

LOGARITHMICALLY COMPLETE MONOTONICITY OF A MATRIX-PARAMETRIZED ANALOGUE OF THE MULTINOMIAL DISTRIBUTION

FRÉDÉRIC OUIMET* AND FENG QI

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Abstract. In the paper, the authors introduce a matrix-parametrized generalization of the multinomial probability mass function that involves a ratio of several multivariate gamma functions. They show the logarithmically complete monotonicity of this generalization and derive several inequalities involving ratios of multivariate gamma functions.

1. Preliminaries

Recall from [9, Chapter XIII], [22, Chapter 1], and [23, Chapter IV], that an infinitely differentiable function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in I$ and $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. Recall from [16, Definition 1] and [22, Definition 5.10] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if

$$(-1)^n \frac{d^n}{dx^n} \ln f(x) \geq 0$$

for all $n \in \mathbb{N} = \{1, 2, \dots\}$ and $x \in I$. The property of being logarithmically completely monotonic is stronger than being completely monotonic, see [3, Theorem 1.1], [16, Theorem 1], and [18, p. 627, (1.4)]. When $I = (0, \infty)$, Bernstein's theorem (see, e.g., Theorem 12b in [23, p. 161]) states that a function f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xs} d\sigma(s) \tag{1}$$

and the integral converges for all $x \in (0, \infty)$, where $\sigma(s)$ is nondecreasing on $(0, \infty)$. The integral representation (1) equivalently says that a function f is completely monotonic on $(0, \infty)$ if and only if it is the Laplace transform of $\sigma(s)$ on $(0, \infty)$. This is one

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* Corresponding author.

of many reasons why researchers have been investigating (logarithmically) completely monotonic functions.

The literature on this topic is far too extensive to cite here, but one specific kind of (logarithmically) completely monotonic functions, functions involving ratios of gamma functions, has attracted a lot of attention lately. Recent contributions for this type of completely monotonic functions include [2, 4, 6, 13, 15, 17, 20, 21, 19, 24].

Let $m, r \in \mathbb{N}$ and let \mathbf{M} be a matrix of order m . We use the notation $\mathbf{M} \geq 0$ (or $\mathbf{M} > 0$, respectively) to indicate that the matrix \mathbf{M} is positive semidefinite (or positive definite, respectively). According to [5, Definition 6.2.1], the r -tuple of real symmetric matrices, $(\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_r)$, is said to have the (type I) $(m \times m)$ -matrix-variate Dirichlet distribution with parameters $(a_1, a_2, \dots, a_{r+1}) \in (0, \infty)^{r+1}$ if its density function is given by

$$\frac{\Gamma_m(\sum_{i=1}^{r+1} a_i)}{\prod_{i=1}^{r+1} \Gamma_m(a_i)} \left| \mathbf{I}_m - \sum_{i=1}^r \mathbf{M}_i \right|^{a_{r+1} - (m+1)/2} \prod_{i=1}^r |\mathbf{M}_i|^{a_i - (m+1)/2},$$

when $\mathbf{M}_i \geq 0$ for all $1 \leq i \leq r$ and $\mathbf{I}_m - \sum_{i=1}^r \mathbf{M}_i \geq 0$, and is equal to 0 otherwise, where

$$\begin{aligned} \Gamma_m(z) &= \int_{\mathbf{S} \in \text{Sym}_m(\mathbb{R}) : \mathbf{S} > 0} |\mathbf{S}|^{z - (m+1)/2} \exp(-\text{tr}(\mathbf{S})) \, d\mathbf{S} \\ &= \pi^{m(m-1)/4} \prod_{j=1}^m \Gamma\left(z - \frac{j-1}{2}\right), \quad \Re(z) > \frac{m-1}{2} \end{aligned} \tag{2}$$

denotes the multivariate gamma function, see [12, Section 35.3] and [10], and

$$\Gamma(z) = \lim_{\ell \rightarrow \infty} \frac{\ell! \ell^z}{\prod_{k=0}^{\ell} (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

is the classical gamma function.

Given the ties between the Dirichlet and multinomial distributions, a function that naturally generalizes the multinomial probability mass function is the following matrix-parametrized analogue

$$P_{n, \mathbf{k}_r}(\mathbf{M}_1, \dots, \mathbf{M}_r) = \frac{\Gamma_m\left(n + \frac{m+1}{2}\right)}{\prod_{i=1}^{r+1} \Gamma_m\left(k_i + \frac{m+1}{2}\right)} \left| \mathbf{I}_m - \sum_{i=1}^r \mathbf{M}_i \right|^{n - \|\mathbf{k}_r\|_1} \prod_{i=1}^r |\mathbf{M}_i|^{k_i},$$

where $n \in \mathbb{N}$, $\mathbf{k}_r = (k_1, k_2, \dots, k_r) \in \mathbb{N}_0^r \cap (n, \mathcal{S}_r)$, $k_{r+1} = n - \|\mathbf{k}_r\|_1$, the matrices \mathbf{M}_i , as well as $\mathbf{I}_m - \sum_{i=1}^r \mathbf{M}_i$, are assumed to be symmetric and positive definite, and

$$\mathcal{S}_r = \left\{ \mathbf{x}_r = (x_1, x_2, \dots, x_r) \in [0, 1]^r : \|\mathbf{x}_r\|_1 = \sum_{\ell=1}^r x_\ell \leq 1 \right\}$$

denotes the r -dimensional unit simplex.

In this paper, we will show that $x \mapsto P_{x, n, \mathbf{k}_r}(\mathbf{M}_1, \dots, \mathbf{M}_r)$ is a logarithmically completely monotonic function on $(0, \infty)$. Hereafter, in Section 4, we will derive some inequalities involving ratios of multivariate gamma functions.

2. A lemma

To reach our goal, we need to prove the following technical lemma.

LEMMA 1. Let \mathcal{S}_r° denote the interior of \mathcal{S}_r , $\mathbf{u}_r = (u_1, u_2, \dots, u_r) \in \mathcal{S}_r^\circ$, and $u_{r+1} = 1 - \|\mathbf{u}_r\|_1 > 0$. Then, for $\beta \geq 0$ and $y > 1$,

$$\mathcal{J}_{\beta, \mathbf{u}_r}(y) = \frac{1}{y^\beta (y-1)} - \sum_{i=1}^{r+1} \frac{1}{y^{\beta/u_i} (y^{1/u_i} - 1)} > 0. \tag{3}$$

First proof. For $y > 1$, $\beta \geq 0$, and $u \in (0, \infty)$, let

$$H_{y, \beta}(u) = \frac{1}{y^{\beta/u} (y^{1/u} - 1)}. \tag{4}$$

Then straightforward calculations yield

$$\begin{aligned} \frac{d^2}{du^2} H_{y, \beta}(u) &= \frac{\ln y}{u^4 (y^{1/u} - 1)^3 y^{\beta/u}} \{ [\beta^2 (y^{1/u} - 1)^2 + 2\beta (y^{1/u} - 1) y^{1/u} \\ &\quad + (y^{1/u} + 1) y^{1/u}] \ln y - 2u (y^{1/u} - 1) [\beta (y^{1/u} - 1) + y^{1/u}] \} \\ &= \frac{\ln y}{u^3 (y^{1/u} - 1)^3 y^{\beta/u}} \{ [\beta^2 (t - 1)^2 + 2\beta (t - 1)t + (t + 1)t] \ln t \\ &\quad - 2(t - 1)[\beta (t - 1) + t] \} \\ &\triangleq \frac{\ln y}{u^3 (y^{1/u} - 1)^3 y^{\beta/u}} h_\beta(t), \end{aligned}$$

where $t = y^{1/u} > 1$. Direct computations show us that

$$\begin{aligned} h'_\beta(t) &= \frac{(t - 1)[\beta^2(t - 1) - 2\beta t - 3t]}{t} + [2\beta^2(t - 1) + \beta(4t - 2) + 2t + 1] \ln t, \\ h''_\beta(t) &= \frac{(t - 1)[\beta^2(3t + 1) + 2\beta t - t]}{t^2} + 2(\beta + 1)^2 \ln t, \\ h'''_\beta(t) &= \frac{2\beta^2(t^2 + t + 1) + 2\beta t(2t + 1) + t(2t - 1)}{t^3} > 0, \end{aligned}$$

and

$$\lim_{t \rightarrow 1^+} h''_\beta(t) = \lim_{t \rightarrow 1^+} h'_\beta(t) = \lim_{t \rightarrow 1^+} h_\beta(t) = 0.$$

Accordingly, for any fixed $y > 1$ and $\beta \geq 0$, the second derivative $H''_{y, \beta}(u)$ is positive on $(0, \infty)$. Hence, the function $u \mapsto H_{y, \beta}(u)$ is strictly convex on $(0, \infty)$ and the function $u \mapsto H_{y, \beta}(u) + H_{y, \beta}(1 - u)$ is strictly convex on $(0, 1)$. From the limits

$$\lim_{u \rightarrow 0^+} H_{y, \beta}(u) = 0 \quad \text{and} \quad \lim_{u \rightarrow 1^-} H_{y, \beta}(u) = \frac{1}{y^\beta (y - 1)},$$

we conclude that

$$H_{y,\beta}(u) + H_{y,\beta}(1 - u) < \frac{1}{y^\beta(y - 1)},$$

which is equivalent to

$$\frac{1}{y^\beta(y - 1)} - \frac{1}{y^{\beta/\|\mathbf{u}_r\|_1}(y^{1/\|\mathbf{u}_r\|_1} - 1)} - \frac{1}{y^{\beta/(1-\|\mathbf{u}_r\|_1)}[y^{1/(1-\|\mathbf{u}_r\|_1)} - 1]} > 0. \tag{5}$$

The general case will follow by induction. Indeed, assume that, for some integer $r \geq 2$, the inequality

$$\frac{1}{z^\beta(z - 1)} - \sum_{i=1}^r \frac{1}{z^{\beta/v_i}(z^{1/v_i} - 1)} > 0 \tag{6}$$

is valid, where $\beta \geq 0$, $z > 1$, $(v_1, v_2, \dots, v_{r-1}) \in \mathcal{S}_{r-1}^\circ$, and $v_r = 1 - \sum_{i=1}^{r-1} v_i > 0$. On the other hand, we can rewrite the function $\mathcal{J}_{\beta, \mathbf{u}_r}(y)$ defined in (3) as

$$\begin{aligned} \mathcal{J}_{\beta, \mathbf{u}_r}(y) = & \left[\frac{1}{y^{\beta/\|\mathbf{u}_r\|_1}(y^{1/\|\mathbf{u}_r\|_1} - 1)} - \sum_{i=1}^r \frac{1}{y^{\beta/u_i}(y^{1/u_i} - 1)} \right] \\ & + \left[\frac{1}{y^\beta(y - 1)} - \frac{1}{y^{\beta/\|\mathbf{u}_r\|_1}(y^{1/\|\mathbf{u}_r\|_1} - 1)} - \frac{1}{y^{\beta/(1-\|\mathbf{u}_r\|_1)}[y^{1/(1-\|\mathbf{u}_r\|_1)} - 1]} \right]. \end{aligned}$$

By the inequality (5), the quantity in the second bracket is positive. By the induction hypothesis (6) with $z = y^{1/\|\mathbf{u}_r\|_1}$ and $v_i = \frac{u_i}{\|\mathbf{u}_r\|_1}$, the quantity in the first bracket is positive. This ends the first proof of Lemma 1. \square

Second proof. A function $\varphi(x)$ is said to be super-additive on an interval I if the inequality $\varphi(x + y) \geq \varphi(x) + \varphi(y)$ holds for all $x, y \in I$ with $x + y \in I$. A function $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is said to be star-shaped if $\varphi(vt) \leq v\varphi(t)$ for $v \in [0, 1]$ and $t \geq 0$. See [8, Chapter 16] and [11, Section 3.4]. Between convex functions, star-shaped functions, and super-additive functions, there are the following relations:

1. if φ is convex on $[0, \infty)$ with $\varphi(0) \leq 0$, then φ is star-shaped;
2. if $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is star-shaped, then φ is super-additive.

See [8, pp. 650–651, Section B.9]. Shortly speaking, if φ is convex on $[0, \infty)$ with $\varphi(0) \leq 0$, then φ is super-additive.

For $y > 1$, $\beta \geq 0$, and $u \in (-\infty, \infty)$, define

$$\varphi_{y,\beta}(u) = \begin{cases} 0, & u = 0; \\ \frac{1}{y^{\beta/u}(y^{1/u} - 1)}, & u \neq 0. \end{cases}$$

From the facts that the function $H_{y,\beta}(u)$ defined in (4) is strictly convex on $(0, \infty)$, that the limit

$$\lim_{u \rightarrow 0^+} H_{y,\beta}(u) = 0$$

is valid, and that $\varphi_{y,\beta}(u) = H_{y,\beta}(u)$ for $u > 0$, we conclude that the function $u \mapsto \varphi_{y,\beta}(u)$ is convex on $[0, \infty)$ with $\varphi_{y,\beta}(0) = 0$. This means that the function $u \in [0, \infty) \mapsto \varphi_{y,\beta}(u)$ is star-shaped and super-additive. Accordingly, it follows that

$$\sum_{i=1}^{r+1} \varphi_{y,\beta}(u_i) \leq \varphi_{y,\beta}\left(\sum_{i=1}^{r+1} u_i\right) = \varphi_{y,\beta}(1),$$

that is,

$$\sum_{i=1}^{r+1} \frac{1}{y^{\beta/u_i}(y^{1/u_i} - 1)} \leq \frac{1}{y^\beta(y - 1)}.$$

The inequality (3) is thus proved. The second proof of Lemma 1 is complete. \square

Third proof (due to Gérard Letac). Since $u_i > 0$ for $1 \leq i \leq r+1$ and $\sum_{i=1}^{r+1} u_i = 1$, it is sufficient to show that

$$\frac{1}{y^{\beta/u_i}(y^{1/u_i} - 1)} < \frac{u_i}{y^\beta(y - 1)}, \quad 1 \leq i \leq r+1.$$

By writing

$$s_i = \frac{1}{u_i} \quad \text{and} \quad g(s_i) = \frac{s_i}{y^{s_i \beta}(y^{s_i} - 1)},$$

it is sufficient to show that

$$g(s) < g(1), \quad s > 1.$$

But this readily follows from

$$\frac{d}{ds} \ln g(s) = \left[\frac{1}{\ln(y^s)} - \frac{y^s}{y^s - 1} - \beta \right] \ln y = \left(\frac{1}{x} - \frac{1}{1 - e^{-x}} - \beta \right) \ln y < 0,$$

where we made the substitution $x = \ln(y^s) > 0$ for $s, y > 1$ and we used the inequality $e^{-x} > 1 - x$ for $x \in (0, \infty)$. The third proof of Lemma 1 is complete. \square

3. Logarithmically complete monotonicity

Now we are in a position to state and prove our main result.

THEOREM 1. *Let $m, n, r \in \mathbb{N}$, $\boldsymbol{\alpha}_r = (\alpha_1, \alpha_2, \dots, \alpha_r) \in n \mathcal{S}_r^\circ$, $\alpha_{r+1} = n - \|\boldsymbol{\alpha}_r\|_1 > 0$, and let \mathbf{M}_i for $0 \leq i \leq r$ be symmetric matrices of order m , satisfying $\mathbf{M}_i \geq 0$ and $\mathbf{M}_{r+1} = \mathbf{I}_m - \sum_{i=1}^r \mathbf{M}_i \geq 0$. Then the function*

$$\mathcal{Q}(x) = \frac{\Gamma_m(xn + \frac{m+1}{2})}{\prod_{i=1}^{r+1} \Gamma_m(x\boldsymbol{\alpha}_i + \frac{m+1}{2})} \prod_{i=1}^{r+1} |\mathbf{M}_i|^{x\boldsymbol{\alpha}_i} \tag{7}$$

is logarithmically completely monotonic on $(0, \infty)$.

Proof. Without loss of generality, we can assume that $\mathbf{M}_i > 0$ for all $0 \leq i \leq r + 1$. By taking the logarithm on both sides of the equation (7), we have

$$\ln \mathcal{Q}(x) = \ln \Gamma_m \left(xn + \frac{m+1}{2} \right) - \sum_{i=1}^{r+1} \ln \Gamma_m \left(x\alpha_i + \frac{m+1}{2} \right) + x \sum_{i=1}^{r+1} \alpha_i \ln |\mathbf{M}_i|.$$

Denote

$$\psi_m(z) = \frac{d}{dz} \ln \Gamma_m(z) = \sum_{j=1}^m \psi \left(z - \frac{j-1}{2} \right).$$

Then a direct differentiation gives

$$\begin{aligned} [\ln \mathcal{Q}(x)]' &= n\psi_m \left(xn + \frac{m+1}{2} \right) - \sum_{i=1}^{r+1} \alpha_i \psi_m \left(x\alpha_i + \frac{m+1}{2} \right) + \sum_{i=1}^{r+1} \alpha_i \ln |\mathbf{M}_i| \\ &= \sum_{j=1}^m \left\{ n\psi \left(xn + \frac{m-j}{2} + 1 \right) - \sum_{i=1}^{r+1} \alpha_i \psi \left(x\alpha_i + \frac{m-j}{2} + 1 \right) \right\} \\ &\quad + \sum_{i=1}^{r+1} \alpha_i \ln |\mathbf{M}_i|. \end{aligned} \tag{8}$$

Using the special case $\ell = 1$ of the integral representation

$$\psi^{(\ell)}(z) = (-1)^{\ell+1} \int_0^\infty \frac{t^\ell e^{-zt}}{1 - e^{-t}} dt, \quad \Re(z) > 0, \quad \ell \in \mathbb{N},$$

listed in [1, p. 260, 6.4.1], we obtain

$$\begin{aligned} [\ln \mathcal{Q}(x)]'' &= \sum_{j=1}^m \left\{ n^2 \psi' \left(xn + \frac{m-j}{2} + 1 \right) - \sum_{i=1}^{r+1} \alpha_i^2 \psi' \left(x\alpha_i + \frac{m-j}{2} + 1 \right) \right\} \\ &= \sum_{j=1}^m \left\{ n^2 \int_0^\infty \frac{t}{e^t - 1} \exp \left[- \left(xn + \frac{m-j}{2} \right) t \right] dt \right. \\ &\quad \left. - \sum_{i=1}^{r+1} \alpha_i^2 \int_0^\infty \frac{t}{e^t - 1} \exp \left[- \left(x\alpha_i + \frac{m-j}{2} \right) t \right] dt \right\} \\ &= \sum_{j=1}^m \left\{ \int_0^\infty \frac{se^{-xs}}{e^{s/n} - 1} \exp \left(- \frac{m-j}{2n} s \right) ds \right. \\ &\quad \left. - \sum_{i=1}^{r+1} \int_0^\infty \frac{se^{-xs}}{e^{s/\alpha_i} - 1} \exp \left(- \frac{m-j}{2\alpha_i} s \right) ds \right\} \\ &= \sum_{j=1}^m \int_0^\infty \mathcal{I}_{(m-j)/2, \alpha_r/n} (e^{s/n}) s e^{-xs} ds, \end{aligned}$$

where $\mathcal{I}_{\beta, \mathbf{u}_r}(y)$ is defined in (3). Applying Lemma 1 yields

$$(-1)^{n+1} \frac{d^{n+1}}{dx^{n+1}} \ln \mathcal{Q}(x) = \sum_{j=1}^m \int_0^\infty \mathcal{I}_{(m-j)/2, \alpha_r/n} (e^{s/n}) s^n e^{-xs} ds > 0 \tag{9}$$

for $n \in \mathbb{N}$ and $x > 0$.

To prove the logarithmically complete monotonicity of $\mathcal{Q}(x)$ on $(0, \infty)$, it remains to verify that the positivity in (9) is also valid for $n = 0$. From the positivity of (9) for $n = 1$, it follows that the function $x \mapsto [-\ln \mathcal{Q}(x)]'$ is decreasing on $(0, \infty)$. Therefore, it is sufficient to show that $\lim_{x \rightarrow \infty} [-\ln \mathcal{Q}(x)]' \geq 0$.

For $k \in \mathbb{N}_0$ and $a \geq 0$, the first result in [14, Lemma 2.4] reads as

$$\lim_{x \rightarrow \infty} (x^{k+1} [\psi^{(k)}(x+a) - \psi^{(k)}(x)]) = (-1)^k k! a. \tag{10}$$

Hence, the case $k = 0$ and $a = \frac{1}{2}$ in (10) implies the asymptotic formula

$$\psi\left(z + \frac{1}{2}\right) \sim \psi(z) + \frac{1}{2z}, \quad z \rightarrow \infty.$$

Therefore, we have

$$\begin{aligned} \psi\left(xn + \frac{m-j}{2} + 1\right) &\sim \psi\left(xn + \frac{m-j+1}{2}\right) + \frac{1}{2n\left(x + \frac{m-j+1}{2n}\right)} \\ &\sim \psi\left(xn + \frac{m-j}{2}\right) + \frac{1}{2n\left(x + \frac{m-j}{2n}\right)} + \frac{1}{2n\left(x + \frac{m-j+1}{2n}\right)} \end{aligned}$$

as $x \rightarrow \infty$. By induction, we obtain

$$\psi\left(xn + \frac{m-j}{2} + 1\right) \sim \psi(xn) + \frac{1}{2n} \sum_{\ell=0}^{m-j+1} \frac{1}{x + \frac{\ell}{2n}}, \quad x \rightarrow \infty. \tag{11}$$

Similarly, we derive

$$\psi\left(x\alpha_i + \frac{m-j}{2} + 1\right) \sim \psi(x\alpha_i) + \frac{1}{2\alpha_i} \sum_{\ell=0}^{m-j+1} \frac{1}{x + \frac{\ell}{2\alpha_i}}, \quad x \rightarrow \infty. \tag{12}$$

Substituting (11) and (12) into (8) and simplifying lead to

$$\begin{aligned} [-\ln \mathcal{Q}(x)]' &\sim m \left[\sum_{i=1}^{r+1} \alpha_i \psi(x\alpha_i) - n \psi(xn) \right] - \sum_{i=1}^{r+1} \alpha_i \ln |\mathbf{M}_i| \\ &\quad + \frac{1}{2} \sum_{j=1}^m \sum_{\ell=0}^{m-j+1} \left(\sum_{i=1}^{r+1} \frac{1}{x + \frac{\ell}{2\alpha_i}} - \frac{1}{x + \frac{\ell}{2n}} \right), \quad x \rightarrow \infty. \end{aligned}$$

Using the asymptotic formula

$$\psi(z) \sim \ln z - \frac{1}{2z}, \quad z \rightarrow \infty,$$

derived from [1, p. 259, 6.3.18], we conclude that

$$\begin{aligned} [-\ln \mathcal{Q}(x)]' &\sim m \left[\sum_{i=1}^{r+1} \alpha_i \ln(x\alpha_i) - n \ln(xn) \right] - \sum_{i=1}^{r+1} \alpha_i \ln |\mathbf{M}_i| \\ &\quad - \frac{mr}{2x} + \frac{1}{2} \sum_{j=1}^m \sum_{\ell=0}^{m-j+1} \left(\sum_{i=1}^{r+1} \frac{1}{x + \frac{\ell}{2\alpha_i}} - \frac{1}{x + \frac{\ell}{2n}} \right), \quad x \rightarrow \infty. \end{aligned}$$

Consequently, we have

$$\begin{aligned}
 \lim_{x \rightarrow \infty} [-\ln \mathcal{Q}(x)]' &= mn \left[\sum_{i=1}^{r+1} \frac{\alpha_i}{n} \ln \frac{\alpha_i}{n} - \sum_{i=1}^{r+1} \frac{\alpha_i}{n} \ln \left(|\mathbf{M}_i|^{1/m} \right) \right] \\
 &= -mn \sum_{i=1}^{r+1} \frac{\alpha_i}{n} \ln \left(\frac{|\mathbf{M}_i|^{1/m}}{\alpha_i/n} \right) \\
 &\geq -mn \ln \left(\sum_{i=1}^{r+1} |\mathbf{M}_i|^{1/m} \right) \\
 &\geq -mn \ln \left(\left| \sum_{i=1}^{r+1} \mathbf{M}_i \right|^{1/m} \right) \\
 &= -mn \ln (|\mathbf{I}_m|^{1/m}) \\
 &= 0,
 \end{aligned} \tag{13}$$

where we used Jensen’s inequality for the concave function $\ln x$ on $(0, \infty)$, see [9, Chapter I, p. 6], and Minkowski’s determinant inequality for the symmetric positive semidefinite matrices \mathbf{M}_i of order m , see [7] or [9, Chapter VIII, p. 214, Theorem 2]. This ends the proof of Theorem 1. \square

A trivial modification at the end of the last proof yields the following corollary.

COROLLARY 1. *Let $m, n, r \in \mathbb{N}$, $\boldsymbol{\alpha}_r = (\alpha_1, \alpha_2, \dots, \alpha_r) \in n\mathcal{S}_r^\circ$, $\alpha_{r+1} = n - \|\boldsymbol{\alpha}_r\|_1 > 0$, and let $(p_1, p_2, \dots, p_r) \in \mathcal{S}_r$ with $p_{r+1} = 1 - \sum_{i=1}^r p_i$. Then the function*

$$\tilde{\mathcal{Q}}(x) = \frac{\Gamma_m \left(xn + \frac{m+1}{2} \right)}{\prod_{i=1}^{r+1} \Gamma_m \left(x\alpha_i + \frac{m+1}{2} \right)} \prod_{i=1}^{r+1} p_i^{x\alpha_i} \tag{14}$$

is logarithmically completely monotonic on $(0, \infty)$.

Proof. In the proof of Theorem 1, replace $|\mathbf{M}_i|$ everywhere by p_i . On the third line of (13), simply use Jensen’s inequality,

$$\sum_{i=1}^{r+1} p_i^{1/m} \leq \left(\sum_{i=1}^{r+1} p_i \right)^{1/m} = 1^{1/m} = 1, \tag{15}$$

to obtain the desired conclusion. \square

4. Inequalities involving ratios of multivariate gamma functions

Considering the factor involving the gamma functions in (7) and making use of the formula (2), we denote and write

$$R(x) = \frac{\Gamma_m \left(xn + \frac{m+1}{2} \right)}{\prod_{i=1}^{r+1} \Gamma_m \left(x\alpha_i + \frac{m+1}{2} \right)} = \frac{\prod_{j=1}^m \Gamma \left(xn + \frac{m-j}{2} + 1 \right)}{\pi^{rm(m-1)/4} \prod_{i=1}^{r+1} \prod_{j=1}^m \Gamma \left(x\alpha_i + \frac{m-j}{2} + 1 \right)}.$$

We are now ready to derive several inequalities for $R(x)$ involving ratios of multivariate gamma functions Γ_m .

THEOREM 2. *Let $\ell \in \mathbb{N}$ and let $(\lambda_1, \dots, \lambda_\ell) \in (0, \infty)^\ell$ such that $\sum_{k=1}^\ell \lambda_k = 1$.*

1. *For $(x_1, \dots, x_\ell) \in (0, \infty)^\ell$, we have*

$$R\left(\sum_{k=1}^\ell \lambda_k x_k\right) \leq \prod_{k=1}^\ell [R(x_k)]^{\lambda_k}, \tag{16}$$

where the equality holds if and only if $x_1 = x_2 = \dots = x_k$.

2. *For $(x_1, \dots, x_\ell) \in (0, \infty)^\ell$, we have*

$$\prod_{k=1}^\ell R(x_k) < R\left(\sum_{k=1}^\ell x_k\right). \tag{17}$$

3. *For $\ell = 3$ and $(x_1, x_2, x_3) \in (0, \infty)^3$, if $x_1 \leq x_3$, then*

$$R(x_1 + x_2)R(x_3) \leq R(x_1)R(x_2 + x_3), \tag{18}$$

where the equality holds if and only if $x_1 = x_3$.

Proof. The logarithmically complete monotonicity in Theorem 1 implies that the function \mathcal{Q} is logarithmically convex on $(0, \infty)$. Consequently, we obtain

$$\begin{aligned} \mathcal{Q}\left(\sum_{k=1}^\ell \lambda_k x_k\right) &= R\left(\sum_{k=1}^\ell \lambda_k x_k\right) \prod_{i=1}^{r+1} |\mathbf{M}_i|^{\alpha_i \sum_{k=1}^\ell \lambda_k x_k} \\ &\leq \prod_{k=1}^\ell [\mathcal{Q}(x_k)]^{\lambda_k} = \prod_{k=1}^\ell [R(x_k)]^{\lambda_k} \prod_{k=1}^\ell \left[\prod_{i=1}^{r+1} |\mathbf{M}_i|^{x_k \alpha_i} \right]^{\lambda_k} \end{aligned}$$

which can be simplified as (16). Due to the logarithmic convexity, it is trivial to see that the equality in (16) holds if and only if $x_1 = x_2 = \dots = x_k$.

Lemma 3 in [2] states that, if $h : [0, \infty) \rightarrow (0, 1]$ is differentiable and the logarithmic derivative $\frac{h'(x)}{h(x)} = [\ln h(x)]'$ is strictly increasing on $(0, \infty)$, then the strict inequality $h(x)h(y) < h(x+y)$ is valid for $x, y \in (0, \infty)$. By the way, we notice that this lemma is a special case $\varphi(x) = \ln h(x)$ of the result concluded in the first paragraph in the second proof of Lemma 1 in this paper. From this, we can inductively derive

$$\prod_{j=1}^\ell h(x_j) < h\left(\sum_{j=1}^\ell x_j\right). \tag{19}$$

The logarithmically complete monotonicity in Theorem 1 implies that $\mathcal{Q}(x)$ is decreasing on $(0, \infty)$ and that the logarithmic derivative $\frac{\mathcal{Q}'(x)}{\mathcal{Q}(x)} = [\ln \mathcal{Q}(x)]'$ is strictly increasing on $(0, \infty)$. Since

$$\lim_{x \rightarrow 0^+} \mathcal{Q}(x) = \frac{\Gamma_m\left(\frac{m+1}{2}\right)}{\prod_{i=1}^{r+1} \Gamma_m\left(\frac{m+1}{2}\right)} = \frac{1}{\left[\Gamma_m\left(\frac{m+1}{2}\right)\right]^r}$$

and

$$\begin{aligned} \Gamma_m\left(\frac{m+1}{2}\right) &= \pi^{(m-1)m/4} \prod_{j=1}^m \Gamma\left(1 + \frac{m-j}{2}\right) \\ &= \pi^{(m-1)m/4} \prod_{k=0}^{m-1} \Gamma\left(1 + \frac{k}{2}\right) \\ &\geq \pi^{(m-1)m/4} \geq 1 \end{aligned}$$

for $m \in \mathbb{N}$, from the decreasing property of $\mathcal{Q}(x)$ on $(0, \infty)$, we deduce that

$$0 < \mathcal{Q}(0) \triangleq \lim_{x \rightarrow 0^+} \mathcal{Q}(x) \leq 1.$$

Consequently, by applying the inequality (19) to $\mathcal{Q}(x)$, we have

$$\begin{aligned} \prod_{j=1}^{\ell} \mathcal{Q}(x_j) &= \prod_{j=1}^{\ell} \left[R(x_j) \prod_{i=1}^{r+1} |\mathbf{M}_i|^{x_j \alpha_i} \right] \\ &= \prod_{j=1}^{\ell} R(x_j) \prod_{i=1}^{r+1} |\mathbf{M}_i|^{\alpha_i \sum_{j=1}^{\ell} x_j} \\ &< \mathcal{Q}\left(\sum_{j=1}^{\ell} x_j\right) \\ &= R\left(\sum_{j=1}^{\ell} x_j\right) \prod_{i=1}^{r+1} |\mathbf{M}_i|^{\alpha_i \sum_{j=1}^{\ell} x_j}, \end{aligned}$$

which can be reformulated as the inequality (17).

As done in the proof of Theorem 3.5 in [20], the inequality (18) and the equality case follow by using the logarithmically complete monotonicity of the function $\mathcal{Q}(x)$ in Theorem 1 and by adapting the proof of Corollary 3 in [2]. The proof of Theorem 2 is complete. \square

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Frédéric Oumet
Department of Mathematics and Statistics
McGill University
Montreal, QC H3A 0B9, Canada
and
Division of Physics, Mathematics and Astronomy
California Institute of Technology
Pasadena, CA 91125, USA
e-mail: frederic.ouimet2@mcgill.ca

Feng Qi
Institute of Mathematics
Henan Polytechnic University
Jiaozuo 454003, Henan, China
and
School of Mathematical Sciences
Tiangong University
Tianjin 300387, China
e-mail: qifeng618@gmail.com