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A NEW CHARACTERIZATION OF DIFFERENCES OF GENERALIZED WEIGHTED COMPOSITION OPERATORS FROM THE BLOCH SPACE INTO WEIGHTED–TYPE SPACES

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Abstract. In this paper, we give a new characterization for the boundedness and compactness of differences of generalized weighted composition operators from the Bloch space into weighted-type spaces. Moreover, we give some estimates for the essential norm of these operators.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. We denote by $S(\mathbb{D})$ the set of analytic self-map of $\mathbb{D}$. For $a \in \mathbb{D}$, let σ_a be the automorphism of $\mathbb{D}$ exchanging 0 for a, i.e., $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$. For z, $w \in \mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z,w) = |\sigma_w(z)| = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

It is well known that $\rho(z,w) \leqslant 1$.

Throughout this paper, every self-map φ induces a linear composition operator C_φ which is defined on $H(\mathbb{D})$ by $C_\varphi(f)(z) = f(\varphi(z))$, $f \in H(\mathbb{D})$, $z \in \mathbb{D}$. Let $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. The weighted composition operator, denoted by uC_φ, is defined as follows.

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

Let n be a nonnegative integer. Let $f^{(n)}$ denote the n-th derivative of f and $f^{(0)} = f$. A linear operator $D^n_{\varphi,u}$ is defined by

$$(D^n_{\varphi,u} f)(z) = u(z)f^{(n)}(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

The operator $D^n_{\varphi,u}$ is called the generalized weighted composition operator. In fact, if $n = 0$ and $u(z) = 1$, then $D^n_{\varphi,u}$ is the operator C_φ. If $u(z) = 1$, then $D^n_{\varphi,u}$ is the

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operator $C_\varphi D^n$, which was studied, for example, in [3, 17, 25]. If $n=0$, then $D^n_{\varphi,u}$ is just the operator uC_φ. If $n=1$ and $u(z)=\varphi'(z)$, then $D^n_{\varphi,u}=DC_\varphi$, which was studied in [3, 6, 7, 8, 9, 17, 18, 22]. The operator $D^n_{\varphi,u}$ was introduced by Zhu in [29], and studied in [5, 10, 19, 20, 21, 24, 29, 30, 31].

Let $0<\beta<\infty$. An $f\in H(\mathbb{D})$ is said to belong to the β-Bloch space, denoted by $\mathscr{B}^\beta$, if

$$\|f\|_{\mathscr{B}^\beta}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f'(z)|<\infty.$$

$\mathscr{B}^\beta$ is a Banach space under the norm $\|\cdot\|_{\mathscr{B}^\beta}$. When $\beta=1$, we write $\mathscr{B}^1$ by $\mathscr{B}$, which is called the Bloch space. We say that an $f\in H(\mathbb{D})$ belongs to the little Bloch space, denoted by $\mathscr{B}_0$, if $\lim_{|z|\to1}|f'(z)|(1-|z|^2)=0$.

Let $0<\alpha<\infty$. The weighted-type space, denoted by H^∞_α, is defined as follows.

$$H^\infty_\alpha=\{f\in H(\mathbb{D}):\|f\|_{H^\infty_\alpha}=\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)|<\infty\}.$$

Let X and Y be Banach spaces. The essential norm of a bounded linear operator $T:X\to Y$ is its distance to the set of compact operators $K:X\to Y$, that is, $\|T\|_{e,X\to Y}=\inf\{\|T-K\|:K \text{ is compact }\}$.

It is well known that $C_\varphi:\mathscr{B}\to\mathscr{B}$ is bounded for any φ by Schwarz-Pick Lemma. The compactness and essential norm of the operator $C_\varphi:\mathscr{B}\to\mathscr{B}$ were studied by many authors (see, e.g., [2, 13, 23, 26, 27]). In particular, Wulan, Zheng and Zhu [26] proved that $C_\varphi:\mathscr{B}\to\mathscr{B}$ is compact if and only if $\lim_{j\to\infty}\|\varphi^j\|_{\mathscr{B}}=0$. Zhao in [27] showed that $\|C_\varphi\|_{e,\mathscr{B}\to\mathscr{B}}=\frac{e}{2}\limsup_{j\to\infty}\|\varphi^j\|_{\mathscr{B}}$.

Many researchers have studied the differences of two composition operators on various function spaces in recent 20 years. See [1, 15] for more information of this study. It is easy to see that the operator $C_\varphi-C_\psi:\mathscr{B}\to\mathscr{B}$ is bounded for any φ and ψ. See [4] and [14] for the study of compact differences of composition operators on the Bloch space. Recently, Shi and Li obtained some estimates for the essential norm of the operator $C_\varphi-C_\psi:\mathscr{B}\to\mathscr{B}$ in [16]. Among others, they showed that

$$\|C_\varphi-C_\psi\|_{e,\mathscr{B}\to\mathscr{B}}\approx\limsup_{j\to\infty}\|\varphi^j-\psi^j\|_{\mathscr{B}}=\limsup_{j\to\infty}\|(C_\varphi-C_\psi)p_j\|_{\mathscr{B}}.$$

Here $p_j(z)=z^j$.

In [31], Zhu studied the boundedness and compactness of $D^n_{\varphi,u}:\mathscr{B}\to H^\infty_\alpha$. See [10] for more characterizations of the operator $D^n_{\varphi,u}:\mathscr{B}\to H^\infty_\alpha$. In [11], Liu and Li studied the operator $D^n_{\varphi,u}-D^n_{\psi,v}:\mathscr{B}\to H^\infty_\alpha$. Among others, they showed that $D^n_{\varphi,u}-D^n_{\psi,v}:\mathscr{B}\to H^\infty_\alpha$ is compact if and only if $D^n_{\varphi,u}-D^n_{\psi,v}:\mathscr{B}\to H^\infty_\alpha$ is bounded and the following equalities hold.

$$\lim_{|\varphi(z)|\to1}\big|M_{u,\varphi}(z)\big|\rho(\varphi(z),\psi(z))=0; \tag{1}$$

$$\lim_{|\psi(z)|\to1}\big|M_{v,\psi}(z)\big|\rho(\varphi(z),\psi(z))=0; \tag{2}$$

$$\lim_{|\varphi(z)|\to 1, |\psi(z)|\to 1} |M_{u,\varphi}(z) - M_{v,\psi}(z)| = 0. \tag{3}$$

Here

$$M_{u,\varphi}(z) = \frac{(1-|z|^2)^\alpha u(z)}{(1-|\varphi(z)|^2)^n}, \qquad M_{v,\psi}(z) = \frac{(1-|z|^2)^\alpha v(z)}{(1-|\psi(z)|^2)^n}. \tag{4}$$

The present paper, motivated by [11, 16], gives a new characterization of the boundedness and compactness of the operator $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B} \to H^\infty_\alpha$. Moreover, we give some estimates for the essential norm of the operator $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B} \to H^\infty_\alpha$.

For two quantities A and B which may depend on φ and ψ, we use the abbreviation $A \lesssim B$ whenever there is a positive constant c such that $A \leqslant cB$. We write $A \approx B$, if $A \lesssim B \lesssim A$.

2. Boundedness of $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B} \to H^\infty_\alpha$

In this section, we give a new characterization of the boundedness of the operator $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B} \to H^\infty_\alpha$. Let $\mathbb{N}$ denote the set of all positive integers. Let $j \in \mathbb{N}$. We define $p_j(z) = z^j$, $z \in \mathbb{D}$. Let $n \in \mathbb{N}$. For any $a \in \mathbb{D}$, we define the following two families test functions:

$$E_a(z) = \frac{1-|a|^2}{(1-\overline{a}z)^{n+1}}, \quad H_a(z) = \frac{1-|a|^2}{(1-\overline{a}z)^{n+1}} \frac{a-z}{1-\overline{a}z}, \quad z \in \mathbb{D}.$$

From [28], we see that $f \in \mathscr{B}$ if and only if

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{n+1} |f^{(n+1)}(z)| < \infty.$$

It is easy to check that $E_a, H_a \in \mathscr{B}^{n+1}$. Thus, there exist $f_a, g_a \in \mathscr{B}$ such that $f_a^{(n)} = E_a, g_a^{(n)} = H_a$.

In order to prove the result in this section, we need the following lemmas.

LEMMA 2.1. [11] *Let $n \in \mathbb{N}$. For all $z, w \in \mathbb{D}$,*

$$\flat_n(z,w) := \sup_{\|f\|_{\mathscr{B}} \leqslant 1} |(1-|z|^2)^n f^{(n)}(z) - (1-|w^2|)^n f^{(n)}(w)| \lesssim \rho(z,w).$$

Let $\alpha > 0$, $n \in \mathbb{N}$, $u, v \in H(\mathbb{D})$. Let φ and ψ be analytic self-maps of $\mathbb{D}$. We denote

$$\mathscr{D}^\#_u \varphi(z) = \frac{(1-|z|^2)^\alpha u(z)}{(1-|\varphi(z)|^2)^n}, \qquad \mathscr{D}^\#_v \psi(z) = \frac{(1-|z|^2)^\alpha v(z)}{(1-|\psi(z)|^2)^n}.$$

LEMMA 2.2. *Let $\alpha > 0$, $n \in \mathbb{N}$, $u, v \in H(\mathbb{D})$. Let φ and ψ be analytic self-maps of $\mathbb{D}$. Then the following inequalities hold.*

(i)

$$\sup_{z\in\mathbb{D}} |\mathscr{D}^\#_u \varphi(z)| \rho(\varphi(z), \psi(z)) \lesssim \sup_{a\in\mathbb{D}} \|(D^n_{\varphi,u} - D^n_{\psi,v}) f_a\|_{H^\infty_\alpha} + \sup_{a\in\mathbb{D}} \|(D^n_{\varphi,u} - D^n_{\psi,v}) g_a\|_{H^\infty_\alpha}.$$

(ii)

$$\sup_{z\in\mathbb{D}} |\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z)) \lesssim \sup_{a\in\mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} + \sup_{a\in\mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_a\|_{H_\alpha^\infty}.$$

(iii)

$$\sup_{z\in\mathbb{D}} |\mathscr{D}_u^{\#}\varphi(z) - \mathscr{D}_v^{\#}\psi(z)| \lesssim \sup_{a\in\mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} + \sup_{a\in\mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_a\|_{H_\alpha^\infty}.$$

Proof. *(i)* For any $z \in \mathbb{D}$, we have

$$\begin{aligned}
\|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\varphi(z)}\|_{H_\alpha^\infty} &\geqslant \left|u(z) f_{\varphi(z)}^{(n)}(\varphi(z)) - v(z) f_{\varphi(z)}^{(n)}(\psi(z))\right|(1-|z|^2)^\alpha \\
&= \left|\mathscr{D}_u^{\#}\varphi(z) - \frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)^n}{(1-\overline{\varphi(z)}\psi(z))^{n+1}} \mathscr{D}_v^{\#}\psi(z)\right| \\
&\geqslant |\mathscr{D}_u^{\#}\varphi(z)| - \frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)^n}{|1-\overline{\varphi(z)}\psi(z)|^{n+1}} |\mathscr{D}_v^{\#}\psi(z)|
\end{aligned}$$

and

$$\begin{aligned}
\|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\varphi(z)}\|_{H_\alpha^\infty} &\geqslant \left|u(z) g_{\varphi(z)}^{(n)}(\varphi(z)) - v(z) g_{\varphi(z)}^{(n)}(\psi(z))\right|(1-|z|^2)^\alpha \\
&= \frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)^n}{|1-\overline{\varphi(z)}\psi(z)|^{n+1}} |\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z)).
\end{aligned}$$

Hence

$$\begin{aligned}
&|\mathscr{D}_u^{\#}\varphi(z)|\rho(\varphi(z),\psi(z)) \\
&\leqslant \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\varphi(z)}\|_{H_\alpha^\infty} \rho(\varphi(z),\psi(z)) + \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\varphi(z)}\|_{H_\alpha^\infty} \\
&\leqslant \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\varphi(z)}\|_{H_\alpha^\infty} + \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\varphi(z)}\|_{H_\alpha^\infty}.
\end{aligned} \tag{5}$$

Similarly,

$$\begin{aligned}
&|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z)) \\
&\leqslant \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\psi(z)}\|_{H_\alpha^\infty} + \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\psi(z)}\|_{H_\alpha^\infty}.
\end{aligned} \tag{6}$$

Therefore, from (5) we obtain

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} |\mathscr{D}_u^{\#}\varphi(z)|\rho(\varphi(z),\psi(z)) \\
&\leqslant \sup_{z\in\mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_{\varphi(z)}\|_{H_\alpha^\infty} + \sup_{z\in\mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_{\varphi(z)}\|_{H_\alpha^\infty} \\
&\leqslant \sup_{a\in\mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) f_a\|_{H_\alpha^\infty} + \sup_{a\in\mathbb{D}} \|(D_{\varphi,u}^n - D_{\psi,v}^n) g_a\|_{H_\alpha^\infty}.
\end{aligned}$$

(ii) From (6) and similarly to the proof of *(i)* we get the desired result.

(iii) By Lemma 2.1,

$$\begin{aligned}
&\|(D^n_{\varphi,u}-D^n_{\psi,v})f_{\varphi(z)}\|_{H^\infty_\alpha}\\
\geqslant\ & \Big|\mathscr{D}^\#_u\varphi(z)-\frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)^n}{(1-\overline{\varphi(z)}\psi(z))^{n+1}}\mathscr{D}^\#_v\psi(z)\Big|\\
\geqslant\ & |\mathscr{D}^\#_u\varphi(z)-\mathscr{D}^\#_v\psi(z)|-\left|1-\frac{(1-|\varphi(z)|^2)(1-|\psi(z)|^2)^n}{(1-\overline{\varphi(z)}\psi(z))^{n+1}}\right||\mathscr{D}^\#_v\psi(z)|\\
\geqslant\ & |\mathscr{D}^\#_u\varphi(z)-\mathscr{D}^\#_v\psi(z)|-\Big|(1-|\varphi(z)|^2)^n f^{(n)}_{\varphi(z)}(\varphi(z))\\
&-(1-|\psi(z)|^2)^n f^{(n)}_{\varphi(z)}(\psi(z))\Big||\mathscr{D}^\#_v\psi(z)|\\
\geqslant\ & |\mathscr{D}^\#_u\varphi(z)-\mathscr{D}^\#_v\psi(z)|-\flat_n(\varphi(z),\psi(z))|\mathscr{D}^\#_v\psi(z)|\\
\gtrsim\ & |\mathscr{D}^\#_u\varphi(z)-\mathscr{D}^\#_v\psi(z)|-|\mathscr{D}^\#_v\psi(z)|\rho(\varphi(z),\psi(z)).
\end{aligned}$$

Thus, by (6) we obtain

$$\begin{aligned}
|\mathscr{D}^\#_u\varphi(z)-\mathscr{D}^\#_v\psi(z)| &\lesssim \|(D^n_{\varphi,u}-D^n_{\psi,v})f_{\varphi(z)}\|_{H^\infty_\alpha}+|\mathscr{D}^\#_v\psi(z)|\rho(\varphi(z),\psi(z))\\
&\lesssim \|(D^n_{\varphi,u}-D^n_{\psi,v})f_{\varphi(z)}\|_{H^\infty_\alpha}+\|(D^n_{\varphi,u}-D^n_{\psi,v})f_{\psi(z)}\|_{H^\infty_\alpha}\\
&\quad+\|(D^n_{\varphi,u}-D^n_{\psi,v})g_{\psi(z)}\|_{H^\infty_\alpha}. \qquad (7)
\end{aligned}$$

Therefore,

$$\sup_{z\in\mathbb{D}}|\mathscr{D}^\#_u\varphi(z)-\mathscr{D}^\#_v\psi(z)|\lesssim\sup_{a\in\mathbb{D}}\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}+\sup_{a\in\mathbb{D}}\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}.$$

The proof is complete. □

LEMMA 2.3. *Let $\alpha>0$, $n\in\mathbb{N}$, $u,v\in H(\mathbb{D})$. Let φ and ψ be analytic self-maps of $\mathbb{D}$. Then the following inequalities hold.*

(i)

$$\sup_{a\in\mathbb{D}}\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}\lesssim\sup_{j\in\mathbb{N}}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.$$

(ii)

$$\sup_{a\in\mathbb{D}}\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}\lesssim\sup_{j\in\mathbb{N}}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.$$

Proof. *(i)* When $a=0$, it is clear that $f^{(n)}_a(z)=1$. Thus,

$$\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}=\|u-v\|_{H^\infty_\alpha}\lesssim\sup_{j\in\mathbb{N}}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.$$

For any $a\in\mathbb{D}$ with $a\neq0$, we have

$$f^{(n)}_a(z)=\frac{1-|a|^2}{(1-\overline{a}z)^{n+1}}=(1-|a|^2)\sum_{k=0}^{\infty}\frac{\Gamma(k+n+1)}{k!\Gamma(n+1)}\overline{a}^kz^k,\quad z\in\mathbb{D}.$$

By Stirling's formula, we have

$$
\begin{aligned}
\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha} &\leqslant (1-|a|^2)\sum_{k=0}^{\infty}\frac{\Gamma(k+n+1)}{k!\Gamma(n+1)}|a|^k\|u\varphi^k-v\psi^k\|_{H^\infty_\alpha}\\
&=(1-|a|^2)\sum_{k=0}^{\infty}\frac{\Gamma(k+n+1)}{k!\Gamma(n+1)}k^{-n}|a|^k k^n\|u\varphi^k-v\psi^k\|_{H^\infty_\alpha}\\
&\lesssim (1-|a|^2)\sum_{k=0}^{\infty}|a|^k\sup_{j\geqslant n}(j-n)^n\|u\varphi^{j-n}-v\psi^{j-n}\|_{H^\infty_\alpha}\\
&\lesssim \sup_{j\in\mathbb{N}}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}. \qquad (8)
\end{aligned}
$$

Since a is arbitrary, we see that *(i)* holds.

(ii) When $a=0$, it is clear that $g_a^{(n)}(z)=-z$. Thus,

$$
\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}=\|u\varphi-v\psi\|_{H^\infty_\alpha}\lesssim\sup_{j\in\mathbb{N}}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.
$$

For any $a\in\mathbb{D}$ with $a\neq 0$, we have

$$
\begin{aligned}
g_a^{(n)}(z)&=\frac{1-|a|^2}{(1-\overline{a}z)^{n+1}}\cdot\frac{a-z}{1-\overline{a}z}\\
&=(1-|a|^2)\Big(\sum_{k=0}^{\infty}\frac{\Gamma(k+n+1)}{k!\Gamma(n+1)}\overline{a}^k z^k\Big)\Big(a-(1-|a|^2)\sum_{k=0}^{\infty}\overline{a}^k z^{k+1}\Big)\\
&=af_a^{(n)}(z)-(1-|a|^2)^2\Big(\sum_{k=0}^{\infty}\frac{\Gamma(k+n+1)}{k!\Gamma(n+1)}\overline{a}^k z^k\Big)\Big(\sum_{k=0}^{\infty}\overline{a}^k z^{k+1}\Big)\\
&=af_a^{(n)}(z)-(1-|a|^2)^2\sum_{k=1}^{\infty}\Big(\sum_{l=0}^{k-1}\frac{\Gamma(l+n+1)}{l!\Gamma(n+1)}\Big)\overline{a}^{k-1}z^k.
\end{aligned}
$$

By Stirling's formula, we have

$$
\sum_{l=0}^{k-1}\frac{\Gamma(l+n+1)}{l!\Gamma(n+1)}\approx\sum_{l=0}^{k-1}l^n\approx k^{n+1},\qquad k\to\infty.
$$

Therefore,

$$
\begin{aligned}
\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}&\leqslant\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}+(1-|a|^2)^2\\
&\quad\times\sum_{k=1}^{\infty}\Big(\sum_{l=0}^{k-1}\frac{\Gamma(l+n+1)}{l!\Gamma(n+1)}\Big)|a|^{k-1}\|u\varphi^k-v\psi^k\|_{H^\infty_\alpha}\\
&\lesssim\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}\\
&\quad+(1-|a|^2)^2\sum_{k=1}^{\infty}\frac{k^{n+1}}{k^n}|a|^{k-1}\sup_{j\geqslant n}(j-n)^n\|u\varphi^{j-n}-v\psi^{j-n}\|_{H^\infty_\alpha}\\
&\lesssim\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}+\sup_{j\in\mathbb{N}}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}\\
&\lesssim\sup_{j\in\mathbb{N}}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.
\end{aligned}
$$

By the arbitrariness of a we see that *(ii)* holds. The proof is complete. □

The following result is the main result in this section.

THEOREM 2.1. *Let* $\alpha>0$, $n\in\mathbb{N}$, $u,v\in H(\mathbb{D})$. *Let* φ *and* ψ *be analytic self-maps of* $\mathbb{D}$. *Then* $D^n_{\varphi,u}-D^n_{\psi,v}:\mathscr{B}\to H^\infty_\alpha$ *is bounded if and only if*

$$\sup_{j\in\mathbb{N}}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}<\infty. \tag{9}$$

Proof. First we assume that $D^n_{\varphi,u}-D^n_{\psi,v}:\mathscr{B}\to H^\infty_\alpha$ is bounded. For any $j\in\mathbb{N}$, $\|p_j\|_{\mathscr{B}}\approx 1$. Thus

$$\infty>\|D^n_{\varphi,u}-D^n_{\psi,v}\|_{\mathscr{B}\to H^\infty_\alpha}\gtrsim\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha},$$

as desired.

Conversely, we assume that (9) holds. Let $f\in\mathscr{B}$ and $\|f\|_{\mathscr{B}}\leqslant 1$. By Lemmas 2.1–2.3 and the proof of Theorem 1 in [11] we have

$$\begin{aligned}\|(D^n_{\varphi,u}-D^n_{\psi,v})f\|_{H^\infty_\alpha}&\lesssim\sup_{z\in\mathbb{D}}|\mathscr{D}^\#_u\varphi(z)-\mathscr{D}^\#_v\psi(z)|+\sup_{z\in\mathbb{D}}|\mathscr{D}^\#_u\varphi(z)|\rho(\varphi(z),\psi(z))\\&\lesssim\sup_{a\in\mathbb{D}}\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}+\sup_{a\in\mathbb{D}}\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}\\&\lesssim\sup_{j\in\mathbb{N}}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}<\infty.\end{aligned}$$

Therefore, $D^n_{\varphi,u}-D^n_{\psi,v}:\mathscr{B}\to H^\infty_\alpha$ is bounded. □

3. Essential norm estimates

In this section we give an estimate for the essential norm of $D^n_{\varphi,u}-D^n_{\psi,v}$ from $\mathscr{B}$ to H^∞_α. For this purpose, we need some auxiliary results as follows.

LEMMA 3.1. *Let* $\alpha>0$, $n\in\mathbb{N}$, $u,v\in H(\mathbb{D})$. *Let* φ *and* ψ *be analytic self-maps of* $\mathbb{D}$. *Then the following statements hold.*

(i)

$$\begin{aligned}&\lim_{s\to1}\sup_{|\varphi(z)|>s}|\mathscr{D}^\#_u\varphi(z)|\rho(\varphi(z),\psi(z))\\&\lesssim\limsup_{|a|\to1}\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}+\limsup_{|a|\to1}\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}.\end{aligned}$$

(ii)

$$\begin{aligned}&\lim_{s\to1}\sup_{|\psi(z)|>s}|\mathscr{D}^\#_v\psi(z)|\rho(\varphi(z),\psi(z))\\&\lesssim\limsup_{|a|\to1}\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}+\limsup_{|a|\to1}\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}.\end{aligned}$$

(iii)

$$\lim_{s\to 1}\sup_{\substack{|\varphi(z)|>s\\|\psi(z)|>s}}|\mathscr{D}_u^{\#}\varphi(z)-\mathscr{D}_v^{\#}\psi(z)|$$
$$\lesssim \limsup_{|a|\to 1}\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}+\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}.$$

Proof. For any $z\in\mathbb{D}$, from the proof of Lemma 2.2 we have

$$|\mathscr{D}_u^{\#}\varphi(z)|\rho(\varphi(z),\psi(z))\leqslant\|(D^n_{\varphi,u}-D^n_{\psi,v})f_{\varphi(z)}\|_{H^\infty_\alpha}+\|(D^n_{\varphi,u}-D^n_{\psi,v})g_{\varphi(z)}\|_{H^\infty_\alpha},$$

$$|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z))\leqslant\|(D^n_{\varphi,u}-D^n_{\psi,v})f_{\psi(z)}\|_{H^\infty_\alpha}+\|(D^n_{\varphi,u}-D^n_{\psi,v})g_{\psi(z)}\|_{H^\infty_\alpha}$$

and

$$|\mathscr{D}_u^{\#}\varphi(z)-\mathscr{D}_v^{\#}\psi(z)|\lesssim\|(D^n_{\varphi,u}-D^n_{\psi,v})f_{\varphi(z)}\|_{H^\infty_\alpha}+\|(D^n_{\varphi,u}-D^n_{\psi,v})f_{\psi(z)}\|_{H^\infty_\alpha}$$
$$+\|(D^n_{\varphi,u}-D^n_{\psi,v})g_{\psi(z)}\|_{H^\infty_\alpha}.$$

From the above inequalities the assertion follows easily. The proof is complete. □

LEMMA 3.2. *Let $\alpha>0$, $n\in\mathbb{N}$, $u,v\in H(\mathbb{D})$. Let φ and ψ be analytic self-maps of $\mathbb{D}$. Suppose that $D^n_{\varphi,u}-D^n_{\psi,v}:\mathscr{B}\to H^\infty_\alpha$ is bounded, then the following inequalities hold.*

(i)

$$\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}\lesssim\limsup_{j\to\infty}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.$$

(ii)

$$\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}\lesssim\limsup_{j\to\infty}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.$$

Proof. For each N and any $a\in\mathbb{D}$ with $|a|>1/2$, from the proof of Lemma 2.3, we have

$$\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}$$
$$\lesssim(1-|a|^2)\sum_{k=0}^{N}\frac{\Gamma(k+n+1)}{k!\Gamma(n+1)}k^{-n}|a|^k k^n\|u\varphi^k-v\psi^k\|_{H^\infty_\alpha}$$
$$+(1-|a|^2)\sum_{k=N+1}^{\infty}|a|^k\sup_{j\geqslant N+n+1}(j-n)^n\|u\varphi^{j-n}-v\psi^{j-n}\|_{H^\infty_\alpha}. \tag{10}$$

Taking the limit $|a|\to 1$ in (10),

$$\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}\lesssim\sup_{j\geqslant N+n+1}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}$$

for any positive integer N. Therefore

$$\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}\lesssim \limsup_{j\to\infty}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.$$

Also for each N and any $a\in\mathbb{D}$ with $|a|>1/2$, from the proof of Lemma 2.3,

$$\begin{aligned}
&\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}\\
\leqslant\ &\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}+(1-|a|^2)^2\\
&\times\sum_{k=1}^{\infty}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+n+1)}{l!\Gamma(n+1)}\right)|a|^{k-1}\|u\varphi^k-v\psi^k\|_{H^\infty_\alpha}\\
\leqslant\ &\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}+\sup_{j\geqslant N+n+1}(j-n)^n\|u\varphi^{j-n}-v\psi^{j-n}\|_{H^\infty_\alpha}\\
&+(1-|a|^2)^2\sum_{k=1}^{N}k|a|^{k-1}k^n\|u\varphi^k-v\psi^k\|_{H^\infty_\alpha}.
\end{aligned}\tag{11}$$

Letting $|a|\to 1$ in (11). We get

$$\begin{aligned}
&\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}\\
\lesssim\ &\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}+\sup_{j\geqslant N+n+1}(j-n)^n\|u\varphi^{j-n}-v\psi^{j-n}\|_{H^\infty_\alpha}\\
\lesssim\ &\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}+\sup_{j\geqslant N+n+1}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}
\end{aligned}$$

for any positive integer N. Thus, by *(i)* we obtain

$$\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}\lesssim \limsup_{j\to\infty}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.$$

The proof is complete. □

THEOREM 3.1. *Let $\alpha>0$, $n\in\mathbb{N}$, $u,v\in H(\mathbb{D})$. Let φ and ψ be analytic self-maps of $\mathbb{D}$. Suppose that $D^n_{\varphi,u}:\mathscr{B}\to H^\infty_\alpha$ and $D^n_{\psi,v}:\mathscr{B}\to H^\infty_\alpha$ are bounded, then*

$$\|D^n_{\varphi,u}-D^n_{\psi,v}\|_{e,\mathscr{B}\to H^\infty_\alpha}\approx \limsup_{j\to\infty}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.$$

Proof. For $r\in[0,1)$, set $K_r:H(\mathbb{D})\to H(\mathbb{D})$ by

$$(K_rf)(z)=f_r(z)=f(rz),\quad f\in H(\mathbb{D}).$$

It is clear that $f_r-f\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $r\to 1$. Moreover, K_r is compact on $\mathscr{B}$ and $\|K_r\|_{\mathscr{B}\to\mathscr{B}}\leqslant 1$. Let $\{r_j\}\subset(0,1)$ be a sequence such that $r_j\to 1$

as $j \to \infty$. Then for each positive integer j, the operator $(D^n_{\varphi,u} - D^n_{\psi,v})K_{r_j} : \mathscr{B} \to H^\infty_\alpha$ is compact. By the definition of the essential norm we have

$$\begin{aligned}
\|D^n_{\varphi,u} - D^n_{\psi,v}\|_{e,\mathscr{B}\to H^\infty_\alpha} &\leqslant \limsup_{j\to\infty} \|D^n_{\varphi,u} - D^n_{\psi,v} - (D^n_{\varphi,u} - D^n_{\psi,v})K_{r_j}\|_{\mathscr{B}\to H^\infty_\alpha} \\
&= \limsup_{j\to\infty} \|(D^n_{\varphi,u} - D^n_{\psi,v})(I - K_{r_j})\|_{\mathscr{B}\to H^\infty_\alpha} \\
&\leqslant \limsup_{j\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \|(D^n_{\varphi,u} - D^n_{\psi,v})(I - K_{r_j})f\|_{H^\infty_\alpha} \\
&= \limsup_{j\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{z\in\mathbb{D}} \Omega^f_j(z),
\end{aligned}$$

where

$$\Omega^f_j(z) := |u(z)(f - f_{r_j})^{(n)}(\varphi(z)) - v(z)(f - f_{r_j})^{(n)}(\psi(z))|(1 - |z|^2)^\alpha.$$

For any $r \in (0,1)$, define

$$\mathbb{D}_1 := \{z \in \mathbb{D} : |\varphi(z)| \leqslant r, |\psi(z)| \leqslant r\}, \quad \mathbb{D}_2 := \{z \in \mathbb{D} : |\varphi(z)| \leqslant r, |\psi(z)| > r\},$$

$$\mathbb{D}_3 := \{z \in \mathbb{D} : |\varphi(z)| > r, |\psi(z)| \leqslant r\}, \quad \mathbb{D}_4 := \{z \in \mathbb{D} : |\varphi(z)| > r, |\psi(z)| > r\}.$$

Then

$$\begin{aligned}
&\limsup_{j\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{z\in\mathbb{D}} \Omega^f_j = \max_{1\leqslant i\leqslant 4} \limsup_{j\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{z\in\mathbb{D}_i} \Omega^f_j \\
&= \max\{\limsup_{j\to\infty} J_1, \limsup_{j\to\infty} J_2, \limsup_{j\to\infty} J_3, \limsup_{j\to\infty} J_4\},
\end{aligned}$$

where $J_i = \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{z\in\mathbb{D}_i} \Omega^f_j$. Using the fact that $u, v \in H^\infty_\alpha$ we get

$$\begin{aligned}
\limsup_{j\to\infty} J_1 &= \limsup_{j\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{z\in\mathbb{D}_1} \Omega^f_j \\
&\leqslant \limsup_{j\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{|\varphi(z)|\leqslant r} |u(z)(f - f_{r_j})^{(n)}(\varphi(z))|(1 - |z|^2)^\alpha \\
&\quad + \limsup_{j\to\infty} \sup_{\|f\|_{\mathscr{B}}\leqslant 1} \sup_{|\psi(z)|\leqslant r} |v(z)(f - f_{r_j})^{(n)}(\psi(z))|(1 - |z|^2)^\alpha \\
&= 0.
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\Omega^f_j(z) &\leqslant |(f - f_{r_j})^{(n)}(\psi(z))|(1 - |\psi(z)|^2)^n |\mathscr{D}^\#_u \varphi(z) - \mathscr{D}^\#_v \psi(z)| + |\mathscr{D}^\#_u \varphi(z)| \\
&\quad \times |(f - f_{r_j})^{(n)}(\varphi(z))(1 - |\varphi(z)|^2)^n - (f - f_{r_j})^{(n)}(\psi(z))(1 - |\psi(z)|^2)^n| \\
&\leqslant |(f - f_{r_j})^{(n)}(\psi(z))|(1 - |\psi(z)|^2)^n |\mathscr{D}^\#_u \varphi(z) - \mathscr{D}^\#_v \psi(z)| \\
&\quad + \flat_n(\varphi(z), \psi(z))|\mathscr{D}^\#_u \varphi(z)| \\
&\lesssim |(f - f_{r_j})^{(n)}(\psi(z))|(1 - |\psi(z)|^2)^n |\mathscr{D}^\#_u \varphi(z) - \mathscr{D}^\#_v \psi(z)| \\
&\quad + |\mathscr{D}^\#_u \varphi(z)| \rho(\varphi(z), \psi(z)).
\end{aligned}$$

Similarly,

$$\begin{aligned}\Omega_j^f(z) \lesssim\ & |(f-f_{r_j})^{(n)}(\varphi(z))|(1-|\varphi(z)|^2)^n|\mathscr{D}_u^{\#}\varphi(z)-\mathscr{D}_v^{\#}\psi(z)| \\ & +|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z)).\end{aligned}$$

Then, we obtain

$$\begin{aligned}\limsup_{j\to\infty} J_2 \lesssim\ & \limsup_{j\to\infty}\sup_{\|f\|_{\mathscr{B}}\leqslant 1}\sup_{z\in\mathbb{D}_2}\Big(|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z)) \\ & +|(f-f_{r_j})^{(n)}(\varphi(z))|(1-|\varphi(z)|^2)^n|\mathscr{D}_u^{\#}\varphi(z)-\mathscr{D}_v^{\#}\psi(z)|\Big) \\ \leqslant\ & \limsup_{j\to\infty}\sup_{\|f\|_{\mathscr{B}}\leqslant 1}\sup_{|\varphi(z)|\leqslant r}|(f-f_{r_j})^{(n)}(\varphi(z))|(1-|\varphi(z)|^2)^n \\ & \times|\mathscr{D}_u^{\#}\varphi(z)-\mathscr{D}_v^{\#}\psi(z)|+\sup_{|\psi(z)|>r}|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z)) \\ =\ & \sup_{|\psi(z)|>r}|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z)),\end{aligned}$$

where we used $\sup_{z\in\mathbb{D}}|\mathscr{D}_u^{\#}\varphi(z)-\mathscr{D}_v^{\#}\psi(z)|<\infty$, since $D_{\varphi,u}^n-D_{\psi,v}^n$ is bounded (see Theorem 1 of [11]), and $(f-f_{r_j})^{(n)}\to 0$ uniformly on compact subset of $\mathbb{D}$ as $j\to\infty$ again in the last inequality. Since r is arbitrary, we have

$$\limsup_{j\to\infty} J_2 \lesssim \lim_{r\to 1}\sup_{|\psi(z)|>r}|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z)).$$

Similarly,

$$\limsup_{j\to\infty} J_3 \lesssim \lim_{r\to 1}\sup_{|\varphi(z)|>r}|\mathscr{D}_u^{\#}\varphi(z)|\rho(\varphi(z),\psi(z)).$$

In addition,

$$\begin{aligned}\limsup_{j\to\infty} J_4 \lesssim\ & \limsup_{j\to\infty}\sup_{\|f\|_{\mathscr{B}}\leqslant 1}\sup_{z\in\mathbb{D}_4}\Big(|(f-f_{r_j})^{(n)}(\varphi(z))|(1-|\varphi(z)|^2)^n \\ & \times|\mathscr{D}_u^{\#}\varphi(z)-\mathscr{D}_v^{\#}\psi(z)|+|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z))\Big) \\ \lesssim\ & \limsup_{j\to\infty}\sup_{\|f\|_{\mathscr{B}}\leqslant 1}\sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}}\|f-f_{r_j}\|_{\mathscr{B}}|\mathscr{D}_u^{\#}\varphi(z)-\mathscr{D}_v^{\#}\psi(z)| \\ & +\sup_{|\psi(z)|>r}|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z)) \\ \lesssim\ & \sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}}|\mathscr{D}_u^{\#}\varphi(z)-\mathscr{D}_v^{\#}\psi(z)|+\sup_{|\psi(z)|>r}|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z)),\end{aligned}$$

where we used the fact $\limsup_{j\to\infty}\|f-f_{r_j}\|_{\mathscr{B}}\leqslant 2$ in the last inequality. Thus,

$$\limsup_{j\to\infty} J_4 \lesssim \lim_{r\to 1}\sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}}|\mathscr{D}_u^{\#}\varphi(z)-\mathscr{D}_v^{\#}\psi(z)|+\lim_{r\to 1}\sup_{|\psi(z)|>r}|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z)).$$

Then we have

$$\begin{aligned}
&\limsup_{j\to\infty}\sup_{\|f\|_{\mathscr{B}}\leqslant 1}\sup_{z\in\mathbb{D}}\Omega_j^f(z)=\max\{\limsup_{j\to\infty}J_1,\limsup_{j\to\infty}J_2,\limsup_{j\to\infty}J_3,\limsup_{j\to\infty}J_4\}\\
\lesssim\ &\lim_{r\to 1}\sup_{|\varphi(z)|>r}|\mathscr{D}_u^{\#}\varphi(z)|\rho(\varphi(z),\psi(z))+\lim_{r\to 1}\sup_{|\psi(z)|>r}|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z))\\
&+\lim_{r\to 1}\sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}}|\mathscr{D}_u^{\#}\varphi(z)-\mathscr{D}_v^{\#}\psi(z)|,
\end{aligned}$$

which together with Lemmas 3.1 and 3.2 imply

$$\begin{aligned}
&\|D^n_{\varphi,u}-D^n_{\psi,v}\|_{e,\mathscr{B}\to H^\infty_\alpha}\\
\lesssim\ &\lim_{r\to 1}\sup_{|\varphi(z)|>r}|\mathscr{D}_u^{\#}\varphi(z)|\rho(\varphi(z),\psi(z))+\lim_{r\to 1}\sup_{|\psi(z)|>r}|\mathscr{D}_v^{\#}\psi(z)|\rho(\varphi(z),\psi(z))\\
&+\lim_{r\to 1}\sup_{\substack{|\varphi(z)|>r\\|\psi(z)|>r}}|\mathscr{D}_u^{\#}\varphi(z)-\mathscr{D}_v^{\#}\psi(z)|\\
\lesssim\ &\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-D^n_{\psi,v})f_a\|_{H^\infty_\alpha}+\limsup_{|a|\to 1}\|(D^n_{\varphi,u}-D^n_{\psi,v})g_a\|_{H^\infty_\alpha}\\
\lesssim\ &\limsup_{j\to\infty}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.
\end{aligned}\tag{12}$$

Next, we prove that

$$\|D^n_{\varphi,u}-D^n_{\psi,v}\|_{e,\mathscr{B}\to H^\infty_\alpha}\gtrsim\limsup_{j\to\infty}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.$$

Let j be any positive integer. Then $\|p_j\|_{\mathscr{B}}\approx 1$ and $p_j\to 0$ weakly in $\mathscr{B}$. This follows since a bounded sequence contained in $\mathscr{B}_0$ which converges uniformly to 0 on compact subsets of $\mathbb{D}$ converges weakly to 0 in $\mathscr{B}$ (see [12]). Thus, if K is any compact operator from $\mathscr{B}$ to H^∞_α, then $\lim_{j\to\infty}\|Kp_j\|_{H^\infty_\alpha}=0$. Hence,

$$\begin{aligned}
\|D^n_{\varphi,u}-D^n_{\psi,v}-K\|&\gtrsim\limsup_{j\to\infty}\|(D^n_{\varphi,u}-D^n_{\psi,v}-K)p_j\|_{H^\infty_\alpha}\\
&\geqslant\limsup_{j\to\infty}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.
\end{aligned}$$

Thus

$$\|D^n_{\varphi,u}-D^n_{\psi,v}\|_{e,\mathscr{B}\to H^\infty_\alpha}\gtrsim\limsup_{j\to\infty}\|(D^n_{\varphi,u}-D^n_{\psi,v})p_j\|_{H^\infty_\alpha}.\tag{13}$$

Combining (12) with (13), we immediately get the desired result. The proof is complete. □

From Theorem 3.1, we immediately get the following result.

THEOREM 3.2. *Let $\alpha > 0$, $n \in \mathbb{N}$, $u, v \in H(\mathbb{D})$. Let φ and ψ be analytic self-maps of $\mathbb{D}$. Suppose that $D^n_{\varphi,u} : \mathscr{B} \to H^\infty_\alpha$ and $D^n_{\psi,v} : \mathscr{B} \to H^\infty_\alpha$ are bounded, then*

$$\begin{aligned}
&\|D^n_{\varphi,u} - D^n_{\psi,v}\|_{e,\mathscr{B}\to H^\infty_\alpha} \\
\approx &\lim_{r\to 1} \sup_{|\varphi(z)|>r} |\mathscr{D}^\#_u \varphi(z)| \rho(\varphi(z), \psi(z)) + \lim_{r\to 1} \sup_{|\psi(z)|>r} |\mathscr{D}^\#_v \psi(z)| \rho(\varphi(z), \psi(z)) \\
&+ \lim_{r\to 1} \sup_{\substack{|\varphi(z)|>r \\ |\psi(z)|>r}} |\mathscr{D}^\#_u \varphi(z) - \mathscr{D}^\#_v \psi(z)| \\
\approx &\limsup_{|a|\to 1} \|(D^n_{\varphi,u} - D^n_{\psi,v}) f_a\|_{H^\infty_\alpha} + \limsup_{|a|\to 1} \|(D^n_{\varphi,u} - D^n_{\psi,v}) g_a\|_{H^\infty_\alpha}.
\end{aligned}$$

From Theorem 3.1, we also immediately get the following corollary.

COROLLARY 3.1. *Let $\alpha > 0$, $n \in \mathbb{N}$, $u, v \in H(\mathbb{D})$. Let φ and ψ be analytic self-maps of $\mathbb{D}$. Suppose that $D^n_{\varphi,u} : \mathscr{B} \to H^\infty_\alpha$ and $D^n_{\psi,v} : \mathscr{B} \to H^\infty_\alpha$ are bounded, then $D^n_{\varphi,u} - D^n_{\psi,v} : \mathscr{B} \to H^\infty_\alpha$ is compact if and only if*

$$\limsup_{j\to\infty} \|(D^n_{\varphi,u} - D^n_{\psi,v}) p_j\|_{H^\infty_\alpha} = 0.$$

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