

AN INFINITE SEQUENCE OF INEQUALITIES INVOLVING SPECIAL VALUES OF THE RIEMANN ZETA FUNCTION

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*Dedicated to the 70th Anniversary
of Professor Constantin P. Niculescu*

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Abstract. In this paper, we give an infinite sequence of inequalities involving the Riemann zeta function with even arguments $\zeta(2n)$ and the Chebyshev-Stirling numbers of the first kind. This result is based on a recent connection between the Riemann zeta function and the complete homogeneous symmetric functions [18]. An interesting asymptotic formula related to the n th complete homogeneous symmetric function is conjectured in this context:

$$h_n \left(1, \left(\frac{k}{k+1} \right)^2, \left(\frac{k}{k+2} \right)^2, \dots \right) \sim \binom{2k}{k}, \quad n \rightarrow \infty.$$

1. Introduction

The main object of our investigation is the Riemann zeta function or Euler-Riemann zeta function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

which is defined over the complex plane when the real part of s is greater than 1. Originally the Riemann zeta function was defined for real arguments by Euler as

$$\zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}, \quad x > 1.$$

Moreover, for real $x > 1$, we have

$$\zeta(x) > \zeta(x+1) \quad \text{and} \quad \lim_{x \rightarrow \infty} \zeta(x) = 1.$$

In spite of its utter simplicity, this function plays a pivotal role in analytic number theory having applications in physics, probability theory, applied statistics and other fields of

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mathematics. The reader should consult the classical papers by Abramowitz and Stegun [1], Apostol [5], Berndt [6], Everest, Röttger and T. Ward [7], Ireland and Rosen [13], Murty and Reece [23], and Weil [24] for the full background on this function.

Being given an infinite set of variables $\{x_1, x_2, x_3, \dots\}$, recall [14] that the n th complete homogeneous symmetric function h_n is the sum of all monomials of total degree n in these variables so that $h_0 = 1$ and for $n > 0$

$$h_n = h_n(x_1, x_2, x_3, \dots) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_n} x_{i_1} x_{i_2} \dots x_{i_n}.$$

In a recent paper [18, Eq. (3.1)], the Riemann zeta function with even arguments, $\zeta(2n)$, was expressed in terms of the n th complete homogeneous symmetric function of the numbers $\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots$, as follows

$$h_n\left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots\right) = 2\left(1 - \frac{2}{2^{2n}}\right) \zeta(2n), \quad n \geq 0. \quad (1)$$

Considering the recurrence relation

$$h_n(x_1, x_2, x_3, \dots) = x_1 h_{n-1}(x_1, x_2, x_3, \dots) + h_n(x_2, x_3, x_4, \dots), \quad (2)$$

we deduce the inequality

$$\left(1 - \frac{2}{2^{2n}}\right) \zeta(2n) - \left(1 - \frac{2}{2^{2n-2}}\right) \zeta(2n-2) > 0. \quad (3)$$

This result seems more interesting if we consider the trivial inequality

$$\zeta(2n) - \zeta(2n-2) < 0.$$

Upon reflection, one expects that there might be an infinite family of such inequalities where (3) is the second entry, and the trivial inequality

$$\left(1 - \frac{2}{2^{2n}}\right) \zeta(2n) > 0$$

is the first.

For all nonnegative integers n and k , we define $S_n(k)$ by

$$S_n(k) = \sum_{i=0}^k (-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \left(1 - \frac{2}{2^{2n-2i}}\right) \zeta(2n-2i),$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_{1/2}$ are the Chebyshev-Stirling numbers of the first kind. Recall that the Chebyshev-Stirling numbers of the first kind are known in the literature [9, 10, 17] as the case $\gamma = 1/2$ of the Jacobi-Stirling numbers of the first kind that can be given through the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\gamma} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{\gamma} + (n-1)(n+2\gamma-2) \begin{bmatrix} n-1 \\ k \end{bmatrix}_{\gamma} \quad (4)$$

with the initial conditions

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_\gamma = \delta_{0,n} \quad \text{and} \quad \begin{bmatrix} 0 \\ k \end{bmatrix}_\gamma = \delta_{0,k},$$

where $\delta_{i,j}$ is the Kronecker delta. The Jacobi-Stirling numbers were discovered in 2007 as a result of a problem involving the spectral theory of powers of the classical second-order Jacobi differential expression. In the last decade, these numbers received considerable attention especially in combinatorics and graph theory, see, e.g., [2, 3, 4, 8, 9, 10, 11, 12, 15, 16, 17, 19, 20, 21, 22].

In this paper, we shall prove the following inequalities.

THEOREM 1. For $n, k \geq 0$,

1. $S_n(k) > 0$;
2. $\frac{S_n(k)}{k!^2} > \frac{S_n(k+1)}{(k+1)!^2}$.

EXAMPLE 1. Having

$$\left[\begin{bmatrix} n \\ k \end{bmatrix}_{1/2} \right]_{n,k=1,5} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 4 & 5 & 1 & & \\ 36 & 49 & 14 & 1 & \\ 576 & 820 & 273 & 30 & 1 \end{bmatrix},$$

we can write the following sequence of inequalities:

$$\begin{aligned} & \left(1 - \frac{2}{2^{2n}}\right) \zeta(2n) \\ & > \left(1 - \frac{2}{2^{2n}}\right) \zeta(2n) - \left(1 - \frac{2}{2^{2n-2}}\right) \zeta(2n-2) \\ & > \left(1 - \frac{2}{2^{2n}}\right) \zeta(2n) - \frac{5}{4} \left(1 - \frac{2}{2^{2n-2}}\right) \zeta(2n-2) + \frac{1}{4} \left(1 - \frac{2}{2^{2n-4}}\right) \zeta(2n-4) \\ & > \left(1 - \frac{2}{2^{2n}}\right) \zeta(2n) - \frac{49}{36} \left(1 - \frac{2}{2^{2n-2}}\right) \zeta(2n-2) \\ & \quad + \frac{7}{18} \left(1 - \frac{2}{2^{2n-4}}\right) \zeta(2n-4) - \frac{1}{36} \left(1 - \frac{2}{2^{2n-6}}\right) \zeta(2n-6) \\ & > \left(1 - \frac{2}{2^{2n}}\right) \zeta(2n) - \frac{205}{144} \left(1 - \frac{2}{2^{2n-2}}\right) \zeta(2n-2) + \frac{91}{192} \left(1 - \frac{2}{2^{2n-4}}\right) \zeta(2n-4) \\ & \quad - \frac{5}{96} \left(1 - \frac{2}{2^{2n-6}}\right) \zeta(2n-6) + \frac{1}{576} \left(1 - \frac{2}{2^{2n-8}}\right) \zeta(2n-8) \\ & > \dots > 0. \end{aligned}$$

Related to the first inequality of Theorem 1, we remark a well-known property of the Chebyshev-Stirling numbers of the first kind, that is,

$$\sum_{i=0}^k (-1)^i \left[\begin{matrix} k+1 \\ i+1 \end{matrix} \right]_{1/2} = 0.$$

2. Proof of Theorem 1

Firstly, we prove the following lemma in two ways.

LEMMA 1. For $n, k \geq 0$,

$$\begin{aligned} h_n \left(\frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}, \frac{1}{(k+3)^2}, \dots \right) \\ = \sum_{i=0}^k \frac{(-1)^i}{k!^2} \left[\begin{matrix} k+1 \\ i+1 \end{matrix} \right]_{1/2} h_{n-i} \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots \right). \end{aligned}$$

Proof 1. This proof invokes the generating functions for the complete and elementary symmetric functions. Being given a set of variables $\{x_1, x_2, \dots, x_n\}$, recall [14] that the k th elementary symmetric function $e_k(x_1, x_2, \dots, x_n)$ is given by

$$e_k(x_1, x_2, \dots, x_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}$$

for $k = 1, 2, \dots, n$. We set $e_0(x_1, x_2, \dots, x_n) = 1$ by convention. For $k < 0$ or $k > n$, we set $e_k(x_1, x_2, \dots, x_n) = 0$. In particular, according to Merca [18], we have

$$e_k \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots, \frac{1}{n^2} \right) = \frac{1}{n!^2} \left[\begin{matrix} n+1 \\ k+1 \end{matrix} \right]_{1/2}.$$

The elementary symmetric functions are characterized by the following identity of formal power series in t :

$$\sum_{k=0}^{\infty} e_k(x_1, x_2, \dots, x_n) t^k = \prod_{k=1}^n (1 + x_k t).$$

For the complete homogeneous symmetric functions in infinitely many variables x_1, x_2, \dots , we have

$$\sum_{k=0}^{\infty} h_k(x_1, x_2, \dots) t^k = \prod_{k=1}^{\infty} (1 - x_k t)^{-1}.$$

Thus, we can write

$$\begin{aligned}
& \sum_{n=0}^{\infty} h_n \left(\frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}, \frac{1}{(k+3)^2}, \dots \right) t^n \\
&= \prod_{i=k+1}^{\infty} \left(1 - \frac{t}{i^2} \right)^{-1} = \prod_{i=1}^k \left(1 - \frac{t}{i^2} \right) \times \prod_{i=1}^{\infty} \left(1 - \frac{t}{i^2} \right)^{-1} \\
&= \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{k!^2} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} t^i \right) \left(\sum_{i=0}^{\infty} h_i \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots \right) t^i \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^k \frac{(-1)^i}{k!^2} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} h_{n-i} \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots \right) \right) t^n,
\end{aligned}$$

where we have invoked the well known Cauchy multiplication of two power series. Equating coefficients of t^n give the result. \square

Proof 2. We are going to prove this lemma by induction on k . For $k=0$, we have

$$h_n \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1/2} h_n \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots \right).$$

The base case of induction is finished. We suppose that the relation

$$h_n \left(\frac{1}{(k'+1)^2}, \frac{1}{(k'+2)^2}, \frac{1}{(k'+3)^2}, \dots \right) = \sum_{i=0}^{k'} \frac{(-1)^i}{k'!^2} \begin{bmatrix} k'+1 \\ i+1 \end{bmatrix}_{1/2} h_{n-i} \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots \right).$$

is true for any integer k' , $0 \leq k' < k$. Taking into account (2), we can write

$$\begin{aligned}
& h_n \left(\frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}, \frac{1}{(k+3)^2}, \dots \right) \\
&= h_n \left(\frac{1}{k^2}, \frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}, \dots \right) - \frac{1}{k^2} h_{n-1} \left(\frac{1}{k^2}, \frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}, \dots \right) \\
&= \sum_{i=0}^{k-1} \frac{(-1)^i}{(k-1)!^2} \begin{bmatrix} k \\ i+1 \end{bmatrix}_{1/2} h_{n-i} \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots \right) \\
&\quad - \frac{1}{k^2} \sum_{i=0}^{k-1} \frac{(-1)^i}{(k-1)!^2} \begin{bmatrix} k \\ i+1 \end{bmatrix}_{1/2} h_{n-1-i} \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots \right) \\
&= \sum_{i=0}^{k-1} \frac{(-1)^i}{(k-1)!^2} \begin{bmatrix} k \\ i+1 \end{bmatrix}_{1/2} h_{n-i} \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots \right) \\
&\quad - \frac{1}{k^2} \sum_{i=1}^k \frac{(-1)^{i-1}}{(k-1)!^2} \begin{bmatrix} k \\ i \end{bmatrix}_{1/2} h_{n-i} \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots \right) \\
&= \sum_{i=0}^k \frac{(-1)^i}{k!^2} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} h_{n-i} \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots \right),
\end{aligned}$$

where we have invoked the recurrence relation (4), with γ replaced by $1/2$. Thus, the proof of the lemma is finished. \square

Theorem 1 follows considering this lemma, the equation (1) and the inequalities

$$h_n \left(\frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}, \frac{1}{(k+3)^2}, \dots \right) > 0$$

and

$$h_n \left(\frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}, \frac{1}{(k+3)^2}, \dots \right) > h_n \left(\frac{1}{(k+2)^2}, \frac{1}{(k+3)^2}, \frac{1}{(k+4)^2}, \dots \right).$$

3. Concluding remarks

A formula for the n th complete homogeneous symmetric function

$$h_n \left(\frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}, \frac{1}{(k+3)^2}, \dots \right)$$

in terms of the complete homogeneous symmetric functions

$$h_i \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \dots \right), \quad i = n-k, \dots, n$$

has been introduced in this paper. Using this result, we derived an infinite sequence of inequalities involving the Riemann zeta function with even arguments $\zeta(2n)$ and the Chebyshev-Stirling numbers of the first kind.

There is a substantial amount of numerical evidence to conjecture that the following inequality is true.

CONJECTURE 1. For $n, k \geq 0$,

$$h_n \left(\frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}, \frac{1}{(k+3)^2}, \dots \right) < \frac{1}{(k+1)^{2n}} \binom{2k+2}{k+1}.$$

Moreover, we conjecture that the sequence

$$\left\{ (k+1)^{2n} h_n \left(\frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}, \frac{1}{(k+3)^2}, \dots \right) \right\}_{n \geq 0}$$

converges to

$$\binom{2k+2}{k+1}.$$

CONJECTURE 2. For $k > 0$,

$$\lim_{n \rightarrow \infty} h_n \left(1, \left(\frac{k}{k+1} \right)^2, \left(\frac{k}{k+2} \right)^2, \dots \right) = \binom{2k}{k}.$$

Finally, assuming Conjecture 1, we can write the following inequality.

CONJECTURE 3. For $n, k \geq 0$,

$$S_n(k) < \frac{(2k+1)!}{(k+1)^{2n+1}}.$$

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