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ON QUASINORMALITY OF THE DILATION OF TRUNCATED TOEPLITZ OPERATORS

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Abstract. An operator $S^u_{\varphi,\psi}$ on L^2 is called the *dilation of a truncated Toeplitz operator* if for two symbols $\varphi,\psi \in L^\infty$ and an inner function u,

$$S^u_{\varphi,\psi} f = \varphi P_u f + \psi Q_u f$$

holds for $f \in L^2$ where P_u denotes the orthogonal projection of L^2 onto $\mathscr{K}^2_u$ and $Q_u = I - P_u$. In this paper, we study characterizations for the dilation of truncated Toeplitz operators ${S_{\varphi,\psi}}^u$ to be quasinormal. As consequences of the results, we investigate the forms of the symbol functions φ and ψ when such operator becomes a quasinormal operator.

1. Introduction and preliminaries

Let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathscr{H}$. For an operator $T \in \mathscr{L}(\mathscr{H})$, T^* denotes the adjoint of T. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *isometry* if $T^*T = I$, *normal* if $T^*T = TT^*$, and *quasinormal* if T^*T and T commute, respectively.

Let $\mathscr{H}$ be a subspace of a Hilbert space $\mathscr{K}$ and let P be the orthogonal projection from $\mathscr{K}$ onto $\mathscr{H}$. Then R is called a (weak) *dilation* of T to $\mathscr{K}$ if $T = PRP$, equivalently, $Tf = PRf$ for each $f \in \mathscr{H}$ (see [1] or [7]). In this case, the operator T is called the *compression* of R to $\mathscr{H}$. Since $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}^\perp$, it follows that R is a dilation of T if and only if the matrix representation of R has the following form

$$\begin{pmatrix} T & X \\ Y & Z \end{pmatrix}.$$

In some books ([1] and [11, page 10]), an additional condition is imposed for which $T^n = PR^nP$ holds for every positive integer n. In this case, R is called a *power dilation*

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of T. The concept of a dilation is related to model theory which means the representation of some class as pieces of operators in a smaller, better-understood, class.

Let L^2 be the Lebesgue (Hilbert) space on the unit circle and let L^∞ be the Banach space of all functions in L^2 essentially bounded on $\partial\mathbb{D}$. The Hilbert Hardy space, denoted by H^2, consists of all analytic functions f on $\mathbb{D}$ having square-summable Taylor coefficients at the origin. Then the Hilbert Hardy space $H^2 = \overline{\text{span}\{z^n : n = 0,1,2,3,\cdots\}}$ in L^2. Let P denote the orthogonal projection of L^2 onto H^2. Then $Q = I - P$ is the orthogonal projection of L^2 onto $(H^2)^\perp := L^2 \ominus H^2 = L^2 \cap (H^2)^\perp$. For any $\varphi \in L^\infty$, let M_φ denote the multiplication operator on L^2 such that $M_\varphi f = \varphi f$ for $f \in L^2$. For any $\varphi \in L^\infty$, the *Toeplitz operator* $T_\varphi : H^2 \to H^2$ is defined by the formula

$$T_\varphi f = P(\varphi f), \ f \in H^2 \tag{1}$$

where P denotes the orthogonal projection of L^2 onto H^2. It is known that T_φ is bounded if and only if $\varphi \in L^\infty$, and in this case, $\|T_\varphi\| = \|\varphi\|_\infty$.

In 2007, Sarason [12] initiated the study of truncated Toeplitz operators which are the compression of Toeplitz operators. A function $u \in H^2$ is called *inner* if $|u| = 1$ a.e. For a nonconstant inner function u, the *model space* $\mathcal{K}_u^2$ is given by $\mathcal{K}_u^2 := H^2 \ominus uH^2$. For any $\varphi \in L^\infty$ and an inner function u, the *truncated Toeplitz operator* $A_\varphi^u : \mathcal{K}_u^2 \to \mathcal{K}_u^2$ is defined by the formula

$$A_\varphi^u f = P_u(\varphi f) \text{ for } f \in \mathcal{K}_u^2 \tag{2}$$

where P_u denotes the orthogonal projection of L^2 onto $\mathcal{K}_u^2$. Several aspects of this operator were studied in [2]–[6], [12], and [13]. For any $\varphi \in L^\infty$ and an inner function u, let $\widetilde{A_\varphi^u}$ denote the operator on $(\mathcal{K}_u^2)^\perp := L^2 \ominus \mathcal{K}_u^2$ such that

$$\widetilde{A_\varphi^u} f = Q_u(\varphi f) \text{ for } f \in (\mathcal{K}_u^2)^\perp \tag{3}$$

where Q_u denotes the orthogonal projection of L^2 onto $(\mathcal{K}_u^2)^\perp$. Let Γ_φ^u be the *truncated Hankel operator* of $\mathcal{K}_u^2$ to $(\mathcal{K}_u^2)^\perp$ such that

$$\Gamma_\varphi^u f = Q_u(\varphi f) \text{ for } f \in \mathcal{K}_u^2. \tag{4}$$

Let $\widetilde{\Gamma_\varphi^u}$ be the operator of $(\mathcal{K}_u^2)^\perp$ to $\mathcal{K}_u^2$ such that

$$\widetilde{\Gamma_\varphi^u} f = P_u(\varphi f) \text{ for } f \in (\mathcal{K}_u^2)^\perp. \tag{5}$$

It is obvious that $A_\varphi^{u\,*} = A_{\overline{\varphi}}^u$ and $\widetilde{A_\varphi^u}^{\,*} = \widetilde{A_{\overline{\varphi}}^u}$.

In light of the function space, we can consider the following dilation of a truncated Toeplitz operator A_φ^u on $\mathcal{K}_u^2 \oplus (\mathcal{K}_u^2)^\perp = L^2$ (see Lemma 3.2 in [8] for more details).

DEFINITION 1. An operator $S_{\varphi,\psi}^u$ on L^2 is called the *dilation of a truncated Toeplitz operator* if for two symbols $\varphi, \psi \in L^\infty$ and an inner function u,

$$S_{\varphi,\psi}^u f = \varphi P_u f + \psi Q_u f$$

holds for $f \in L^2$ where P_u denotes the orthogonal projection of L^2 onto $\mathscr{K}_u^2$ and $Q_u = I - P_u$. In particular, if $\varphi = \psi$, then $S^u_{\varphi,\varphi} = M_\varphi$ is a multiplication operator on L^2.

We remark that a bounded operator $S^u_{\varphi,\psi}$ on L^2 satisfying Definition 1 has the following block matrix representation:

$$S^u_{\varphi,\psi} = \begin{pmatrix} A^u_\varphi & \widetilde{\Gamma^u_\psi} \\ \Gamma^u_\varphi & \widetilde{A^u_\psi} \end{pmatrix} \tag{6}$$

on $\mathscr{K}_u^2 \oplus (\mathscr{K}_u^2)^\perp = L^2$ where $A^u_\varphi, \widetilde{\Gamma^u_\psi}, \Gamma^u_\varphi$, and $\widetilde{A^u_\psi}$ are defined as before. For example, if $u(z) = z^n$, $\varphi(z) = \sum_{k=-\infty}^{\infty} a_k z^k$, and $\psi(z) = \sum_{k=-\infty}^{\infty} b_k z^k$, then $\mathscr{B} = \{1, z, z^2, \cdots, z^{n-1}\}$ is a basis for $\mathscr{K}^2_{z^n}$ and

$$[A^{z^n}_\varphi]_{\mathscr{B}} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \ddots & a_{-n+2} \\ \cdots & \cdots & \ddots & \ddots & \vdots \\ a_{n-2} & a_{n-3} & a_{n-4} & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix}.$$

Therefore the operator matrix of $\begin{pmatrix} A^{z^n}_\varphi & \widetilde{\Gamma^{z^n}_\psi} \\ \Gamma^{z^n}_\varphi & \widetilde{A^{z^n}_\psi} \end{pmatrix}$ with respect to $\mathscr{B} \oplus \mathscr{B}^\perp$ is a dilation of $[A^{z^n}_\varphi]_{\mathscr{B}}$ (see [8]–[10]).

In 2016, the authors in [8] introduced the dilation of truncated Toeplitz operators on L^2 using the concept of singular integral operators. Moreover, the authors in ([8]–[10]) have studied normality and hyponormality of the dilation of a truncated Toeplitz operator. In 2018, Gu and Kang [5] gave a complete characterization of self-adjoint, isometric, coisometric and normal truncated singular integral operators. We concentrate on the following questions;

When do the dilation of truncated Toeplitz operators $S^u_{\varphi,\psi}$ become quasinormal?

In this paper, we give necessary and sufficient conditions for the dilation of truncated Toeplitz operators to be quasinormal. As applications for such operators, we investigate the forms of the symbol functions φ and ψ when such operator becomes a quasinormal operator.

2. Main results

In this section, we study the quasinormality of the dilation of truncated Toeplitz operators $S^u_{\varphi,\psi}$ for $\varphi, \psi \in L^\infty$.

LEMMA 1. ([9]) *Let $\varphi \in L^\infty$ and let u be a nonconstant inner function. Then the following statements hold.*

(i) $\Gamma^u_\varphi = 0$ *if and only if* $\varphi \in \mathbb{C}$.
(ii) $\widetilde{\Gamma^u_\varphi} = 0$ *if and only if* $\varphi \in \mathbb{C}$.
(iii) $\widetilde{A^u_\varphi} = 0$ *if and only if* $\varphi = 0$.

LEMMA 2. ([8]) *Let* $\varphi, \psi \in L^\infty$ *and* $\nu = \varphi - \psi$. *Then*

$$[(S^u_{\varphi,\psi})^*, S^u_{\varphi,\psi}] = \begin{pmatrix} \widetilde{\Gamma^u_\varphi}\Gamma^u_{\overline{\varphi}} - \widetilde{\Gamma^u_\psi}\Gamma^u_{\overline{\psi}} & \widetilde{\Gamma^u_\psi}\widetilde{A^u_{\overline{\nu}}} - A^u_\nu\widetilde{\Gamma^u_{\overline{\varphi}}} \\ \widetilde{A^u_\nu}\Gamma^u_{\overline{\psi}} - \Gamma^u_\varphi A^u_{\overline{\nu}} & \Gamma^u_\psi\widetilde{\Gamma^u_{\overline{\psi}}} - \Gamma^u_\varphi\widetilde{\Gamma^u_{\overline{\varphi}}} \end{pmatrix} \tag{7}$$

where $[A,B] = AB - BA$.

LEMMA 3. *Let* $\varphi, \psi \in L^\infty$ *and* $\nu = \varphi - \psi$. *Then* $S^u_{\varphi,\psi}$ *is quasinormal if and only if the following identities hold.*
(i) $(\widetilde{\Gamma^u_\varphi}\Gamma^u_{\overline{\varphi}} - \widetilde{\Gamma^u_\psi}\Gamma^u_{\overline{\psi}})A^u_\varphi + (\widetilde{\Gamma^u_\psi}\widetilde{A^u_{\overline{\nu}}} - A^u_\nu\widetilde{\Gamma^u_{\overline{\varphi}}})\Gamma^u_\varphi = 0$,
(ii) $(\widetilde{\Gamma^u_\varphi}\Gamma^u_{\overline{\varphi}} - \widetilde{\Gamma^u_\psi}\Gamma^u_{\overline{\psi}})\widetilde{\Gamma^u_\psi} + (\widetilde{\Gamma^u_\psi}\widetilde{A^u_{\overline{\nu}}} - A^u_\nu\widetilde{\Gamma^u_{\overline{\varphi}}})\widetilde{A^u_\psi} = 0$,
(iii) $(\widetilde{A^u_\nu}\Gamma^u_{\overline{\psi}} - \Gamma^u_\varphi A^u_{\overline{\nu}})A^u_\varphi + (\Gamma^u_\psi\widetilde{\Gamma^u_{\overline{\psi}}} - \Gamma^u_\varphi\widetilde{\Gamma^u_{\overline{\varphi}}})\Gamma^u_\varphi = 0$, *and*
(iv) $(\widetilde{A^u_\nu}\Gamma^u_{\overline{\psi}} - \Gamma^u_\varphi A^u_{\overline{\nu}})\widetilde{\Gamma^u_\psi} + (\Gamma^u_\psi\widetilde{\Gamma^u_{\overline{\psi}}} - \Gamma^u_\varphi\widetilde{\Gamma^u_{\overline{\varphi}}})\widetilde{A^u_\psi} = 0$.

Proof. Since $S^u_{\varphi,\psi}$ is quasinormal if and only if $[(S^u_{\varphi,\psi})^*, S^u_{\varphi,\psi}]S^u_{\varphi,\psi} = 0$, the proof follows from Lemma 2. □

THEOREM 1. *Let* $\varphi, \psi \in L^\infty$ *and* $\nu = \varphi - \psi$. *Assume that* $\varphi(\mathscr{K}^2_u) \subset (\mathscr{K}^2_u)$ *or* $\psi(\mathscr{K}^2_u)^\perp \subset (\mathscr{K}^2_u)^\perp$. *Then* $S^u_{\varphi,\psi}$ *is quasinormal if and only if* $\varphi \in \mathbb{C}$ *and* $\psi \in \mathbb{C}$. *Furthermore, if* $S^u_{\varphi,\psi}$ *is quasinormal, then the model space* $\mathscr{K}^2_u$ *is a reducing subspace of* $S^u_{\varphi,\psi}$.

Proof. If $\varphi(\mathscr{K}^2_u) \subset (\mathscr{K}^2_u)$, then $\varphi \in \mathbb{C}$ by [9]. If $S^u_{\varphi,\psi}$ is quasinormal, then from Lemma 3, we get that $\widetilde{\Gamma^u_\psi}\Gamma^u_{\overline{\psi}}A^u_\varphi = 0$. For all $f \in \mathscr{K}^2_u$,

$$0 = \widetilde{\Gamma^u_\psi}\Gamma^u_{\overline{\psi}}A^u_\varphi f = \varphi(\Gamma^u_{\overline{\psi}})^*\Gamma^u_{\overline{\psi}}f.$$

Thus $\Gamma^u_{\overline{\psi}} = 0$ on $\mathscr{K}^2_u$ or $\varphi = 0$. By Lemma 1, $\psi \in \mathbb{C}$ or $\varphi = 0$. Conversely, if $\varphi, \psi \in \mathbb{C}$, then $\Gamma^u_\varphi = \widetilde{\Gamma^u_\varphi} = 0$ and $\Gamma^u_\psi = \widetilde{\Gamma^u_\psi} = 0$. Hence $S^u_{\varphi,\psi}$ is quasinormal.

Similarly, if $\psi(\mathscr{K}^2_u)^\perp \subset (\mathscr{K}^2_u)^\perp$, then $\psi \in \mathbb{C}$ by [9]. If $S^u_{\varphi,\psi}$ is quasinormal, then from Lemma 3, we get that $\Gamma^u_\varphi\widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_\psi} = 0$. For all $g \in (\mathscr{K}^2_u)^\perp$,

$$0 = \Gamma^u_\varphi\widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_\psi}g = \psi\Gamma^u_\varphi(\Gamma^u_\varphi)^*g.$$

Thus $\Gamma^u_\varphi = 0$ on $(\mathscr{K}^2_u)^\perp$ or $\psi = 0$. By Lemma 1, $\psi \in \mathbb{C}$ or $\psi = 0$. Conversely, if $\varphi, \psi \in \mathbb{C}$, then $\Gamma^u_\varphi = \widetilde{\Gamma^u_\varphi} = 0$ and $\Gamma^u_\psi = \widetilde{\Gamma^u_\psi} = 0$. Hence $S^u_{\varphi,\psi}$ is quasinormal.

On the other hand, If $S^u_{\varphi,\psi}$ is quasinormal and $\Gamma^u_\varphi = 0$ or $\widetilde{\Gamma^u_\psi} = 0$, then $\varphi \in \mathbb{C}$ and $\psi \in \mathbb{C}$. Thus $\Gamma^u_\varphi = \widetilde{\Gamma^u_\psi} = 0$. Hence the model space $\mathscr{K}^2_u$ reduces $S^u_{\varphi,\psi}$. So, the model space $\mathscr{K}^2_u$ is a reducing subspace of $S^u_{\varphi,\psi}$. □

COROLLARY 1. *Let* $\varphi, \psi \in L^\infty$ *and* $\nu = \varphi - \psi$. *If* $\varphi(\mathscr{K}_u^2) \subset (\mathscr{K}_u^2)$ *or* $\psi(\mathscr{K}_u^2)^\perp \subset (\mathscr{K}_u^2)^\perp$, *then* $S^u_{\varphi,\psi}$ *is quasinormal if and only if* A^u_φ *and* $\widetilde{A^u_\psi}$ *are normal and* $S^u_{\varphi,\psi} = A^u_\varphi \oplus \widetilde{A^u_\psi}$.

Proof. By Theorem 1, $S^u_{\varphi,\psi}$ is quasinormal if and only if $\varphi \in \mathbb{C}$ and $\psi \in \mathbb{C}$. By Lemma 1, $S^u_{\varphi,\psi}$ is quasinormal if and only if $\Gamma^u_\varphi = \widetilde{\Gamma^u_\psi} = 0$. This is equivalent to $S^u_{\varphi,\psi} = A^u_\varphi \oplus \widetilde{A^u_\psi}$ and A^u_φ and $\widetilde{A^u_\psi}$ are normal since $\varphi, \psi \in \mathbb{C}$. □

THEOREM 2. *Let* $\varphi, \psi \in L^\infty$ *and* $\nu = \varphi - \psi$. *If* $\widetilde{A^u_\psi} = 0$, *then* $S^u_{\varphi,\psi}$ *is quasinormal if and only if* $A^u_{|\varphi|^2} A^u_\varphi = A^u_\varphi A^u_{|\varphi|^2}$ *and* $\Gamma^u_\varphi A^u_{|\varphi|^2} = 0$.

Proof. If $\widetilde{A^u_\psi} = 0$, then $\psi = 0$. From Lemma 3, $S^u_{\varphi,\psi}$ is quasinormal if and only if $\widetilde{\Gamma^u_\varphi}\Gamma^u_{\overline{\varphi}} A^u_\varphi - A^u_\varphi \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_\varphi = 0$ and $\Gamma^u_\varphi A^u_{\overline{\varphi}} A^u_\varphi + \Gamma^u_\varphi \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_\varphi = 0$. For any $f \in \mathscr{K}_u^2$,

$$\begin{aligned}
0 &= \widetilde{\Gamma^u_\varphi}\Gamma^u_{\overline{\varphi}} A^u_\varphi f - A^u_\varphi \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_\varphi f \\
&= P_u[\varphi Q_u\{\overline{\varphi} P_u(\varphi f)\} - \varphi P_u\{\overline{\varphi} Q_u(\varphi f)\}] \\
&= P_u[|\varphi|^2 P_u(\varphi f) - \varphi P_u\{\overline{\varphi} P_u(\varphi f)\} - \varphi P_u\{\overline{\varphi} Q_u(\varphi f)\}] \\
&= P_u[|\varphi|^2 P_u(\varphi f) - \varphi P_u(|\varphi|^2 f)] \\
&= [A^u_{|\varphi|^2} A^u_\varphi - A^u_\varphi A^u_{|\varphi|^2}] f.
\end{aligned}$$

Thus $A^u_{|\varphi|^2} A^u_\varphi = A^u_\varphi A^u_{|\varphi|^2}$. Similarly, for any $f \in \mathscr{K}_u^2$, we obtain that

$$\begin{aligned}
0 &= \Gamma^u_\varphi A^u_{\overline{\varphi}} A^u_\varphi f + \Gamma^u_\varphi \widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_\varphi f \\
&= \Gamma^u_\varphi [P_u\{\overline{\varphi} P_u(\varphi f)\} + P_u\{\overline{\varphi} Q_u(\varphi f)\}] \\
&= \Gamma^u_\varphi [P_u\{\overline{\varphi}(\varphi f)\}] \\
&= \Gamma^u_\varphi [P_u(|\varphi|^2 f)] \\
&= \Gamma^u_\varphi A^u_{|\varphi|^2} f.
\end{aligned}$$

Hence $\Gamma^u_\varphi A^u_{|\varphi|^2} = 0$. Therefore $S^u_{\varphi,\psi}$ is quasinormal if and only if $A^u_{|\varphi|^2} A^u_\varphi = A^u_\varphi A^u_{|\varphi|^2}$ and $\Gamma^u_\varphi A^u_{|\varphi|^2} = 0$. □

COROLLARY 2. *Let* $\varphi \in L^\infty$ *be an inner function and* $\psi \in L^\infty$. *If* $\widetilde{A^u_\psi} = 0$, *then* $S^u_{\varphi,\psi}$ *is quasinormal if and only if* $\varphi \in \mathbb{C}$.

Proof. By Theorem 2, $S^u_{\varphi,\psi}$ is quasinormal if and only if $A^u_{|\varphi|^2} A^u_\varphi = A^u_\varphi A^u_{|\varphi|^2}$ and $\Gamma^u_\varphi A^u_{|\varphi|^2} = 0$. Since $|\varphi| = 1$, it follows that $S^u_{\varphi,\psi}$ is quasinormal if and only if $\Gamma^u_\varphi = 0$ if and only if $\varphi \in \mathbb{C}$ by Lemma 1. □

COROLLARY 3. *Let* $\varphi, \psi \in L^\infty$ *and* $\nu = \varphi - \psi$. *Assume that* $\widetilde{A^u_\psi} = 0$. *If* $S^u_{\varphi,\psi}$ *is quasinormal, then* $S^u_{\varphi,\psi} = \begin{pmatrix} A^u_\varphi & 0 \\ \Gamma^u_\varphi & 0 \end{pmatrix}$. *In particular, if* φ *is an inner function, then* $S^u_{\varphi,\psi} = A^u_\varphi \oplus 0$.

Proof. If $\widetilde{A^u_\psi}=0$, then $\psi=0$ and hence $\widetilde{\Gamma^u_\psi}=0$ by Lemma 1. Thus $S^u_{\varphi,\psi}=\begin{pmatrix} A^u_\varphi & 0 \\ \Gamma^u_\varphi & 0 \end{pmatrix}$. In particular, if φ is an inner function, then $\varphi\in\mathbb{C}$ by Corollary 2. Then by Lemma 1, $\Gamma^u_\varphi=0$. Hence $S^u_{\varphi,\psi}=A^u_\varphi\oplus 0$. □

We next give an equivalent statement for quasinormality of the dilation of truncated Toeplitz operators under some conditions.

THEOREM 3. *Let $\varphi,\psi\in L^\infty$ and $\varphi-\lambda\psi=c\in\mathbb{C}$ where $|\lambda|=1$. Then $S^u_{\varphi,\psi}$ is quasinormal if and only if*

$$\begin{pmatrix} 0 & \widetilde{\Gamma^u_{\overline{\Phi}}} \\ \Gamma^u_\Phi & 0 \end{pmatrix} S^u_{\varphi,\psi}=0$$

where $\Phi=(\lambda-1)|\varphi|^2+c\overline{\varphi}-\overline{c}\lambda\varphi$.

Proof. Suppose that $\varphi-\lambda\psi=c\in\mathbb{C}$ where $|\lambda|=1$. Since $\Gamma^u_{\varphi-\lambda\psi}=\widetilde{\Gamma^u_{\varphi-\lambda\psi}}=0$ and $|\lambda|=1$ by [9], we get that

$$\widetilde{\Gamma^u_\varphi}\Gamma^u_{\overline{\varphi}}=|\lambda|^2\widetilde{\Gamma^u_\psi}\Gamma^u_{\overline{\psi}}=\widetilde{\Gamma^u_\psi}\Gamma^u_{\overline{\psi}} \text{ and } \Gamma^u_\psi\widetilde{\Gamma^u_{\overline{\psi}}}=\Gamma^u_\varphi\widetilde{\Gamma^u_{\overline{\varphi}}}.$$

By Lemma 3, $S^u_{\varphi,\psi}$ is quasinormal if and only if the following identities hold;
$(\widetilde{\Gamma^u_\psi}\widetilde{A^u_{\overline{\nu}}}-A^u_\nu\widetilde{\Gamma^u_{\overline{\varphi}}})\Gamma^u_\varphi=0$, $(\widetilde{\Gamma^u_\psi}\widetilde{A^u_{\overline{\nu}}}-A^u_\nu\widetilde{\Gamma^u_{\overline{\varphi}}})\widetilde{A^u_\psi}=0$, $(\widetilde{A^u_\nu}\Gamma^u_{\overline{\psi}}-\Gamma^u_\varphi A^u_{\overline{\nu}})A^u_\varphi=0$, and $(\widetilde{A^u_\nu}\Gamma^u_{\overline{\psi}}-\Gamma^u_\varphi A^u_{\overline{\nu}})\widetilde{\Gamma^u_\psi}=0$, i.e.,

$$\begin{pmatrix} 0 & (\widetilde{A^u_\nu}\Gamma^u_{\overline{\psi}}-\Gamma^u_\varphi A^u_{\overline{\nu}})^* \\ \widetilde{A^u_\nu}\Gamma^u_{\overline{\psi}}-\Gamma^u_\varphi A^u_{\overline{\nu}} & 0 \end{pmatrix}\begin{pmatrix} A^u_\varphi & \widetilde{\Gamma^u_\psi} \\ \Gamma^u_\varphi & \widetilde{A^u_\psi} \end{pmatrix}=0 \tag{8}$$

where $\nu=\varphi-\psi$. On the other hand, for any $f\in\mathcal{K}^2_u$, we get that

$$\begin{aligned}
&(\widetilde{A^u_\nu}\Gamma^u_{\overline{\psi}}-\Gamma^u_\varphi A^u_{\overline{\nu}})f\\
&=Q_u[(\varphi-\psi)Q_u(\overline{\psi}f)-\varphi P_u((\overline{\varphi}-\overline{\psi})f)]\\
&=Q_u[(\lambda\psi+c-\psi)Q_u(\overline{\psi}f)-(\lambda\psi+c)P_u((\overline{\lambda\psi+c}-\overline{\psi})f)]\\
&=Q_u[((\lambda-1)\psi+c)Q_u(\overline{\psi}f)-\lambda\psi P_u(((\overline{\lambda}-1)\overline{\psi}+\overline{c})f)]\\
&=Q_u[((\lambda-1)\psi Q_u(\overline{\psi}f)+cQ_u(\overline{\psi}f)-\lambda(\overline{\lambda}-1)\psi P_u(\overline{\psi}f)-\lambda\overline{c}\psi P_uf]\\
&=(\lambda-1)Q_u[\psi Q_u(\overline{\psi}f)]+cQ_u(\overline{\psi}f)-(|\lambda|^2-\lambda)Q_u[\psi P_u(\overline{\psi}f)]-\lambda\overline{c}Q_u[\psi P_uf]\\
&=(\lambda-1)Q_u[|\psi|^2f]+cQ_u(\overline{\psi}f)-\lambda\overline{c}Q_u[\psi P_uf]\\
&=Q_u[(\lambda-1)|\psi|^2+c\overline{\psi}-\lambda\overline{c}\psi]f.
\end{aligned}$$

Since $\varphi=\lambda\psi+c$, we have

$$(\lambda-1)|\psi|^2+c\overline{\psi}-\lambda\overline{c}\psi=(\lambda-1)|\varphi|^2+c\overline{\varphi}-\lambda\overline{c}\varphi.$$

Hence we obtain that

$$\begin{aligned}
(\widetilde{A^u_\nu}\Gamma^u_{\overline{\psi}}-\Gamma^u_\varphi A^u_{\overline{\nu}})f&=Q_u[(\lambda-1)|\psi|^2+c\overline{\psi}-\lambda\overline{c}\psi]f\\
&=Q_u[(\lambda-1)|\varphi|^2+c\overline{\varphi}-\lambda\overline{c}\varphi]f=\Gamma^u_\Phi f
\end{aligned}$$

where $\Phi=(\lambda-1)|\varphi|^2+c\overline{\varphi}-\lambda\overline{c}\varphi$. Therefore we know from (8) that $S^u_{\varphi,\psi}$ is quasinormal if and only if

$$\begin{pmatrix} 0 & (\Gamma^u_\Phi)^* \\ \Gamma^u_\Phi & 0 \end{pmatrix} S^u_{\varphi,\psi}=0$$

where $\Phi=(\lambda-1)|\varphi|^2+c\overline{\varphi}-\overline{c}\lambda\varphi$. Since $(\Gamma^u_\Phi)^*=\widetilde{\Gamma^u_{\overline{\Phi}}}$, we complete the proof. $\square$

COROLLARY 4. *Let $\varphi,\psi\in L^\infty$ and $\varphi-\lambda\psi=c\in\mathbb{C}$ where $|\lambda|=1$. Then the following statements hold.*

(i) $S^u_{\varphi,\psi}$ *is quasinormal.*

(ii) $S^u_{\varphi,\psi}$ *is normal.*

(iii) $\Phi:=(\lambda-1)|\varphi|^2+c\overline{\varphi}-\overline{c}\lambda\varphi\in\mathbb{C}$.

Proof. (i) $\Rightarrow$ (ii): Set $D=\begin{pmatrix} 0 & \widetilde{\Gamma^u_{\overline{\Phi}}} \\ \Gamma^u_\Phi & 0 \end{pmatrix}$ where $\Phi=(\lambda-1)|\varphi|^2+c\overline{\varphi}-\overline{c}\lambda\varphi$. If $S^u_{\varphi,\psi}$ is quasinormal, then we get from Theorem 3 that $DS^u_{\varphi,\psi}=0$. Hence $D(\overline{\operatorname{ran}(S^u_{\varphi,\psi})})=\{0\}$. Since $L^2=\ker(S^u_{\varphi,\psi})^*+\overline{\operatorname{ran}(S^u_{\varphi,\psi})}$, we want to show that $D\ker(S^u_{\varphi,\psi})^*=0$. For any $s\in\ker(S^u_{\varphi,\psi})$ and any $t\in\overline{\operatorname{ran}(S^u_{\varphi,\psi})}$, we know that $\langle Dt,t\rangle=0$, $\langle Ds,t\rangle=\langle s,Dt\rangle=0$, and $\langle Dt,s\rangle=0$. Hence for any $x,y\in L^2$, write $x=s_x+t_x$ and $y=s_y+t_y$ for any $s_x,s_y\in\ker(S^u_{\varphi,\psi})^*$ and $t_x,t_y\in\overline{\operatorname{ran}(S^u_{\varphi,\psi})}$. Then $\langle Dx,y\rangle=\langle Ds_x,s_y\rangle=0$. So it suffices to show that $\langle Ds_x,s_y\rangle=0$. If $x=y$, then

$$\begin{aligned}\langle Dx,x\rangle &= \langle Ds_x,s_x\rangle \\ &= \langle (S^u_{\varphi,\psi})^*S^u_{\varphi,\psi}s_x,s_x\rangle-\langle S^u_{\varphi,\psi}(S^u_{\varphi,\psi})^*s_x,s_x\rangle \\ &= \langle (S^u_{\varphi,\psi})^*S^u_{\varphi,\psi}s_x,s_x\rangle=\|S^u_{\varphi,\psi}s_x\|^2\geqslant 0.\end{aligned}$$

Hence $D\geqslant 0$. So by the positivity of D(see [14]), there exists a contraction S such that $\Gamma^u_\Phi=0^{\frac12}S0^{\frac12}=0$. Hence $\Phi\in\mathbb{C}$ by Lemma 1 and $D=0$. Thus, in this case, $S^u_{\varphi,\psi}$ is normal. The converse statement (ii) $\Rightarrow$ (i) is trivial. The statement (ii) $\Leftrightarrow$ (iii) follows from [9]. $\square$

EXAMPLE 1. *If $\varphi(z)=z^n-e^{i\theta}$ for $\theta\in\mathbb{R}$, $\psi(z)=z^n$, and $\lambda=1$, then $\varphi-\lambda\psi=-e^{i\theta}=c\in\mathbb{C}$. Since*

$$\begin{aligned}\Phi &= (\lambda-1)|\varphi|^2+c\overline{\varphi}-\overline{c}\lambda\varphi \\ &= -e^{i\theta}(\overline{z}^n-e^{-i\theta})+e^{-i\theta}(z^n-e^{i\theta}) \\ &= -e^{i\theta}\overline{z}^n-e^{-i\theta}z^n\notin\mathbb{C},\end{aligned}$$

we get from Corollary 4 that $S^u_{\varphi,\psi}$ is not quasinormal.

COROLLARY 5. *Let $\varphi\in L^\infty$ be such that $\varphi\mathscr{K}^2_u\subset\mathscr{K}^2_u$, $\psi\in L^\infty$, and $\varphi-\lambda\psi\in\mathbb{C}$ where $|\lambda|=1$. Then $S^u_{\varphi,\psi}$ is quasinormal if and only if $S^u_{\varphi,\psi}$ is normal.*

Proof. Since $\varphi\mathscr{K}_u^2 \subset \mathscr{K}_u^2$, it follows from [9] that $\Gamma_\varphi^u = 0$ and so $\varphi \in \mathbb{C}$. If $S_{\varphi,\psi}^u$ is quasinormal, then

$$\begin{pmatrix} 0 & \widetilde{\Gamma_{\overline{\Phi}}^u} \\ \Gamma_\Phi^u & 0 \end{pmatrix} S_{\varphi,\psi}^u = 0$$

by Theorem 3 where $\Phi = (\lambda-1)|\varphi|^2 + c\overline{\varphi} - \overline{c}\lambda\varphi$. Therefore,

$$0 = \begin{pmatrix} 0 & \widetilde{\Gamma_{\overline{\Phi}}^u} \\ \Gamma_\Phi^u & 0 \end{pmatrix} \begin{pmatrix} A_\varphi^u & \widetilde{\Gamma_\psi^u} \\ 0 & \widetilde{A_\psi^u} \end{pmatrix} = \begin{pmatrix} 0 & \widetilde{\Gamma_\Phi^u}\widetilde{A_\psi^u} \\ \Gamma_\Phi^u A_\varphi^u & \Gamma_\Phi^u \widetilde{\Gamma_\psi^u} \end{pmatrix}.$$

Thus $\widetilde{\Gamma_\Phi^u}\widetilde{A_\psi^u} = 0$, $\Gamma_\Phi^u A_\varphi^u = 0$, and $\Gamma_\Phi^u\widetilde{\Gamma_\psi^u} = 0$. Since $\widetilde{\Gamma_\Phi^u}\widetilde{A_\psi^u} = 0$ and $\varphi \in \mathbb{C}$, it follows that for all $f \in \mathscr{K}_u^2$,

$$0 = \widetilde{\Gamma_\Phi^u}\widetilde{A_\psi^u} f = Q_u[\Phi P_u(\varphi f)] = Q_u[\Phi\varphi f] = \varphi Q_u(\Phi f) = \varphi\Gamma_\Phi f.$$

Hence $\Gamma_\Phi = 0$ or $\varphi = 0$. By Lemma 1 or [9], $\Phi \in \mathbb{C}$. By Corollary 4, $S_{\varphi,\psi}^u$ is normal. The converse statement is obvious. □

COROLLARY 6. *Let $\varphi \in L^\infty$ be such that $\varphi\mathscr{K}_u^2 \subset \mathscr{K}_u^2$, $\psi \in L^\infty$, and $\varphi - \lambda\psi \in \mathbb{C}$ where $|\lambda| = 1$. If $S_{\varphi,\psi}^u$ is quasinormal, then A_φ^u and $\widetilde{A_\psi^u}$ are normal and $S_{\varphi,\psi}^u = A_\varphi^u \oplus \widetilde{A_\psi^u}$.*

Proof. By Corollary 5, $S_{\varphi,\psi}^u$ is normal. Hence the proof follows from [9, Theorem 5.5]. □

Finally, we give another characterization of a quasinormal dilation of truncated Toeplitz operators $S_{\varphi,\psi}^u$.

THEOREM 4. *Let $\varphi, \psi \in L^\infty$. Then $S_{\varphi,\psi}^u$ is quasinormal if and only if the following identities hold.*

(i) $[A_{|\varphi|^2}^u, A_\varphi^u] = \widetilde{\Gamma_\psi^u}\Gamma_{\varphi\overline{\psi}}^u - \widetilde{\Gamma_{\overline{\varphi}\psi}^u}\Gamma_\varphi^u$,

(ii) $A_{|\varphi|^2}^u\widetilde{\Gamma_\psi^u} - \widetilde{\Gamma_\psi^u}\widetilde{A_{|\psi|^2}^u} = A_\varphi^u\widetilde{\Gamma_{\overline{\varphi}\psi}^u} - \widetilde{\Gamma_{\overline{\varphi}\psi}^u}\widetilde{A_\psi^u}$,

(iii) $\widetilde{A_{|\psi|^2}^u}\Gamma_\varphi^u - \Gamma_\varphi^u A_{|\varphi|^2}^u = \widetilde{A_\psi^u}\Gamma_{\varphi\overline{\psi}}^u - \Gamma_{\varphi\overline{\psi}}^u A_\varphi^u$, *and*

(iv) $[\widetilde{A_{|\psi|^2}^u}, \widetilde{A_\psi^u}] = \Gamma_\varphi^u\widetilde{\Gamma_{\overline{\varphi}\psi}^u} - \Gamma_{\varphi\overline{\psi}}^u\widetilde{\Gamma_\psi^u}$.

Proof. Set $v = \varphi - \psi$. Assume that $S_{\varphi,\psi}^u$ is quasinormal. From Lemma 3-(i), we get that for any $f \in \mathscr{K}_u^2$,

$$\begin{aligned} 0 &= (\widetilde{\Gamma_\varphi^u}\Gamma_{\overline{\varphi}}^u - \widetilde{\Gamma_\psi^u}\Gamma_{\overline{\psi}}^u)A_\varphi^u f + (\widetilde{\Gamma_\psi^u}\widetilde{A_{\overline{v}}^u} - A_v^u\widetilde{\Gamma_{\overline{\varphi}}^u})\Gamma_\varphi^u f \\ &= P_u[\varphi Q_u(\overline{\varphi}A_\varphi^u f) - \psi Q_u(\overline{\psi}A_\varphi^u f) + \psi\{P_u(\overline{\varphi}\Gamma_\varphi^u f) + Q_u(\overline{\varphi}\Gamma_\varphi^u f) - Q_u(\overline{\psi}\Gamma_\varphi^u f)\} \\ &\quad - \varphi P_u(\overline{\varphi}\Gamma_\varphi^u f)] \\ &= P_u[\varphi Q_u\{\overline{\varphi}P_u(\varphi f)\} - \psi Q_u\{\overline{\psi}P_u(\varphi f)\} + \overline{\varphi}\psi Q_u(\varphi f) \\ &\quad - \psi Q_u\{\overline{\psi}Q_u(\varphi f)\} - \varphi P_u\{\overline{\varphi}Q_u(\varphi f)\}] \end{aligned}$$

$$
\begin{aligned}
&= P_u[\varphi Q_u\{\overline{\varphi}P_u(\varphi f)\} - \psi Q_u\{\varphi\overline{\psi}f\} + \overline{\varphi}\psi Q_u(\varphi f) - \varphi P_u\{\overline{\varphi}Q_u(\varphi f)\}] \\
&= P_u[|\varphi|^2 P_u(\varphi f) - \varphi P_u\{\overline{\varphi}P_u(\varphi f)\} - \psi Q_u\{\varphi\overline{\psi}f\} + \overline{\varphi}\psi Q_u(\varphi f) - \varphi P_u\{\overline{\varphi}Q_u(\varphi f)\}] \\
&= P_u[|\varphi|^2 P_u(\varphi f) - \varphi P_u\{|\varphi|^2 f\} - \psi Q_u\{\varphi\overline{\psi}f\} + \overline{\varphi}\psi Q_u(\varphi f)] \\
&= [A^u_{|\varphi|^2}A^u_\varphi - A^u_\varphi A^u_{|\varphi|^2} - \widetilde{\Gamma^u_\psi}\Gamma^u_{\varphi\overline{\psi}} + \widetilde{\Gamma^u_{\overline{\varphi}\psi}}\Gamma^u_\varphi]f.
\end{aligned}
$$

Hence we complete the proof for (i). From Lemma 3-(ii), we get that for any $g \in (\mathscr{K}_u^2)^\perp$,

$$
\begin{aligned}
0 &= (\widetilde{\Gamma^u_\varphi}\Gamma^u_{\overline{\varphi}} - \widetilde{\Gamma^u_\psi}\Gamma^u_{\overline{\psi}})\widetilde{\Gamma^u_\psi}g + (\widetilde{\Gamma^u_\psi}\widetilde{A^u_{\overline{\nu}}} - A^u_\nu\widetilde{\Gamma^u_{\overline{\varphi}}})\widetilde{A^u_\psi}g \\
&= P_u[\varphi Q_u(\overline{\varphi}\widetilde{\Gamma^u_\psi}g) - \psi Q_u(\overline{\psi}\widetilde{\Gamma^u_\psi}g)] + P_u[\psi Q_u(\overline{\nu}\widetilde{A^u_\psi}g) - \nu P_u(\overline{\varphi}\widetilde{A^u_\psi}g)] \\
&= P_u[\varphi Q_u(\overline{\varphi}P_u\psi g) - \psi Q_u(\overline{\psi}P_u\psi g)] + P_u[\psi Q_u(\overline{\nu}Q_u\psi g) - \nu P_u(\overline{\varphi}Q_u\psi g)] \\
&= P_u[\varphi Q_u(\overline{\varphi}P_u\psi g) - \psi Q_u(\overline{\psi}P_u\psi g) + \psi Q_u(\overline{\varphi}Q_u\psi g) - \psi Q_u(\overline{\psi}Q_u(\psi g)) \\
&\quad -\varphi P_u(\overline{\varphi}Q_u\psi g) + \psi P_u(\overline{\varphi}Q_u\psi g)] \\
&= P_u[\varphi Q_u(\overline{\varphi}P_u\psi g) - \psi Q_u(|\psi|^2 g) + \overline{\varphi}\psi Q_u(\psi g) - \varphi P_u\{\overline{\varphi}Q_u(\psi g)\}] \\
&= P_u[|\varphi|^2 P_u(\psi g) - \varphi P_u\{\overline{\varphi}P_u(\psi g)\} - \psi Q_u(|\psi|^2 g) + \overline{\varphi}\psi Q_u(\psi g) - \varphi P_u\{\overline{\varphi}Q_u(\psi g)\}] \\
&= P_u[|\varphi|^2 P_u(\psi g) - \varphi P_u(\overline{\varphi}\psi g) - \psi Q_u(|\psi|^2 g) + \overline{\varphi}\psi Q_u(\psi g)] \\
&= (A^u_{|\varphi|^2}\widetilde{\Gamma^u_\psi} - A^u_\varphi\widetilde{\Gamma^u_{\overline{\varphi}\psi}} - \widetilde{\Gamma^u_\psi}\widetilde{A^u_{|\psi|^2}} + \widetilde{\Gamma^u_{\overline{\varphi}\psi}}\widetilde{A^u_\psi})g.
\end{aligned}
$$

Hence (ii) is proved. From Lemma 3-(iii), we get that for any $f \in \mathscr{K}_u^2$,

$$
\begin{aligned}
0 &= [(\widetilde{A^u_\nu}\Gamma^u_{\overline{\psi}} - \Gamma^u_\varphi A^u_{\overline{\nu}})A^u_\varphi + (\Gamma^u_\psi\widetilde{\Gamma^u_{\overline{\psi}}} - \Gamma^u_\varphi\widetilde{\Gamma^u_{\overline{\varphi}}})\Gamma^u_\varphi]f \\
&= Q_u[\nu Q_u\{\overline{\psi}P_u(\varphi f)\} - \varphi P_u\{\overline{\nu}P_u(\varphi f)\} + \psi P_u\{\overline{\psi}Q_u(\varphi f)\} - \varphi P_u\{\overline{\varphi}Q_u(\varphi f)\}] \\
&= Q_u[\varphi\overline{\psi}P_u(\varphi f) - \psi Q_u\{\overline{\psi}P_u(\varphi f)\} - \varphi P_u\{|\varphi|^2\} + |\psi|^2 Q_u(\varphi f) - \psi Q_u\{\overline{\psi}Q_u(\varphi f)\}] \\
&= Q_u[\varphi\overline{\psi}P_u(\varphi f) - \psi Q_u\{\overline{\psi}\varphi f\} - \varphi P_u\{|\varphi|^2\} + |\psi|^2 Q_u(\varphi f)] \\
&= (\Gamma^u_{\varphi\overline{\psi}}A^u_\varphi - \widetilde{A^u_\psi}\Gamma^u_{\varphi\overline{\psi}} - \Gamma^u_\varphi A^u_{|\varphi|^2} + \widetilde{A^u_{|\psi|^2}}\Gamma^u_\varphi)f.
\end{aligned}
$$

From Lemma 3-(iv), we get that for any $g \in (\mathscr{K}_u^2)^\perp$,

$$
\begin{aligned}
0 &= [(\widetilde{A^u_\nu}\Gamma^u_{\overline{\psi}} - \Gamma^u_\varphi A^u_{\overline{\nu}})\widetilde{\Gamma^u_\psi} + (\Gamma^u_\psi\widetilde{\Gamma^u_{\overline{\psi}}} - \Gamma^u_\varphi\widetilde{\Gamma^u_{\overline{\varphi}}})\widetilde{A^u_\psi}]g \\
&= Q_u[\nu Q_u\{\overline{\psi}P_u(\psi g)\} - \varphi P_u\{\overline{\nu}P_u(\psi g)\} + \psi P_u(\overline{\psi}Q_u(\psi g)) - \varphi P_u\{\overline{\varphi}Q_u(\psi g)\}] \\
&= Q_u[\varphi\overline{\psi}P_u(\psi g) - \psi Q_u\{\overline{\psi}P_u(\psi g)\} - \varphi P_u\{\overline{\varphi}\psi g\} + \psi P_u\{\overline{\psi}Q_u(\psi g)\}] \\
&= Q_u[\varphi\overline{\psi}P_u(\psi g) - \psi Q_u\{\overline{\psi}P_u(\psi g)\} - \varphi P_u\{\overline{\varphi}\psi g\} + |\psi|^2 Q_u(\psi g) - \psi Q_u\{\overline{\psi}Q_u(\psi g)\}] \\
&= Q_u[\varphi\overline{\psi}P_u(\psi g) - \psi Q_u\{|\psi|^2 g\} - \varphi P_u\{\overline{\varphi}\psi g\} + |\psi|^2 Q_u(\psi g)] \\
&= (\Gamma^u_{\varphi\overline{\psi}}\widetilde{\Gamma^u_\psi} - \widetilde{A^u_\psi}\widetilde{A^u_{|\psi|^2}} - \Gamma^u_\varphi\widetilde{\Gamma^u_{\overline{\varphi}\psi}} + \widetilde{A^u_{|\psi|^2}}\widetilde{A^u_\psi})g.
\end{aligned}
$$

So we complete the proof. $\square$

As some consequences of Theorem 4, we obtain the following corollaries.

COROLLARY 7. *Let $\varphi,\psi\in L^\infty$ such that $\overline{\varphi}\psi=c\in\mathbb{C}$ with $c\neq 0$. Then $S^u_{\varphi,\psi}$ is quasinormal if and only if $A^u_{|\varphi|^2}A^u_\varphi=A^u_\varphi A^u_{|\varphi|^2}$, $A^u_{|\varphi|^2}\widetilde{\Gamma^u_\psi}=\widetilde{\Gamma^u_\psi}\widetilde{A^u_{|\psi|^2}}$, $\widetilde{A^u_{|\psi|^2}}\Gamma^u_\varphi=\Gamma^u_\varphi A^u_{|\varphi|^2}$, and $\widetilde{A^u_{|\psi|^2}}\widetilde{A^u_\psi}=\widetilde{A^u_\psi}\widetilde{A^u_{|\psi|^2}}$ hold.*

Proof. If $\overline{\varphi}\psi=c\in\mathbb{C}$ where $c\neq 0$, then we get from Lemma 1 that $\Gamma^u_{\varphi\overline{\psi}}=\widetilde{\Gamma^u_{\overline{\varphi}\psi}}=0$. Hence the proof follows from Theorem 4. □

COROLLARY 8. *Let $\varphi,\psi\in L^\infty$ be such that $|\varphi|^2=|\psi|^2=k$ for some constant k and $\overline{\varphi}\psi\in\mathbb{C}$. Then $S^u_{\varphi,\psi}$ is quasinormal. Moreover, if $k\neq 1$, then $S^u_{\varphi,\psi}$ is not an isometry.*

Proof. Since $\overline{\varphi}\psi\in\mathbb{C}$, $\Gamma^u_{\varphi\overline{\psi}}=\widetilde{\Gamma^u_{\overline{\varphi}\psi}}=0$ by Lemma 1 and so $A^u_kA^u_\varphi=A^u_\varphi A^u_k$, $A^u_k\widetilde{\Gamma^u_\psi}=\widetilde{\Gamma^u_\psi}\widetilde{A^u_k}$, and $\widetilde{A^u_k}\Gamma^u_\varphi=\Gamma^u_\varphi A^u_k$, and $\widetilde{A^u_k}\widetilde{A^u_\psi}=\widetilde{A^u_\psi}\widetilde{A^u_k}$ hold. Hence $S^u_{\varphi,\psi}$ is quasinormal from Corollary 7, but $S^u_{\varphi,\psi}$ is not an isometry. In fact, note that $S^u_{\varphi,\psi}$ is an isometry if and only if $(S^u_{\varphi,\psi})^*S^u_{\varphi,\psi}=\begin{pmatrix}A^u_{\overline{\varphi}}A^u_\varphi+\widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_\varphi & A^u_{\overline{\varphi}}\widetilde{\Gamma^u_\psi}+\widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_\psi}\\ \Gamma^u_{\overline{\psi}}A^u_\varphi+\widetilde{A^u_{\overline{\psi}}}\Gamma^u_\varphi & \Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_\psi}+\widetilde{A^u_{\overline{\psi}}}\widetilde{A^u_\psi}\end{pmatrix}=I$, which means that

$$\begin{cases}A^u_{\overline{\varphi}}A^u_\varphi+\widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_\varphi=I\\ A^u_{\overline{\varphi}}\widetilde{\Gamma^u_\psi}+\widetilde{\Gamma^u_{\overline{\varphi}}}\widetilde{A^u_\psi}=0,\ \text{and}\\ \Gamma^u_{\overline{\psi}}\widetilde{\Gamma^u_\psi}+\widetilde{A^u_{\overline{\psi}}}\widetilde{A^u_\psi}=I.\end{cases}$$

Therefore for any $f\in\mathscr{K}^2_u$,

$$\begin{aligned}f&=(A^u_{\overline{\varphi}}A^u_\varphi+\widetilde{\Gamma^u_{\overline{\varphi}}}\Gamma^u_\varphi)f=A^u_{\overline{\varphi}}P_u(\varphi f)+\widetilde{\Gamma^u_{\overline{\varphi}}}Q_u(\varphi f)\\&=P_u[\overline{\varphi}\{P_u(\varphi f)+Q_u(\varphi f)\}]=P_u[\overline{\varphi}(\varphi f)]=P_u(|\varphi|^2f).\end{aligned}$$

Then for any $f\in\mathscr{K}^2_u$, $P_u[(|\varphi|^2-1)f]=(|\varphi|^2-1)f=0$ implies that $|\varphi|=1$, which is a contradiction. So $S^u_{\varphi,\psi}$ is not an isometry. □

EXAMPLE 2. *Let $\varphi(z)=2z^n$ and $\psi(z)=2\lambda z^n$ where $|\lambda|=1$. Then $|\varphi|^2=4=|\psi|^2$ and $\overline{\varphi}\psi=4\overline{z}^n\lambda z^n=4\lambda\in\mathbb{C}$. Hence $S^u_{2z^n,2\lambda z^n}$ is quasinormal, but it is not an isometry.*

COROLLARY 9. *Let $\varphi,\psi\in L^\infty$ be nonconstant inner functions. If $S^u_{\varphi,\psi}$ is quasinormal, then*

$$\widetilde{\Gamma^u_\psi}\Gamma^u_{\varphi\overline{\psi}}=\widetilde{\Gamma^u_{\overline{\varphi}\psi}}\Gamma^u_\varphi,\Gamma^u_\varphi\widetilde{\Gamma^u_{\overline{\varphi}\psi}}=\Gamma^u_{\varphi\overline{\psi}}\widetilde{\Gamma^u_\psi},\ \textit{and}\ A^u_{\varphi-\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}}=\widetilde{\Gamma^u_{\overline{\varphi}\psi}}\widetilde{A^u_{\psi-\overline{\psi}}}. \tag{9}$$

Conversely, if equations (9) hold, then $A^u_{|\varphi|^2}A^u_\varphi=A^u_\varphi A^u_{|\varphi|^2}$ and $\widetilde{A^u_{|\psi|^2}}\widetilde{A^u_\psi}=\widetilde{A^u_\psi}\widetilde{A^u_{|\psi|^2}}$.

Proof. Since $A^u_{|\varphi|^2} = I$ and $\widetilde{A^u_{|\psi|^2}} = I$, we have that the first and second equations hold from Theorem 4-(i) and (iv). We also get from Theorem 4-(ii) and (iii) that $A^u_\varphi \widetilde{\Gamma^u_{\overline{\varphi}\psi}} - \widetilde{\Gamma^u_{\overline{\varphi}\psi}} \widetilde{A^u_\psi} = 0$ and $\widetilde{A^u_\psi}\Gamma^u_{\varphi\overline{\psi}} - \Gamma^u_{\varphi\overline{\psi}} A^u_\varphi = 0$. If we take the adjoint of $\widetilde{A^u_\psi}\Gamma^u_{\varphi\overline{\psi}} - \Gamma^u_{\varphi\overline{\psi}} A^u_\varphi = 0$, then $\widetilde{\Gamma^u_{\overline{\varphi}\psi}}\widetilde{A^u_{\overline{\psi}}} - A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}} = 0$. Hence we obtain that

$$A^u_{\varphi-\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}} - \widetilde{\Gamma^u_{\overline{\varphi}\psi}}\widetilde{A^u_{\psi-\overline{\psi}}} = [A^u_\varphi\widetilde{\Gamma^u_{\overline{\varphi}\psi}} - \widetilde{\Gamma^u_{\overline{\varphi}\psi}}\widetilde{A^u_\psi}] + [\widetilde{\Gamma^u_{\overline{\varphi}\psi}}\widetilde{A^u_{\overline{\psi}}} - A^u_{\overline{\varphi}}\widetilde{\Gamma^u_{\overline{\varphi}\psi}}] = 0.$$

Conversely, if equations (9) hold, then $A^u_{|\varphi|^2}A^u_\varphi = A^u_\varphi A^u_{|\varphi|^2}$ and $\widetilde{A^u_{|\psi|^2}}\widetilde{A^u_\psi} = \widetilde{A^u_\psi}\widetilde{A^u_{|\psi|^2}}$. So we complete the proof. □

As an application of Theorem 4, we want to find ψ for which $S^u_{\varphi,\psi}$ becomes quasinormal when $\varphi \in L^\infty$ and u are given.

COROLLARY 10. *Let $\varphi, \psi \in L^\infty$ and let $u(z) = zB(z)$ where $B(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$ for $\alpha \in \mathbb{D}$. If $\varphi(z) = d\overline{B(z)}$ for some nonzero constant d, then $S^u_{\varphi,\psi}$ is quasinormal if and only if $\psi(z) = c\overline{B(z)}$ for some constant c with $|c| = |d|$.*

Proof. Assume that $u(z) = zB(z)$ for $\alpha \in \mathbb{D}$. If $\psi(z) = c\overline{B(z)}$ for some constant c with $|c| = |d|$, then $|\varphi|^2 = |\overline{B}|^2 = |d|^2$ and $|\psi|^2 = |c|^2$. Thus $A^u_{|\varphi|^2} = |d|^2 = |c|^2 = \widetilde{A^u_{|\psi|^2}}$. Since $\varphi\overline{\psi} = \overline{cB}dB = \overline{c}d$ is constant, it follows from Lemma 1 that $\widetilde{\Gamma^u_{\overline{\varphi}\psi}} = 0 = \Gamma^u_{\varphi\overline{\psi}}$. Hence $S^u_{\varphi,\psi}$ is quasinormal by Theorem 4.

Conversely, assume that $S^u_{\varphi,\psi}$ is quasinormal. We know that $1, B$ are orthogonal to uH^2 and $\dim \mathscr{K}^2_u = \text{degree}(u)$. Since $u(z) = zB(z)$, $\dim(\vee\{1,B\}) = 2$. Hence $\mathscr{K}^2_u = \vee\{1,B\}$. Since $A^u_{|\varphi|^2} = |d|^2 I$, by Theorem 4-(i), $\widetilde{\Gamma^u_\psi}\Gamma^u_{d\overline{B\psi}} - \widetilde{\Gamma^u_{\overline{dB}\psi}}\Gamma^u_{d\overline{B}} = 0$. Since $\mathscr{K}^2_u = \vee\{1,B\}$, it follows that $\Gamma^u_{d\overline{B}}B = Q_u(d) = 0$ and so

$$0 = \widetilde{\Gamma^u_\psi}\Gamma^u_{d\overline{B\psi}}B - \widetilde{\Gamma^u_{\overline{dB}\psi}}\Gamma^u_{d\overline{B}}B = \widetilde{\Gamma^u_\psi}Q_u(d\overline{\psi}) = dP_u[\psi Q_u(\overline{\psi})].$$

Thus $\psi Q_u(\overline{\psi}) \in (\mathscr{K}^2_u)^\perp$ and so $\langle\overline{\psi}, Q_u(\overline{\psi})\rangle = \langle 1, \psi Q_u(\overline{\psi})\rangle = 0$. Therefore $\overline{\psi} \in \mathscr{K}^2_u = \vee\{1,B\}$. So, $\overline{\psi} = a + bB = a + \frac{b}{d}\overline{\varphi}$ for some constant a, b. From Theorem 4-(iii),

$$\widetilde{A^u_{|\psi|^2}}\Gamma^u_\varphi(B) - \Gamma^u_\varphi A^u_{|\varphi|^2}(B) = \widetilde{A^u_\psi}\Gamma^u_{\varphi\overline{\psi}}(B) - \Gamma^u_{\varphi\overline{\psi}}A^u_\varphi(B). \tag{10}$$

Since $\varphi = d\overline{B}$, it follows that $\Gamma^u_\varphi(B) = \Gamma^u_{d\overline{B}}(B) = Q_u(d) = 0$ and $\Gamma^u_\varphi A^u_{|\varphi|^2}(B) = 0$. Thus we get that (10) implies

$$\begin{aligned} 0 &= \widetilde{A^u_\psi}Q_u(d\overline{\psi}) - \Gamma^u_{d\overline{B\psi}}P_u(d) \\ &= Q_u[\psi Q_u(d\overline{\psi}) - d^2\overline{B\psi}] \\ &= -Q_u[d^2\overline{B}(a+bB)] \\ &= -d^2Q_u(a\overline{B}+b) = -ad^2\overline{B}. \end{aligned}$$

Therefore $a = 0$ and so $\overline{\psi} = bB$. Hence $\psi = c\overline{B}$ where $c = \overline{b}$. □

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