpha-1} 1_{\{x_2 > \pi/2\}}. \tag{7}$$

LEMMA 3. *If* $0 < \alpha \leqslant 1$, *then*

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{T}^2} |K_n^\alpha| \, d\lambda \leqslant C.$$

4. Convergence of the Cesàro means at Lebesgue points

Now we are ready to prove our main theorem.

THEOREM 2. *Suppose that* $0 < \alpha < \infty$, $0 < \omega < \min(\alpha,1)/2$ *and* $\mathscr{M}^\omega f(x)$ *is finite. If* $f \in L_1(\mathbb{T}^2)$ *is periodic with respect to* π *and* x *is a strong* $(1,\omega)$*-Lebesgue point of* f, *then*

$$\lim_{n\to\infty} \sigma_n^\alpha f(x) = f(x).$$

Proof. By Lemma 1, it is enough to prove the theorem for $0 < \alpha \leqslant 1$. Since

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^2} K_n^\alpha(t)\, dt = 1,$$

we have

$$|\sigma_n^\alpha f(x) - f(x)| \leqslant \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^2} |f(x-t) - f(x)| |K_n^\alpha(t)| \, dt.$$

Let $0 < \omega < \alpha/2$ and fix a number $r < 1$ such that $U_r^\omega f(x_1,x_2) < \varepsilon$. Suppose that $2/n < r/2$. Let us introduce the sets

$$S_{r/2} := \left[-\frac{r}{2}, \frac{r}{2}\right] \times \left[-\frac{r}{2}, \frac{r}{2}\right], \qquad S'_{r/2} := \left[\pi - \frac{r}{2}, \pi + \frac{r}{2}\right] \times \left[\pi - \frac{r}{2}, \pi + \frac{r}{2}\right]$$

and

$$\begin{aligned}
A_1 &:= \{(x_1,x_2) : 0 < x_1 \leqslant 2/n, 0 < x_2 < x_1 < \pi, x_2 \leqslant \pi/2\},\\
A_2 &:= \{(x_1,x_2) : 2/n < x_1 < \pi, 0 < x_2 \leqslant 1/n, x_2 \leqslant \pi/2\},\\
A_3 &:= \{(x_1,x_2) : 2/n < x_1 < \pi, 1/n < x_2 \leqslant x_1/2, x_2 \leqslant \pi/2\},\\
A_4 &:= \{(x_1,x_2) : 2/n < x_1 < \pi, x_1/2 < x_2 \leqslant x_1 - 1/n, x_2 \leqslant \pi/2\},\\
A_5 &:= \{(x_1,x_2) : 2/n < x_1 < \pi, x_1 - 1/n < x_2 < x_1, x_2 \leqslant \pi/2\}\\
A_6 &:= \{(x_1,x_2) : x_2 > \pi/2, \pi - 2/n \leqslant x_2 < \pi, 0 < x_2 < x_1 < \pi\},\\
A_7 &:= \{(x_1,x_2) : \pi/2 < x_2 < \pi - 2/n, \pi - 1/n < x_1 < \pi\},\\
A_8 &:= \{(x_1,x_2) : \pi/2 < x_2 < \pi - 2/n, (\pi + x_2)/2 < x_1 \leqslant \pi - 1/n\},\\
A_9 &:= \{(x_1,x_2) : \pi/2 < x_2 < \pi - 2/n, x_2 + 1/n < x_1 \leqslant (\pi + x_2)/2\},\\
A_{10} &:= \{(x_1,x_2) : \pi/2 < x_2 < \pi - 2/n, x_2 < x_1 \leqslant x_2 + 1/n\}
\end{aligned}$$

(see Figure 1). It is enough to integrate on the set $\{(t_1,t_2) : 0 < t_2 < t_1 < \pi\}$,

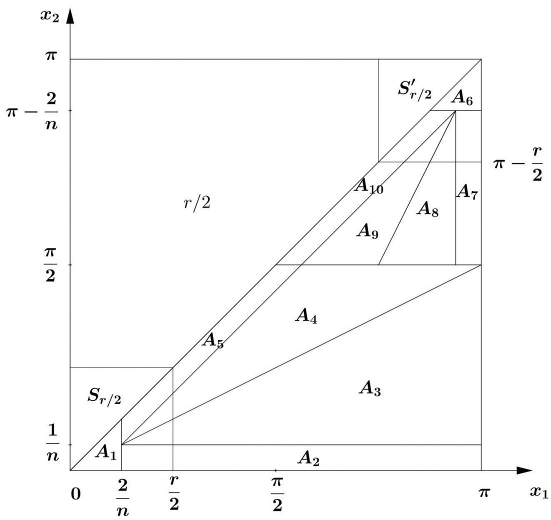

Figure 1: *The sets* A_i.

in other words, on

$$\bigcup_{i=1}^{5}(A_i \cap S_{r/2}), \qquad \bigcup_{i=1}^{5}(A_i \cap S^c_{r/2}), \qquad \bigcup_{i=6}^{10}(A_i \cap S'_{r/2}), \qquad \bigcup_{i=6}^{10}(A_i \cap (S'_{r/2})^c).$$

Since $A_1 \subset S_{r/2}$ and $A_6 \subset S'_{r/2}$, inequality (2) implies

$$\begin{aligned}
\int_{A_1} |f(x-t) - f(x)|\,|K_n^{\alpha}(t)|\,dt &\leqslant Cn^2 \int_0^{2/n}\int_0^{2/n} |f(x-t) - f(x)|\,dt_2\,dt_1\\
&\leqslant CU_r^{\omega,1} f(x) < C\varepsilon
\end{aligned}$$

and

$$\int_{A_6} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt \leqslant Cn^2 \int_{\pi-2/n}^{\pi}\int_{\pi-2/n}^{\pi} |f(x-t)-f(x)|\,dt_2\,dt_1$$
$$\leqslant Cn^2 \int_{-2/n}^{0}\int_{-2/n}^{0} |f(x-u-\pi)-f(x)|\,du_2\,du_1$$
$$\leqslant CU_r^{\omega,1} f(x) < C\varepsilon.$$

Let us denote by r_0 the largest number i, for which $r/2 \leqslant 2^{i+1}/n < r$. By (3),

$$\int_{A_2\cap S_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt$$
$$\leqslant C\sum_{i=1}^{r_0}\left(\frac{2^i}{n}-\frac{1}{n}\right)^{-1}\left(\frac{2^i}{n}\right)^{-1}\int_{2^i/n}^{2^{i+1}/n}\int_0^{1/n} |f(x-t)-f(x)|\,dt_2\,dt_1$$
$$\leqslant C\sum_{i=1}^{r_0} 2^{(\omega-1)i}2^{-\omega i}\left(\frac{n^2}{2^i}\right)\int_{2^i/n}^{2^{i+1}/n}\int_0^{1/n} |f(x-t)-f(x)|\,dt_2\,dt_1$$
$$\leqslant C\sum_{i=1}^{r_0} 2^{(\omega-1)i} U_r^{\omega,1} f(x) < C\varepsilon.$$

Similarly,

$$\int_{A_7\cap S'_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt$$
$$\leqslant C\sum_{i=1}^{r_0}\left(\frac{2^i}{n}-\frac{1}{n}\right)^{-1}\left(\frac{2^i}{n}\right)^{-1}\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{\pi-1/n}^{\pi} |f(x-t)-f(x)|\,dt_1\,dt_2$$
$$\leqslant C\sum_{i=1}^{r_0} 2^{(\omega-1)i}2^{-\omega i}\left(\frac{n^2}{2^i}\right)\int_{-2^{i+1}/n}^{-2^i/n}\int_{-1/n}^{0} |f(x-t-\pi)-f(x)|\,dt_1\,dt_2$$
$$\leqslant C\sum_{i=1}^{r_0} 2^{(\omega-1)i} U_r^{\omega,1} f(x_1,x_2) < C\varepsilon.$$

It follows in the same way that

$$\int_{A_2\cap S^c_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt + \int_{A_7\cap (S'_{r/2})^c} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt$$
$$\leqslant C\sum_{i=r_0}^{\infty} 2^{(\omega-1)i}\mathscr{M}^{\omega,1} f(x_1,x_2) + C\sum_{i=r_0}^{\infty} 2^{-i}|f(x_1,x_2)|$$
$$\leqslant C2^{(\omega-1)r_0}\mathscr{M}^{\omega,1} f(x_1,x_2) + C2^{-r_0}|f(x_1,x_2)|$$
$$\leqslant C(nr)^{\omega-1}\mathscr{M}^{\omega,1} f(x_1,x_2) + C(nr)^{-1}|f(x_1,x_2)| \to 0$$

as $n\to\infty$. Observe that $t_1-t_2 > t_1/2$ and $t_1-t_2 > t_2$ on A_3. We use (4) to estimate the kernel as follows:

$$|K_n^\alpha(t)| \leqslant Cn^{-\alpha}(t_1-t_2)^{-1-\alpha/2}(t_1-t_2)^{-\beta+\alpha/2}t_2^{\beta-\alpha-1} \leqslant Cn^{-\alpha}t_1^{-1-\alpha/2}t_2^{-1-\alpha/2}$$

if $\beta > \alpha/2$. Hence

$$\begin{aligned}
&\int_{A_3\cap S_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt \\
&\quad\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} n^{-\alpha}\left(\frac{2^i}{n}\right)^{-1-\alpha/2}\left(\frac{2^j}{n}\right)^{-1-\alpha/2}\int_{2^i/n}^{2^{i+1}/n}\int_{2^j/n}^{2^{j+1}/n}|f(x-t)-f(x)|\,dt_2\,dt_1 \\
&\quad\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)}2^{-\omega(i+j)}\left(\frac{n^2}{2^{i+j}}\right)\int_{2^i/n}^{2^{i+1}/n}\int_{2^j/n}^{2^{j+1}/n}|f(x-t)-f(x)|\,dt_2\,dt_1 \\
&\quad\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)}U_r^{\omega,1}f(x_1,x_2) < C\varepsilon.
\end{aligned}$$

Similarly, $t_1-t_2 > (\pi-t_2)/2 \geqslant (\pi-t_1)/2$ on A_8. By inequality (6),

$$\begin{aligned}
|K_n^\alpha(t)| &\leqslant Cn^{-\alpha}(t_1-t_2)^{-1-\alpha/2}(t_1-t_2)^{-\beta+\alpha/2}(\pi-t_2)^{\beta-\alpha-1} \\
&\leqslant Cn^{-\alpha}(\pi-t_1)^{-1-\alpha/2}(\pi-t_2)^{-1-\alpha/2}
\end{aligned}$$

if $\beta > \alpha/2$, which implies

$$\begin{aligned}
&\int_{A_8\cap S'_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt \\
&\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} n^{-\alpha}\left(\frac{2^i}{n}\right)^{-1-\alpha/2}\left(\frac{2^j}{n}\right)^{-1-\alpha/2}\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{\pi-2^{j+1}/n}^{\pi-2^j/n}|f(x-t)-f(x)|\,dt_1\,dt_2 \\
&\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)}2^{-\omega(i+j)}\left(\frac{n^2}{2^{i+j}}\right)\int_{-2^{i+1}/n}^{-2^i/n}\int_{-2^{j+1}/n}^{-2^j/n}|f(x-t-\pi)-f(x)|\,dt_1\,dt_2 \\
&\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)}U_r^{\omega,1}f(x_1,x_2) < C\varepsilon.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{A_3\cap S_{r/2}^c} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt + \int_{A_8\cap (S'_{r/2})^c} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt \\
&\qquad\leqslant C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)}\mathscr{M}^{\omega,1}f(x) + C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1} 2^{-\alpha(i+j)/2}|f(x)| \\
&\qquad\leqslant C2^{(\omega-\alpha/2)r_0}\mathscr{M}^{\omega,1}f(x) + C2^{-\alpha r_0/2}|f(x)| \to 0
\end{aligned}$$

as $n\to\infty$. It is easy to see that $t_2 > t_1/2$ on A_4. We apply (4) with $\beta = \alpha/2$ and conclude

$$|K_n^\alpha(t)| \leqslant Cn^{-\alpha}(t_1-t_2)^{-1-\alpha/2}t_2^{-1-\alpha/2} \leqslant Cn^{-\alpha}t_1^{-1-\alpha/2}(t_1-t_2)^{-1-\alpha/2}$$

and so

$$\begin{aligned}
&\int_{A_4\cap S_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt \\
&\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} n^{-\alpha}\left(\frac{2^i}{n}\right)^{-1-\alpha/2}\left(\frac{2^j}{n}\right)^{-1-\alpha/2}\int_{2^i/n}^{2^{i+1}/n}\int_{t_1-2^{j+1}/n}^{t_1-2^j/n}|f(x-t)-f(x)|\,dt_2\,dt_1 \\
&\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)}2^{-\omega(i+j)}\left(\frac{n^2}{2^{i+j}}\right)\int_{2^i/n}^{2^{i+1}/n}\int_{t_1-2^{j+1}/n}^{t_1-2^j/n}|f(x-t)-f(x)|\,dt_2\,dt_1 \\
&\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)}U_r^{\omega,2}f(x_1,x_2) < C\varepsilon.
\end{aligned}$$

The inequality

$$|K_n^\alpha(t)| \leqslant Cn^{-\alpha}(t_1-t_2)^{-1-\alpha/2}(\pi-t_2)^{-1-\alpha/2}$$

follows from (6) with $\beta=\alpha/2$. Hence

$$\begin{aligned}
&\int_{A_9\cap S'_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt \\
&\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} n^{-\alpha}\left(\frac{2^i}{n}\right)^{-1-\alpha/2}\left(\frac{2^j}{n}\right)^{-1-\alpha/2}\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{t_2+2^j/n}^{t_2+2^{j+1}/n}|f(x-t)-f(x)|\,dt_1\,dt_2 \\
&\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)}2^{-\omega(i+j)}\left(\frac{n^2}{2^{i+j}}\right) \\
&\quad \int_{-2^{i+1}/n}^{-2^i/n}\int_{t_2+2^j/n}^{t_2+2^{j+1}/n}|f(x-t-\pi)-f(x)|\,dt_1\,dt_2 \\
&\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)}U_r^{\omega,4}f(x_1,x_2) < C\varepsilon.
\end{aligned}$$

We conclude in the same way that

$$\begin{aligned}
&\int_{A_4\cap S^c_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt + \int_{A_9\cap (S'_{r/2})^c} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt \\
&\quad\leqslant C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)}\left(\mathscr{M}^{\omega,2}f(x)+\mathscr{M}^{\omega,4}f(x)\right) + C\sum_{i=r_0}^{\infty}\sum_{j=0}^{i-1} 2^{-\alpha(i+j)/2}|f(x)| \\
&\quad\leqslant C2^{(\omega-\alpha/2)r_0}\mathscr{M}^{\omega}f(x) + C2^{-\alpha r_0/2}|f(x)| \to 0
\end{aligned}$$

as $n\to\infty$. On A_5, we get

$$|K_n^\alpha(t)| \leqslant Cn^{1-\alpha}t_1^{-\alpha-1}$$

from (5). Then

$$\begin{aligned}&\int_{A_5\cap S_{r/2}} |f(x-t)-f(x)||K_n^\alpha(t)|\,dt\\&\quad\leqslant C\sum_{i=1}^{r_0} n^{1-\alpha}\left(\frac{2^i}{n}\right)^{-\alpha-1}\int_{2^i/n}^{2^{i+1}/n}\int_{t_1-1/n}^{t_1}|f(x-t)-f(x)|\,dt_1\,dt_2\\&\quad\leqslant C\sum_{i=1}^{r_0} 2^{(\omega-\alpha)i}2^{-\omega i}\left(\frac{n^2}{2^i}\right)\int_{2^i/n}^{2^{i+1}/n}\int_{t_1-1/n}^{t_1}|f(x-t)-f(x)|\,dt_1\,dt_2\\&\quad\leqslant C\sum_{i=1}^{r_0} 2^{(\omega-1)i}U_r^{\omega,2}f(x)<C\varepsilon.\end{aligned}$$

Similarly, by (7),

$$\begin{aligned}&\int_{A_{10}\cap S'_{r/2}} |f(x-t)-f(x)||K_n^\alpha(t)|\,dt\\&\quad\leqslant C\sum_{i=1}^{r_0} n^{1-\alpha}\left(\frac{2^i}{n}\right)^{-1-\alpha}\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{t_2}^{t_2+1/n}|f(x-t)-f(x)|\,dt_1\,dt_2\\&\quad\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\omega-\alpha)i}2^{-\omega i}\left(\frac{n^2}{2^i}\right)\int_{-2^{i+1}/n}^{-2^i/n}\int_{t_2}^{t_2+1/n}|f(x-t-\pi)-f(x)|\,dt_1\,dt_2\\&\quad\leqslant C\sum_{i=1}^{r_0}\sum_{j=0}^{i-1} 2^{(\omega-\alpha)i}U_r^{\omega,4}f(x_1,x_2)<C\varepsilon.\end{aligned}$$

Finally,

$$\begin{aligned}&\int_{A_5\cap S^c_{r/2}} |f(x-t)-f(x)||K_n^\alpha(t)|\,dt+\int_{A_{10}\cap(S'_{r/2})^c}|f(x-t)-f(x)||K_n^\alpha(t)|\,dt\\&\quad\leqslant C\sum_{i=r_0}^{\infty}2^{(\omega-\alpha)i}\left(\mathscr{M}^{\omega,2}f(x)+\mathscr{M}^{\omega,4}f(x)\right)+C\sum_{i=r_0}^{\infty}2^{-\alpha i}|f(x)|\\&\quad\leqslant C2^{(\omega-\alpha)r_0}\mathscr{M}^{\omega}f(x)+C2^{-\alpha r_0}|f(x)|\to 0\end{aligned}$$

as $n\to\infty$, which finishes the proof. □

Theorems 1, 2 and (1) imply

COROLLARY 1. *If $0<\alpha<\infty$ and $f\in L_1(\mathbb{T}^2)$ is periodic with respect to π, then*

$$\lim_{n\to\infty}\sigma_n^\alpha f=f \qquad a.e.$$

This corollary was also obtained in Weisz [19]. For $1<p<\infty$, we can prove the theorem for (p,ω)-Lebesgue points.

THEOREM 3. *Suppose that* $0<\alpha<\infty$, $1/(\min(\alpha,1)<p<\infty$, $1/p+1/q=1$, $0<\omega<(1+q\min(\alpha,1)-q)/2q$ *and* $\mathscr{M}_p^{\omega,1}f(x)$ *is finite. If* $f\in L_p(\mathbb{T}^2)$ *is periodic with respect to* π *and* x *is a* (p,ω)*-Lebesgue point of* f*, then*

$$\lim_{n\to\infty}\sigma_n^\alpha f(x)=f(x).$$

Proof. We prove the theorem again for $0<\alpha\leqslant 1$. Note that $1/\alpha<p<\infty$ implies $1<q<1/(1-\alpha)$ and so $1+\alpha q-q>0$. Moreover, $(1+\alpha q-q)/(2q)<\alpha/2$. Fix a number $r<1$ such that $U_{r,p}^{\omega,1}f(x_1,x_2)<\varepsilon$. We have seen in in Theorem 2 that

$$\int_{A_i}|f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt\to 0,$$

for $i=1,2,3,6,7,8$, as $n\to\infty$ and $\omega<\alpha/2$. So we have to estimate the integrals on A_4, A_5, A_9 and A_{10}.

Since $t_2>t_1/2$ on A_4, (4) with $\beta=0$ implies

$$|K_n^\alpha(t)|\leqslant Cn^{-\alpha}(t_1-t_2)^{-1}t_1^{-\alpha-1}.$$

By Hölder's inequality,

$$\begin{aligned}
&\int_{A_4\cap S_{r/2}}|f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt\\
&\leqslant\sum_{i=1}^{r_0}\sum_{j=i-1}^{i}\int_{2^i/n}^{2^{i+1}/n}\int_{2^j/n}^{2^{j+1}/n}|f(x-t)-f(x)|\,|K_n^\alpha(t)|\,1_{A_4}(t)\,dt_2\,dt_1\\
&\leqslant\sum_{i=1}^{r_0}\sum_{j=i-1}^{i}\left(\int_{2^i/n}^{2^{i+1}/n}\int_{2^j/n}^{2^{j+1}/n}|f(x-t)-f(x)|^p\,dt_2\,dt_1\right)^{1/p}\\
&\qquad\left(\int_{2^i/n}^{2^{i+1}/n}\int_{2^{i-1}/n}^{t_1-1/n}n^{-\alpha q}(t_1-t_2)^{-q}t_1^{-q(1+\alpha)}1_{A_4}(t)\,dt_2\,dt_1\right)^{1/q}.
\end{aligned}$$

We compute the last integral as follows:

$$\begin{aligned}
&\int_{2^i/n}^{2^{i+1}/n}\int_{2^{i-1}/n}^{t_1-1/n}n^{-\alpha q}(t_1-t_2)^{-q}t_1^{-q(1+\alpha)}1_{A_4}(t)\,dt_2\,dt_1\\
&\leqslant Cn^{-\alpha q}\left(\frac{1}{n}\right)^{1-q}\int_{2^i/n}^{2^{i+1}/n}t_1^{-q(1+\alpha)}\,dt_1\\
&\leqslant Cn^{-\alpha q}\left(\frac{1}{n}\right)^{1-q}\left(\frac{2^i}{n}\right)^{1-q(1+\alpha)}\\
&\leqslant C\left(\frac{n}{2^i}\right)^{2q-2}2^{-i(1+\alpha q-q)}
\end{aligned}$$

and so

$$\int_{A_4\cap S_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt$$

$$\leqslant C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)}$$

$$2^{-\omega(i+j)}\left(\frac{n^2}{2^{i+j}}\int_{2^i/n}^{2^{i+1}/n}\int_{2^j/n}^{2^{j+1}/n}|f(x-t)-f(x)|^p\,dt_2\,dt_1\right)^{1/p}$$

$$\leqslant C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)}U_{r,p}^{\omega,1}f(x) < C_p\varepsilon.$$

By (6) with $\beta=0$,

$$\int_{A_9\cap S'_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt$$

$$\leqslant \sum_{i=1}^{r_0}\sum_{j=i-1}^{i}\left(\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{\pi-2^{j+1}/n}^{\pi-2^j/n}|f(x-t)-f(x)|^p\,dt_1\,dt_2\right)^{1/p}$$

$$\left(\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{t_2+1/n}^{\pi-2^{i-1}/n} n^{-\alpha q}(t_1-t_2)^{-q}(\pi-t_2)^{-q(1+\alpha)}1_{A_4}(t)\,dt_1\,dt_2\right)^{1/q}.$$

The last integral is equal to

$$\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{t_2+1/n}^{\pi-2^{i-1}/n} n^{-\alpha q}(t_1-t_2)^{-q}(\pi-t_2)^{-q(1+\alpha)}1_{A_4}(t)\,dt_1\,dt_2$$

$$\leqslant Cn^{-\alpha q}\left(\frac{1}{n}\right)^{1-q}\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}(\pi-t_2)^{-q(1+\alpha)}\,dt_2$$

$$\leqslant Cn^{-\alpha q}\left(\frac{1}{n}\right)^{1-q}\left(\frac{2^i}{n}\right)^{1-q(1+\alpha)} \leqslant C\left(\frac{n}{2^i}\right)^{2q-2}2^{-i(1+\alpha q-q)}.$$

Consequently,

$$\int_{A_9\cap S'_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt$$

$$\leqslant C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)}2^{-\omega(i+j)}$$

$$\left(\frac{n^2}{2^{i+j}}\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{\pi-2^{j+1}/n}^{\pi-2^j/n}|f(x-t)-f(x)|^p\,dt_2\,dt_1\right)^{1/p}$$

$$\leqslant C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)}U_{r,p}^{\omega,1}f(x) < C_p\varepsilon.$$

Similarly, we can see that

$$\begin{aligned}
&\int_{A_4\cap S^c_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt + \int_{A_9\cap (S'_{r/2})^c} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt \\
&\quad \leqslant C_p \sum_{i=r_0}^{\infty}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} \mathscr{M}_p^{\omega,1} f(x) \\
&\qquad + C_p \sum_{i=r_0}^{\infty}\sum_{j=i-1}^{i} 2^{-(1+\alpha q-q)(i+j)/2q} |f(x)| \\
&\quad \leqslant C_p 2^{r_0(2\omega-(1+\alpha q-q)/q)} \mathscr{M}_p^{\omega,1} f(x) + C_p 2^{-r_0(1+\alpha q-q)/q} |f(x)| \\
&\quad \leqslant C(nr)^{2\omega-(1+\alpha q-q)/q} \mathscr{M}_p^{\omega,1} f(x) + C(nr)^{-(1+\alpha q-q)} |f(x)| \to 0
\end{aligned}$$

as $n\to\infty$. The fact that $t_2 > t_1/2$ on A_5 and (5) imply

$$\begin{aligned}
&\int_{A_5\cap S_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt \\
&\quad \leqslant \sum_{i=1}^{r_0}\sum_{j=i-1}^{i} \left(\int_{2^i/n}^{2^{i+1}/n}\int_{2^j/n}^{2^{j+1}/n} |f(x-t)-f(x)|^p\,dt_2\,dt_1\right)^{1/p} \\
&\qquad \left(\int_{2^i/n}^{2^{i+1}/n}\int_{t_1-1/n}^{t_1} n^{q(1-\alpha)} t_1^{-q(\alpha+1)}\,dt_2\,dt_1\right)^{1/q}.
\end{aligned}$$

For the last integral, we have

$$\begin{aligned}
\int_{2^i/n}^{2^{i+1}/n}\int_{t_1-1/n}^{t_1} n^{q(1-\alpha)} t_1^{-q(\alpha+1)}\,dt_2\,dt_1 &\leqslant n^{q(1-\alpha)-1}\left(\frac{2^i}{n}\right)^{1-q(\alpha+1)} \\
&\leqslant C\left(\frac{n}{2^i}\right)^{2q-2} 2^{-i(1+\alpha q-q)}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{A_5\cap S_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt \\
&\quad \leqslant \sum_{i=1}^{r_0}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} 2^{-\omega(i+j)} \\
&\qquad \left(\frac{n^2}{2^{i+j}}\int_{2^i/n}^{2^{i+1}/n}\int_{2^j/n}^{2^{j+1}/n} |f(x-t)-f(x)|^p\,dt_2\,dt_1\right)^{1/p} \\
&\quad \leqslant C_p \sum_{i=1}^{r_0}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} U_{r,p}^{\omega,1} f(x) < C\varepsilon.
\end{aligned}$$

By (7),

$$\int_{A_{10}\cap S'_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt$$
$$\leqslant \sum_{i=1}^{r_0}\sum_{j=i-1}^{i}\left(\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{\pi-2^{j+1}/n}^{\pi-2^j/n}|f(x-t)-f(x)|^p\,dt_1\,dt_2\right)^{1/p}$$
$$\left(\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{t_2}^{t_2+1/n} n^{q(1-\alpha)}(\pi-t_2)^{-q(\alpha+1)}\,dt_1\,dt_2\right)^{1/q}.$$

Then

$$\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{t_2}^{t_2+1/n} n^{q(1-\alpha)}(\pi-t_2)^{-q(\alpha+1)}\,dt_1\,dt_2 \leqslant n^{-1}n^{q(1-\alpha)}\left(\frac{2^i}{n}\right)^{1-q(\alpha+1)}$$
$$\leqslant C\left(\frac{n}{2^i}\right)^{2q-2}2^{-i(1+\alpha q-q)}$$

and

$$\int_{A_{10}\cap S'_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt$$
$$\leqslant \sum_{i=1}^{r_0}\sum_{j=i-1}^{i} 2^{(\omega-(1+\alpha q-q)/2q)(i+j)}2^{-\omega(i+j)}$$
$$\left(\frac{n^2}{2^{i+j}}\int_{\pi-2^{i+1}/n}^{\pi-2^i/n}\int_{\pi-2^{j+1}/n}^{\pi-2^j/n}|f(x-t)-f(x)|^p\,dt_2\,dt_1\right)^{1/p}$$
$$\leqslant C_p\sum_{i=1}^{r_0}\sum_{j=i-1}^{i}2^{(\omega-(1+\alpha q-q)/2q)(i+j)}U_{r,p}^{\omega,1}f(x) < C\varepsilon.$$

Finally,

$$\int_{A_5\cap S^c_{r/2}} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt + \int_{A_{10}\cap (S'_{r/2})^c} |f(x-t)-f(x)|\,|K_n^\alpha(t)|\,dt$$
$$\leqslant C_p\sum_{i=r_0}^{\infty}\sum_{j=i-1}^{i}2^{(\omega-(1+\alpha q-q)/2q)(i+j)}\mathscr{M}_p^{\omega,1}f(x)$$
$$+C_p\sum_{i=r_0}^{\infty}\sum_{j=i-1}^{i}2^{-(1+\alpha q-q)(i+j)/2q}|f(x)|$$
$$\leqslant C_p2^{r_0(2\omega-(1+\alpha q-q)/q)}\mathscr{M}_p^{\omega,1}f(x)+C_p2^{-r_0(1+\alpha q-q)/q}|f(x)| \to 0$$

as $n\to\infty$. The proof of the theorem is complete. □

Note that for the Fejér means, i.e., if $\alpha = 1$ or even $\alpha \geqslant 1$, in Theorem 3 we have $1<p<\infty$ and $0<\omega<1/2q$.

REFERENCES

[1] H. BERENS, Z. LI, AND Y. XU, *On l_1 Riesz summability of the inverse Fourier integral*, Indag. Math. (N.S.), 12: 41–53, 2001.

[2] H. BERENS AND Y. XU, *l-1 summability of multiple Fourier integrals and positivity*, Math. Proc. Cambridge Philos. Soc., 122: 149–172, 1997.

[3] H. G. FEICHTINGER AND F. WEISZ, *Wiener amalgams and pointwise summability of Fourier transforms and Fourier series*, Math. Proc. Cambridge Philos. Soc., 140: 509–536, 2006.

[4] G. GÁT, *Pointwise convergence of cone-like restricted two-dimensional $(C,1)$ means of trigonometric Fourier series*, J. Approx. Theory., 149: 74–102, 2007.

[5] G. GÁT, *Almost everywhere convergence of sequences of Cesàro and Riesz means of integrable functions with respect to the multidimensional Walsh system*, Acta Math. Sin., Engl. Ser., 30 (2): 311–322, 2014.

[6] G. GÁT, U. GOGINAVA, AND K. NAGY, *On the Marcinkiewicz-Fejér means of double Fourier series with respect to Walsh-Kaczmarz system*, Studia Sci. Math. Hungar., 46: 399–421, 2009.

[7] U. GOGINAVA, *Marcinkiewicz-Fejér means of d-dimensional Walsh-Fourier series*, J. Math. Anal. Appl., 307: 206–218, 2005.

[8] U. GOGINAVA, *Almost everywhere convergence of (C,α)-means of cubical partial sums of d-dimensional Walsh-Fourier series*, J. Approx. Theory, 141: 8–28, 2006.

[9] U. GOGINAVA, *The maximal operator of the Marcinkiewicz-Fejér means of d-dimensional Walsh-Fourier series*, East J. Approx., 12: 295–302, 2006.

[10] L. GRAFAKOS, *Classical and Modern Fourier Analysis*, Pearson Education, New Jersey, 2004.

[11] H. LEBESGUE, *Recherches sur la convergence des séries de Fourier*, Math. Ann., 61: 251–280, 1905.

[12] Z. LI AND Y. XU, *Summability of product Jacobi expansions*, J. Approx. Theory, 104: 287–301, 2000.

[13] K. NAGY AND G. TEPHNADZE, *The Walsh-Kaczmarz-Marcinkiewicz means and Hardy spaces*, Acta Math. Hungar., 149: 346–374, 2016.

[14] L. E. PERSSON, G. TEPHNADZE, AND P. WALL, *Maximal operators of Vilenkin-Nörlund means*, J. Fourier Anal. Appl., 21 (1): 76–94, 2015.

[15] M. RIESZ, *Sur la sommation des séries de Fourier*, Acta Sci. Math. (Szeged), 1: 104–113, 1923.

[16] P. SIMON, *Cesàro summability with respect to two-parameter Walsh systems*, Monatsh. Math., 131: 321–334, 2000.

[17] P. SIMON, *(C,α) summability of Walsh-Kaczmarz-Fourier series*, J. Approx. Theory, 127: 39–60, 2004.

[18] L. SZILI AND P. VÉRTESI, *On multivariate projection operators*, J. Approx. Theory, 159: 154–164, 2009.

[19] F. WEISZ, *Triangular Cesàro summability of two-dimensional Fourier series*, Acta Math. Hungar., 132: 27–41, 2011.

[20] F. WEISZ, *Summability of multi-dimensional trigonometric Fourier series*, Surv. Approx. Theory, 7: 1–179, 2012.

[21] F. WEISZ, *Lebesgue points of two-dimensional Fourier transforms and strong summability*, J. Fourier Anal. Appl., 21: 885–914, 2015.

[22] F. WEISZ, *Triangular summability and Lebesgue points of two-dimensional Fourier transforms*, Banach J. Math. Anal., 11: 223–238, 2017.

[23] F. WEISZ, *Lebesgue points of ℓ_1-Cesàro summability of d-dimensional Fourier series*, Adv. Oper. Theory., 6: 48, 2021.

[24] F. WEISZ, *Unrestricted Cesàro summability of d-dimensional Fourier series and Lebesgue points*, Constr. Math. Anal., 4: 179–185, 2021.

[25] Y. XU, *Christoffel functions and Fourier series for multivariate orthogonal polynomials*, J. Approx. Theory, 82: 205–239, 1995.

[26] A. ZYGMUND, *Trigonometric Series*, Cambridge Press, London, 3rd edition, 2002.

(Received January 12, 2021)

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