

CONTINUITY AND APPROXIMATE DIFFERENTIABILITY OF MULTISUBLINEAR FRACTIONAL MAXIMAL FUNCTIONS

FENG LIU

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Abstract. In this note we investigate the continuity and approximate differentiability of the m -sublinear fractional maximal operator

$$\mathfrak{M}_\alpha(\vec{f})(x) = \sup_{r>0} |B(x,r)|^{\alpha/d-m} \prod_{i=1}^m \int_{B(x,r)} |f_i(y)| dy,$$

where $m \geq 1$, $0 \leq \alpha < md$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^1_{\text{loc}}(\mathbb{R}^d)$. More precisely, we prove that \mathfrak{M}_α maps $W^{1,p_1}(\mathbb{R}^d) \times \dots \times W^{1,p_m}(\mathbb{R}^d)$ into $W^{1,q}(\mathbb{R}^d)$ continuously, provided that $1 < p_1, \dots, p_m < \infty$ and $0 < \sum_{i=1}^m 1/p_i - \alpha/d = 1/q \leq 1$. We also show that the multisublinear fractional maximal functions $\mathfrak{M}_\alpha(\vec{f})$ are approximately differentiable a.e. if $\vec{f} = (f_1, f_2, \dots, f_m)$ with each $f_j \in L^1(\mathbb{R}^d)$ being approximately differentiable a.e. As applications, the corresponding results for fractional maximal operators are established.

1. Introduction

Let $d \geq 1$ and M denote the centered Hardy-Littlewood maximal operator on \mathbb{R}^d , i.e. for $f \in L^1_{\text{loc}}(\mathbb{R}^d)$,

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

for any $x \in \mathbb{R}^d$, where $B(x,r)$ is the ball in \mathbb{R}^d centered at x with radius r and $|B(x,r)|$ denotes the volume of $B(x,r)$. One of the cornerstones of harmonic analysis is the celebrated theorem of Hardy-Littlewood-Wiener that asserts that $M : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ is bounded for $1 < p \leq \infty$. For $p = 1$ we have $M : L^1(\mathbb{R}^d) \rightarrow L^{1,\infty}(\mathbb{R}^d)$ bounded. During the last several years, a considerable amount of attention has been given to investigate the behavior of differentiability under a maximal operator. The first work in this direction is due to Kinnunen [15] who proved that M is bounded on $W^{1,p}(\mathbb{R}^d)$ for $1 < p \leq \infty$, where $W^{1,p}(\mathbb{R}^d)$ is the first order Sobolev space, which consists of

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functions $f \in L^p(\mathbb{R}^d)$, whose first weak partial derivatives $D_i f$, $i = 1, 2, \dots, d$, belong to $L^p(\mathbb{R}^d)$. We endow $W^{1,p}(\mathbb{R}^d)$ with the norm

$$\|f\|_{1,p} = \|f\|_{L^p(\mathbb{R}^d)} + \|\nabla f\|_{L^p(\mathbb{R}^d)},$$

where $\nabla f = (D_1 f, \dots, D_d f)$ is the weak gradient of f . Later on, this paradigm has been extended to a local version in [16], to a fractional version in [17], to a bilinear version in [11], to a multisublinear fractional version in [25] and to a one-sided version in [24]. Due to the lack of reflexivity of L^1 , results for $p = 1$ are subtler. A crucial question on whether the operator $f \mapsto |\nabla M f|$ is bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, posed by Hajlasz and Onninen in [14], has been restricted to dimension $d = 1$. For example, see [4, 23, 34] for the uncentered Hardy-Littlewood maximal operator, [18] for the centered Hardy-Littlewood maximal operator, [10] for the fractional maximal operators, [26] for the multisublinear fractional maximal operators. Other interesting works related to this topic are [3, 5, 6, 12, 13, 22, 28].

In general, bounded non-sublinear operators need not be continuous (see [7] for a famous example). Continuity of the maximal operator in $L^p(\mathbb{R}^d)$ follows easily from its sublinearity and boundedness. Since we do not have sublinearity for the weak derivatives of the Hardy-Littlewood maximal function, the result of Kinnunen now leads us to another question: is $M : W^{1,p}(\mathbb{R}^d) \rightarrow W^{1,p}(\mathbb{R}^d)$ continuous for $1 < p < \infty$? This question was posed in [14, Question 3] where it was attributed to T. Iwaniec and has been studied by many authors. In 2007, Luiro first proved that M is continuous on $W^{1,p}(\mathbb{R}^d)$ for $1 < p < \infty$. Subsequently, Carneiro and Moreira [11] extended above result to the bilinear case. Later on, Luiro [30] extended the result of [29] to the local case. Recently, Luiro and Nuutinen [31] established the continuity of a class of discrete maximal operators in Sobolev space $W^{1,p}(\mathbb{R}^d)$ under certain sufficient assumptions.

As well known, the multilinear Calderón-Zygmund theory and the boundedness of these maximal operators on various function spaces have been extensively studied, for example, see [8, 9, 19, 20, 21, 25] and therein references. In this paper we focus on the regularity of the multisublinear fractional maximal operator. More precisely, let $m \geq 1$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^1_{\text{loc}}(\mathbb{R}^d)$. For $0 \leq \alpha < md$, we define the multisublinear fractional maximal operator \mathfrak{M}_α by

$$\mathfrak{M}_\alpha(\vec{f})(x) = \sup_{r>0} |B(x,r)|^{\alpha/d-m} \prod_{i=1}^m \int_{B(x,r)} |f_i(y)| dy,$$

for any $x \in \mathbb{R}^d$. For $\alpha = 0$, the operator \mathfrak{M}_α recovers the classical multisublinear Hardy-Littlewood maximal operator. Specially, the centered Hardy-Littlewood maximal operator M corresponds to the special case of \mathfrak{M}_α for $m = 1$ and $\alpha = 0$. For $m = 1$ and $0 < \alpha < d$, the operator \mathfrak{M}_α reduces to the classical fractional maximal operator denoted by M_α , which has extensive applications in potential theory and partial differential equations (see [1, 2, 32, 33] for example).

Recently, it was shown in [25] that

$$\mathfrak{M}_\alpha : W^{1,p_1}(\mathbb{R}^d) \times \dots \times W^{1,p_m}(\mathbb{R}^d) \rightarrow W^{1,q}(\mathbb{R}^d)$$

is bounded for $1/q = \sum_{i=1}^m 1/p_i - \alpha/d$ with $1 < p_1, \dots, p_m < \infty$ and $1 \leq q < \infty$. A question that arises naturally is

QUESTION A. *Is the operator \mathfrak{M}_α continuous from $W^{1,p_1}(\mathbb{R}^d) \times \dots \times W^{1,p_m}(\mathbb{R}^d)$ to $W^{1,q}(\mathbb{R}^d)$, when $1/q = \sum_{i=1}^m 1/p_i - \alpha/d$, $1 < p_1, \dots, p_m < \infty$ and $1 \leq q < \infty$?*

Question A is the main motivation for this work. This problem is solved by our next theorem.

THEOREM 1. *Let $0 \leq \alpha < md$. Then \mathfrak{M}_α maps $W^{1,p_1}(\mathbb{R}^d) \times \dots \times W^{1,p_m}(\mathbb{R}^d)$ into $W^{1,q}(\mathbb{R}^d)$ continuously, where $1/q = \sum_{i=1}^m 1/p_i - \alpha/d$, $1 < p_1, \dots, p_m < \infty$ and $1 \leq q < \infty$.*

REMARK 1. Theorem 1 extends the continuity result in [29], which corresponds to the case $m = 1$ and $\alpha = 0$. In [17], Kinnunen and Saksman observed that $M_\alpha : W^{1,p}(\mathbb{R}^d) \rightarrow W^{1,q}(\mathbb{R}^d)$ is bounded if $1/q = 1/p - \alpha/d$ with $1 < p < \infty$ and $0 \leq \alpha < d/p$. As an application of Theorem 1, it is known that $M_\alpha : W^{1,p}(\mathbb{R}^d) \rightarrow W^{1,q}(\mathbb{R}^d)$ is continuous if $1/q = 1/p - \alpha/d$ with $1 < p < \infty$ and $0 \leq \alpha < d/p$.

To the best of my knowledge Hajlasz and Onninen's question remains open in dimension $d \geq 2$. Motivated by this challenging problem, Hajlasz and Maly [13] established the approximate differentiability of the Hardy-Littlewood maximal operator, which can be listed as follows.

THEOREM B. ([13]) *Let $f \in L^1(\mathbb{R}^d)$ be approximately differentiable a.e., then Mf is also approximately differentiable a.e.*

For the endpoint regularity of \mathfrak{M}_α , it was shown in [26, 27] that if $d \geq 2$, $1 \leq \alpha < m(d-1) + 1$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in W^{1,1}(\mathbb{R}^d)$, then $\mathfrak{M}_\alpha(\vec{f})$ is weakly differentiable and

$$\|\nabla \mathfrak{M}_\alpha(\vec{f})\|_{L^{\frac{d}{m(d-1)-\alpha+1}}(\mathbb{R}^d)} \leq C \prod_{j=1}^m \|\nabla f_j\|_{L^1(\mathbb{R}^d)}.$$

A natural and interesting question is that what about the smoothness properties of the maximal function $\mathfrak{M}_\alpha(\vec{f})$ if $0 \leq \alpha < 1$ or $m(d-1) + 1 \leq \alpha < md$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in W^{1,1}(\mathbb{R}^d)$. Motivated by Theorem B, we shall establish the following result.

THEOREM 2. *Let $0 \leq \alpha \leq md - 1$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^1(\mathbb{R}^d)$ being approximately differentiable a.e., then $\mathfrak{M}_\alpha(\vec{f})$ is approximately differentiable a.e.*

REMARK 2. Clearly, Theorem 2 extends Theorem B, which corresponds to the case $m = 1$ and $\alpha = 0$. Since every function $f \in W^{1,1}(\mathbb{R}^d)$ is approximately differentiable a.e., Theorem 2 implies that if $0 \leq \alpha \leq md - 1$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in W^{1,1}(\mathbb{R}^d)$, then $\mathfrak{M}_\alpha(\vec{f})$ is approximately differentiable a.e. Moreover, we know from Theorem 2 that the fractional maximal function $M_\alpha f$ is approximately differentiable a.e. if $f \in L^1(\mathbb{R}^d)$ is approximately differentiable a.e. and $0 \leq \alpha \leq d - 1$.

The rest of this paper is organized as follows. In Section 2 we prove Theorem 1, the proof of Theorem 2 will be given in Section 3. We remark that the main ideas in our proofs are greatly motivated by [13, 29], but our methods and techniques are more subtle and complex than that of [13, 29]. The main ingredient of our proof of Theorem 1 is to present an explicit formula for the derivative of the multisublinear fractional maximal functions (see Lemma 2). In what follows, we use the following conventions

$$\prod_{i \in \emptyset} a_i = 1 \quad \text{and} \quad \sum_{i \in \emptyset} a_i = 0.$$

2. Proof of Theorem 1

In this section we follow carefully the proof for the continuity in $W^{1,p}(\mathbb{R}^d)$ of the Hardy-Littlewood maximal operator in [29] and simply adjust the notation to our context.

For $R > 0$, we denote by B_R the ball of radius R centered at the origin. For $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we define

$$d(x, A) := \inf_{a \in A} |x - a| \quad \text{and} \quad A_{(\lambda)} := \{x \in \mathbb{R}^d; d(x, A) \leq \lambda\} \quad \text{for } \lambda \geq 0.$$

We denote by $\|f\|_{p,A}$ the L^p -norm of $f\chi_A$ for all measurable sets $A \subset \mathbb{R}^d$. Let $\vec{f} = (f_1, \dots, f_m) \in L^{p_1}(\mathbb{R}^d) \times \dots \times L^{p_m}(\mathbb{R}^d)$ with $1 < p_1, \dots, p_m < \infty$, $1 \leq q < \infty$ and $1/q = \sum_{i=1}^m 1/p_i - \alpha/d$. For a fixed point $x \in \mathbb{R}^d$, we define the set $\mathcal{R}(\vec{f})(x)$ by

$$\begin{aligned} & \mathcal{R}(\vec{f})(x) \\ & := \left\{ r \geq 0 : \mathfrak{M}_\alpha(\vec{f})(x) = \limsup_{r_k \rightarrow r} |B(x, r_k)|^{\alpha/d-m} \prod_{i=1}^m \int_{B(x, r_k)} |f_i(y)| dy \text{ for some } r_k > 0 \right\}. \end{aligned}$$

We also define the function $u_{x, \vec{f}} : [0, \infty) \rightarrow \mathbb{R}$ by

$$\begin{aligned} u_{x, \vec{f}}(0) &= \begin{cases} \prod_{i=1}^m |f_i(x)|, & \text{if } \alpha = 0; \\ 0, & \text{if } 0 < \alpha < md, \end{cases} \\ u_{x, \vec{f}}(r) &= |B(x, r)|^{\alpha/d-m} \prod_{i=1}^m \int_{B(x, r)} |f_i(y)| dy \quad \text{when } r \in (0, \infty). \end{aligned}$$

We notice that the following facts are valid: (i) $u_{x, \vec{f}}$ is continuous on $(0, \infty)$ for all $x \in \mathbb{R}^d$ and at $r = 0$ for a.e. $x \in \mathbb{R}^d$; (ii) $\lim_{r \rightarrow \infty} u_{x, \vec{f}}(r) = 0$ since $u_{x, \vec{f}}(r) \leq \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^d)} |B(x, r)|^{-1/q}$ for any $r > 0$ and $x \in \mathbb{R}^d$; (iii) $\mathcal{R}(\vec{f})(x)$ is nonempty and closed for all $x \in \mathbb{R}^d$ and

$$\mathfrak{M}_\alpha(\vec{f})(x) = u_{x, \vec{f}}(r) \text{ if } 0 < r \in \mathcal{R}(\vec{f})(x), \quad \forall x \in \mathbb{R}^d,$$

$$\mathfrak{M}_\alpha(\vec{f})(x) = u_{x, \vec{f}}(0) \text{ for a.e. } x \in \mathbb{R}^d \text{ such that } 0 \in \mathcal{R}(\vec{f})(x).$$

Motivated by the idea in [29], we can get the following

LEMMA 1. Let $\vec{f}_j = (f_{1,j}, f_{2,j}, \dots, f_{m,j})$. Suppose that $f_{i,j} \rightarrow f_i$ in $L^{p_i}(\mathbb{R}^d)$ when $j \rightarrow \infty$ for all $i = 1, 2, \dots, m$, where $1 < p_1, \dots, p_m < \infty$, $1 \leq q < \infty$ and $1/q = \sum_{i=1}^m 1/p_i - \alpha/d$. Then for all $R > 0$ and $\lambda > 0$, we have

$$\lim_{j \rightarrow \infty} |\{x \in B_R; \mathcal{R}(\vec{f}_j)(x) \not\leq \mathcal{R}(\vec{f})(x)_\lambda\}| = 0. \quad (1)$$

Proof. Without loss of generality we may assume that all $f_{i,j} \geq 0$ and $f_i \geq 0$. By the similar argument as in the proof of [29, Lemma 2.2], we can conclude that the set $\{x \in \mathbb{R}^d; \mathcal{R}(\vec{f}_j)(x) \not\leq \mathcal{R}(\vec{f})(x)_\lambda\}$ is measurable for any $j \in \mathbb{Z}$. Let $\lambda > 0$, $R > 0$ and $\varepsilon \in (0, 1)$. We can claim that for a.e. $x \in B_R$, there exists $\gamma(x) \in \mathbb{N} \setminus \{0\}$ such that

$$u_{x, \vec{f}}(r) < \mathfrak{M}_\alpha(\vec{f})(x) - \frac{1}{\gamma(x)}, \quad \text{when } d(r, \mathcal{R}(\vec{f})(x)) > \lambda. \quad (2)$$

Otherwise, for a.e. $x \in B_R$, there exists a bounded sequence of radii $\{r_k\}_{k=1}^\infty$ such that

$$\lim_{k \rightarrow \infty} u_{x, \vec{f}}(r_k) = \mathfrak{M}_\alpha(\vec{f})(x) \quad \text{and} \quad d(r_k, \mathcal{R}(\vec{f})(x)) > \lambda.$$

We can choose a subsequence $\{s_k\}_{k=1}^\infty$ of $\{r_k\}_{k=1}^\infty$ such that $s_k \rightarrow r$ as $k \rightarrow \infty$. It follows that $r \in \mathcal{R}(\vec{f})(x)$ and $d(r, \mathcal{R}(\vec{f})(x)) \geq \lambda$, which is a contradiction. Thus, (2) holds. From (2) we can conclude that there exists $\gamma = \gamma(R, \lambda, \varepsilon) \in \mathbb{N} \setminus \{0\}$ and a measurable set E with $|E| < \varepsilon$ such that

$$\begin{aligned} B_R &\subset \{x \in \mathbb{R}^d : u_{x, \vec{f}}(r) < \mathfrak{M}_\alpha(\vec{f})(x) - \gamma^{-1} \text{ if } d(r, \mathcal{R}(\vec{f})(x)) > \lambda\} \cup E \\ &\subset A_{1,j} \cup A_{2,j} \cup A_{3,j} \cup E, \end{aligned} \quad (3)$$

where

$$A_{1,j} := \{x \in \mathbb{R}^d : |\mathfrak{M}_\alpha(\vec{f}_j)(x) - \mathfrak{M}_\alpha(\vec{f})(x)| \geq (4\gamma)^{-1}\},$$

$$A_{2,j} := \{x \in \mathbb{R}^d : |u_{x, \vec{f}_j}(r) - u_{x, \vec{f}}(r)| \geq (2\gamma)^{-1} \text{ for some } r \text{ such that } d(r, \mathcal{R}(\vec{f})(x)) > \lambda\},$$

$$A_{3,j} := \{x \in \mathbb{R}^d : u_{x, \vec{f}_j}(r) < \mathfrak{M}_\alpha(\vec{f}_j)(x) - (4\gamma)^{-1} \text{ if } d(r, \mathcal{R}(\vec{f})(x)) > \lambda\}.$$

Let \bar{A} be the set of all points x such that x is a Lebesgue point of all f_j . Note that $|\mathbb{R}^d \setminus \bar{A}| = 0$. One can easily check that $A_{3,j} \cap \bar{A} \subset \cup \{x \in \mathbb{R}^d : \mathcal{R}(\vec{f}_j)(x) \subset \mathcal{R}(\vec{f})(x)_\lambda\}$. This together with (3) yields that

$$\{x \in B_R; \mathcal{R}(\vec{f}_j)(x) \not\leq \mathcal{R}(\vec{f})(x)_\lambda\} \subset A_{1,j} \cup A_{2,j} \cup E \cup (\mathbb{R}^d \setminus \bar{A}).$$

It follows that

$$|\{x \in B_R; \mathcal{R}(\vec{f}_j)(x) \not\leq \mathcal{R}(\vec{f})(x)_\lambda\}| \leq |A_{1,j}| + |A_{2,j}| + \varepsilon. \quad (4)$$

Since $f_{i,j} \rightarrow f_i$ in $L^{p_i}(\mathbb{R}^d)$ when $j \rightarrow \infty$ for all $i = 1, 2, \dots, m$, then for any fixed $\varepsilon \in (0, 1)$, there exists $N_0 = N_0(\varepsilon) \in \mathbb{N}$ such that

$$\|f_{i,j} - f_i\|_{L^{p_i}(\mathbb{R}^d)} < \frac{\varepsilon}{\gamma} \quad \text{and} \quad \|f_{i,j}\|_{L^{p_i}(\mathbb{R}^d)} \leq \|f_i\|_{L^{p_i}(\mathbb{R}^d)} + 1 \quad (5)$$

for any $j \geq N_0$ and $i = 1, 2, \dots, m$. We can write

$$\begin{aligned}
& |\mathfrak{M}_\alpha(\vec{f}_j)(x) - \mathfrak{M}_\alpha(\vec{f})(x)| \\
& \leq \sup_{r>0} |B(x,r)|^{\alpha/d-m} \left| \prod_{i=1}^m \int_{B(x,r)} f_{i,j}(y) dy - \prod_{i=1}^m \int_{B(x,r)} f_i(y) dy \right| \\
& \leq \sum_{i=1}^m \sup_{r>0} |B(x,r)|^{\alpha/d-m} \prod_{\mu=1}^{i-1} \int_{B(x,r)} f_{\mu}(y) dy \prod_{v=i+1}^m \int_{B(x,r)} f_{v,j}(y) dy \\
& \quad \times \int_{B(x,r)} |f_{i,j}(y) - f_i(y)| dy \\
& = \sum_{i=1}^m \mathfrak{M}_\alpha(\vec{F}_j^i)(x)
\end{aligned} \tag{6}$$

for any $x \in \mathbb{R}^d$, where $\vec{F}_j^i = (f_1, \dots, f_{i-1}, f_{i,j} - f_i, f_{i+1,j}, \dots, f_{m,j})$. It follows from (5) and (6) that

$$\begin{aligned}
|A_{1,j}| & \leq \left| \left\{ x \in \mathbb{R}^d : \sum_{i=1}^m \mathfrak{M}_\alpha(\vec{F}_j^i)(x) \geq (4\gamma)^{-1} \right\} \right| \\
& \leq \sum_{i=1}^m \left| \left\{ x \in \mathbb{R}^d : \mathfrak{M}_\alpha(\vec{F}_j^i)(x) \geq (4m\gamma)^{-1} \right\} \right| \\
& \leq (4m\gamma)^q \sum_{i=1}^m \|\mathfrak{M}_\alpha(\vec{F}_j^i)\|_{L^q(\mathbb{R}^d)}^q \\
& \leq C(m, \gamma, q, \alpha, d, p_1, \dots, p_m) \varepsilon
\end{aligned} \tag{7}$$

for any $j \geq N_0$. Similarly, we can get

$$|A_{2,j}| \leq C(m, \gamma, q, \alpha, d, p_1, \dots, p_m) \varepsilon \tag{8}$$

for any $j \geq N_0$. (1) follows from (4), (7) and (8). \square

Let $e_l = (0, \dots, 0, 1, 0, \dots, 0)$ be the canonical l -th base vector in \mathbb{R}^d for $l = 1, 2, \dots, d$. For any fixed $i = 1, 2, \dots, m$, $h > 0$ and $f_i \in L^p(\mathbb{R}^d)$ with $p \geq 1$, define

$$f_{i,h}^l(x) = \frac{f_{\tau(h)}^{i,l}(x) - f_i(x)}{h} \quad \text{and} \quad f_{\tau(h)}^{i,l}(x) = f_i(x + he_l).$$

It is well known that for $p \geq 1$, $f_{\tau(h)}^{i,l} \rightarrow f_i$ in $L^p(\mathbb{R}^d)$ when $h \rightarrow 0$, and if $f_i \in W^{1,p}(\mathbb{R}^d)$ we have $f_{i,h}^l \rightarrow D_l f_i$ in $L^p(\mathbb{R}^d)$ when $h \rightarrow 0$. Let A, B be two subsets of \mathbb{R}^d , we define the Hausdorff distance of A and B by

$$\pi(A, B) := \inf\{\delta > 0 : A \subset B_{(\delta)} \text{ and } B \subset A_{(\delta)}\}.$$

Applying Lemma 1 and the argument similar to that in the proof of [29, Corollary 2.3], we can get the following result. The details are omitted.

COROLLARY 1. *Let $\vec{f} = (f_1, \dots, f_m) \in L^{p_1}(\mathbb{R}^d) \times \dots \times L^{p_m}(\mathbb{R}^d)$ with $1 \leq q < \infty$, $1 < p_1, \dots, p_m < \infty$ and $1/q = \sum_{i=1}^m 1/p_i - \alpha/d$. Then for all $R > 0$, $\lambda > 0$ and $l = 1, 2, \dots, d$, we have*

$$|\{x \in B_R; \pi(\mathcal{R}(\vec{f})(x), \mathcal{R}(\vec{f})(x + he_l)) > \lambda\}| \rightarrow 0 \quad \text{when } h \rightarrow 0.$$

We now state several formulas for the derivative of the multisublinear maximal fractional functions, which play a key role in the proof of Theorem 1.

LEMMA 2. Let $\vec{f} = (f_1, f_2, \dots, f_m) \in W^{1,p_1}(\mathbb{R}^d) \times \dots \times W^{1,p_m}(\mathbb{R}^d)$ with $1 \leq q < \infty$, $1 < p_1, \dots, p_m < \infty$ and $1/q = \sum_{i=1}^m 1/p_i - \alpha/d$. Then for any $l = 1, 2, \dots, d$ and a.e. $x \in \mathbb{R}^d$, we have

$$D_l \mathfrak{M}_\alpha(\vec{f})(x) = \sum_{i=1}^m |B(x, r)|^{\alpha/d-m} \left(\prod_{1 \leq j \neq i \leq m} \int_{B(x, r)} |f_j(y)| dy \right) \\ \times \int_{B(x, r)} D_l(|f_i|)(y) dy \text{ for all } 0 < r \in \mathcal{R}(\vec{f})(x),$$

$$D_l \mathfrak{M}_\alpha(\vec{f})(x) = \begin{cases} \sum_{i=1}^m D_l |f_i|(x) \prod_{1 \leq j \neq i \leq m} |f_j(x)|, & \text{if } \alpha = 0 \text{ and } 0 \in \mathcal{R}(\vec{f})(x), \\ 0, & \text{if } 0 < \alpha < md \text{ and } 0 \in \mathcal{R}(\vec{f})(x). \end{cases}$$

Proof. We may assume without loss of generality that all $f_i \geq 0$, since $|f_i| \in W^{1,p_i}(\mathbb{R}^d)$ if $f_i \in W^{1,p_i}(\mathbb{R}^d)$. Let $R > 0$. Invoking Corollary 1, we can choose a sequence $\{s_k\}_{k=1}^\infty$, $s_k > 0$ and $s_k \rightarrow 0$ such that $\lim_{k \rightarrow \infty} \pi(\mathcal{R}(\vec{f})(x), \mathcal{R}(\vec{f})(x + s_k e_l)) = 0$ for a.e. $x \in B_R$. Then for any $i = 1, 2, \dots, m$ and $l = 1, 2, \dots, d$ we have

$$\|f_{\tau(s_k)}^{i,l} - f_i\|_{L^{p_i}(\mathbb{R}^d)} \rightarrow 0 \text{ as } k \rightarrow \infty, \\ \|f_{i,s_k}^l - D_l f_i\|_{L^{p_i}(\mathbb{R}^d)} \rightarrow 0 \text{ as } k \rightarrow \infty, \\ \|M(f_{\tau(s_k)}^{i,l} - f_i)\|_{L^{p_i}(\mathbb{R}^d)} \rightarrow 0 \text{ as } k \rightarrow \infty, \\ \|M(f_{i,s_k}^l - D_l f_i)\|_{L^{p_i}(\mathbb{R}^d)} \rightarrow 0 \text{ as } k \rightarrow \infty, \\ \|(\mathfrak{M}_\alpha(\vec{f}))_{s_k}^l - D_l \mathfrak{M}_\alpha(\vec{f})\|_{L^q(\mathbb{R}^d)} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Furthermore, there exists a subsequence $\{h_k\}_{k=1}^\infty$ of $\{s_k\}_{k=1}^\infty$ and a measurable set $A_1 \subset B_R$ such that $|B_R \setminus A_1| = 0$ and

(i) $f_{\tau(h_k)}^{i,l}(x) \rightarrow f_i(x)$, $f_{i,h_k}^l(x) \rightarrow D_l f_i(x)$, $M(f_{\tau(h_k)}^{i,l} - f_i)(x) \rightarrow 0$, $M(f_{i,h_k}^l - D_l f_i)(x) \rightarrow 0$ and $(\mathfrak{M}_\alpha(\vec{f}))_{h_k}^l(x) \rightarrow D_l \mathfrak{M}_\alpha(\vec{f})(x)$ when $k \rightarrow \infty$ for any $x \in A_1$, $i = 1, 2, \dots, m$ and $l = 1, \dots, d$;

(ii) $\lim_{k \rightarrow \infty} \pi(\mathcal{R}(\vec{f})(x), \mathcal{R}(\vec{f})(x + h_k e_l)) = 0$ for any $x \in A_1$.

Let

$$A_2 := \bigcap_{k=1}^\infty \{x \in \mathbb{R}^d : \mathfrak{M}_\alpha(\vec{f})(x + h_k e_l) \geq u_{x+h_k e_l, \vec{f}}(0)\},$$

$$A_3 := \{x \in \mathbb{R}^d : \mathfrak{M}_\alpha(\vec{f})(x) = u_{x, \vec{f}}(0) \text{ if } 0 \in \mathcal{R}(\vec{f})(x)\},$$

$$A_4 := \bigcap_{k=1}^\infty \{x \in \mathbb{R}^d : \mathfrak{M}_\alpha(\vec{f})(x + h_k e_l) = u_{x+h_k e_l, \vec{f}}(0) \text{ if } 0 \in \mathcal{R}(\vec{f})(x + h_k e_l)\}.$$

One can easily check that $|B_R \setminus A_i| = 0$ for any $i = 2, 3, 4$. Let $x \in A_1 \cap A_2 \cap A_3 \cap A_4$ be a Lebesgue point of all f_i and $D_l f_i$ and $r \in \mathcal{R}(\vec{f})(x)$, there exists radii $r_k \in \mathcal{R}(\vec{f})(x + h_k e_l)$ such that $\lim_{k \rightarrow \infty} r_k = r$. We now complete the rest of proof by considering the following two cases:

Case A ($r > 0$). Without loss of generality we may assume that all $r_k > 0$. We can write

$$\begin{aligned}
D_l \mathfrak{M}_\alpha(\vec{f})(x) &= \lim_{k \rightarrow \infty} \frac{1}{h_k} (\mathfrak{M}_\alpha(\vec{f})(x + h_k e_l) - \mathfrak{M}_\alpha(\vec{f})(x)) \\
&\leq \lim_{k \rightarrow \infty} \frac{1}{h_k} (u_{x+h_k e_l, \vec{f}}(r_k) - u_{x, \vec{f}}(r_k)) \\
&= \lim_{k \rightarrow \infty} \frac{1}{h_k} \sum_{j=1}^m |B(x, r_k)|^{\alpha/d-m} \int_{B(x, r_k)} (f_j(y + h_k e_l) - f_j(y)) dy \\
&\quad \times \prod_{\mu=1}^{j-1} \int_{B(x, r_k)} f_\mu(y + h_k e_l) dy \prod_{v=j+1}^m \int_{B(x, r_k)} f_v(y) dy \\
&= \sum_{j=1}^m \lim_{k \rightarrow \infty} |B(x, r_k)|^{\alpha/d-m} \int_{B(x, r_k)} f_{j, h_k}^l(y) dy \\
&\quad \times \prod_{\mu=1}^{j-1} \int_{B(x, r_k)} f_{\tau(h_k)}^{\mu, l}(y) dy \prod_{v=j+1}^m \int_{B(x, r_k)} f_v(y) dy.
\end{aligned} \tag{9}$$

Since $\lim_{k \rightarrow \infty} |B(x, r_k)| = |B_r|$, $f_{\tau(h_k)}^{\mu, l} \chi_{B(x, r_k)} \rightarrow f_i \chi_{B(x, r)}$ and $f_{j, h_k}^l \chi_{B(x, r_k)} \rightarrow D_l f_i \chi_{B(x, r)}$ in $L^1(\mathbb{R}^d)$ as $k \rightarrow \infty$. It follows that

$$D_l \mathfrak{M}_\alpha(\vec{f})(x) \leq \sum_{i=1}^m |B(x, r)|^{\alpha/d-m} \left(\prod_{1 \leq j \neq i \leq m} \int_{B(x, r)} f_j(y) dy \right) \int_{B(x, r)} D_l f_i(y) dy.$$

On the other hand, we have

$$\begin{aligned}
D_l \mathfrak{M}_\alpha(\vec{f})(x) &= \lim_{k \rightarrow \infty} \frac{1}{h_k} (\mathfrak{M}_\alpha(\vec{f})(x + h_k e_l) - \mathfrak{M}_\alpha(\vec{f})(x)) \\
&\geq \lim_{k \rightarrow \infty} \frac{1}{h_k} (u_{x+h_k e_l, \vec{f}}(r) - u_{x, \vec{f}}(r)) \\
&= \lim_{k \rightarrow \infty} \frac{1}{h_k} \sum_{j=1}^m |B(x, r)|^{\alpha/d-m} \int_{B(x, r)} (f_j(y + h_k e_l) - f_j(y)) dy \\
&\quad \times \prod_{\mu=1}^{j-1} \int_{B(x, r)} f_\mu(y + h_k e_l) dy \prod_{v=j+1}^m \int_{B(x, r)} f_v(y) dy \\
&= \sum_{j=1}^m |B(x, r)|^{\alpha/d-m} \lim_{k \rightarrow \infty} \int_{B(x, r)} f_{j, h_k}^l(y) dy \\
&\quad \times \prod_{\mu=1}^{j-1} \int_{B(x, r)} f_{\tau(h_k)}^{\mu, l}(y) dy \prod_{v=j+1}^m \int_{B(x, r)} f_v(y) dy \\
&= \sum_{i=1}^m |B(x, r)|^{\alpha/d-m} \int_{B(x, r)} D_l f_i(y) dy \left(\prod_{1 \leq j \neq i \leq m} \int_{B(x, r)} f_j(y) dy \right).
\end{aligned}$$

Case B ($r = 0$). When $0 < \alpha < md$. Since $\mathfrak{M}_\alpha(\vec{f})(x) = 0$ and then $\prod_{i=1}^m f_i(y) = 0$ for a.e. $y \in \mathbb{R}^d$. Thus $\mathfrak{M}_\alpha(\vec{f}) \equiv 0$ and $D_l \mathfrak{M}_\alpha(\vec{f})(x) = 0$. We now consider the case $\alpha = 0$. Let us begin with estimating the lower bound of $D_l \mathfrak{M}_\alpha(\vec{f})(x)$. We can write

$$\begin{aligned}
 D_l \mathfrak{M}_\alpha(\vec{f})(x) &= \lim_{k \rightarrow \infty} \frac{1}{h_k} (\mathfrak{M}_\alpha(\vec{f})(x + h_k e_l) - \mathfrak{M}_\alpha(\vec{f})(x)) \\
 &\geq \lim_{k \rightarrow \infty} \frac{1}{h_k} \left(\prod_{i=1}^m f_i(x + h_k e_l) - \prod_{i=1}^m f_i(x) \right) \\
 &= \lim_{k \rightarrow \infty} \frac{1}{h_k} \sum_{j=1}^m (f_j(x + h_k e_l) - f_j(x)) \prod_{i=1}^{j-1} f_i(x + h_k e_l) \prod_{i=j+1}^m f_i(x) \\
 &= \sum_{i=1}^m D_l f_i(x) \prod_{1 \leq j \neq i \leq m} f_j(x).
 \end{aligned} \tag{10}$$

Below we estimate the upper bound of $D_l \mathfrak{M}_\alpha(\vec{f})(x)$. If we have $r_k = 0$ for infinitely many k , then

$$\begin{aligned}
 D_l \mathfrak{M}_\alpha(\vec{f})(x) &= \lim_{k \rightarrow \infty} \frac{1}{h_k} (\mathfrak{M}_\alpha(\vec{f})(x + h_k e_l) - \mathfrak{M}_\alpha(\vec{f})(x)) \\
 &= \lim_{k \rightarrow \infty} \frac{1}{h_k} \left(\prod_{i=1}^m f_i(x + h_k e_l) - \prod_{i=1}^m f_i(x) \right) \\
 &= \lim_{k \rightarrow \infty} \frac{1}{h_k} \sum_{j=1}^m (f_j(x + h_k e_l) - f_j(x)) \prod_{\mu=1}^{j-1} f_\mu(x + h_k e_l) \prod_{v=j+1}^m f_v(x) \\
 &= \sum_{i=1}^m D_l f_i(x) \prod_{1 \leq j \neq i \leq m} f_j(x).
 \end{aligned}$$

If there exists $k_0 \in \mathbb{N}$ such that $r_k > 0$ when $k \geq k_0$. We get from (9) that

$$\begin{aligned}
 D_l \mathfrak{M}_\alpha(\vec{f})(x) &\leq \sum_{j=1}^m \lim_{k \rightarrow \infty} |B(x, r_k)|^{-m} \int_{B(x, r_k)} f_{j, h_k}^l(y) dy \\
 &\quad \times \prod_{\mu=1}^{j-1} \int_{B(x, r_k)} f_{\tau(h_k)}^{\mu, l}(y) dy \prod_{v=j+1}^m \int_{B(x, r_k)} f_v(y) dy.
 \end{aligned} \tag{11}$$

Since

$$\begin{aligned}
 &\lim_{k \rightarrow \infty} \left| \frac{1}{|B(x, r_k)|} \int_{B(x, r_k)} f_{j, h_k}^l(y) dy - D_l f_j(x) \right| \\
 &= \lim_{k \rightarrow \infty} \left| \frac{1}{|B(x, r_k)|} \int_{B(x, r_k)} (f_{j, h_k}^l(y) - D_l f_j(y)) dy \right| \\
 &\leq \lim_{k \rightarrow \infty} M(f_{j, h_k}^l - D_l f_j)(x) = 0.
 \end{aligned}$$

It follows that

$$\lim_{k \rightarrow \infty} \frac{1}{|B(x, r_k)|} \int_{B(x, r_k)} f_{j, h_k}^l(y) dy = D_l f_j(x). \tag{12}$$

By argument similar to those used to derive (11),

$$\lim_{k \rightarrow \infty} \frac{1}{|B(x, r_k)|} \int_{B(x, r_k)} f_{\tau(h_k)}^{\mu, l}(y) dy = f_{\mu}(x). \quad (13)$$

It follows from (11)–(13) that

$$D_l \mathfrak{M}_{\alpha}(\vec{f})(x) \leq \sum_{i=1}^m D_l f_i(x) \prod_{1 \leq j \neq i \leq m} f_j(x),$$

which together with (10) yields

$$D_l \mathfrak{M}_{\alpha}(\vec{f})(x) = \sum_{i=1}^m D_l f_i(x) \prod_{1 \leq j \neq i \leq m} f_j(x).$$

Since R was arbitrary and $|B_R \setminus (A_1 \cap A_2 \cap A_3 \cap A_4)| = 0$. This proves Lemma 2. \square

We are now in the position of proving Theorem 1.

Proof of Theorem 1. Let $p_1, \dots, p_m, q, \alpha$ be given as in Theorem 1 and $\vec{f} = (f_1, \dots, f_m) \in W^{1, p_1}(\mathbb{R}^d) \times \dots \times W^{1, p_m}(\mathbb{R}^d)$. For $i = 1, 2, \dots, m$, let $\vec{f}_j = (f_{1,j}, \dots, f_{m,j})$ and $f_{i,j} \rightarrow f_i$ in $W^{1, p_i}(\mathbb{R}^d)$ when $j \rightarrow \infty$. We get from (6) that

$$\begin{aligned} \|\mathfrak{M}_{\alpha}(\vec{f}_j) - \mathfrak{M}_{\alpha}(\vec{f})\|_{L^q(\mathbb{R}^d)} &\leq \sum_{i=1}^m \|\mathfrak{M}_{\alpha}(\vec{F}_j^i)\|_{L^q(\mathbb{R}^d)} \\ &\leq \sum_{i=1}^m \|f_{i,j} - f_i\|_{L^{p_i}(\mathbb{R}^d)} \prod_{\mu=1}^{i-1} \|f_{\mu}\|_{L^{p_{\mu}}(\mathbb{R}^d)} \prod_{v=i+1}^m \|f_{v,j}\|_{L^{p_v}(\mathbb{R}^d)}, \end{aligned}$$

where \vec{F}_j^i is given as in (6). It follows that

$$\|\mathfrak{M}_{\alpha}(\vec{f}_j) - \mathfrak{M}_{\alpha}(\vec{f})\|_{L^q(\mathbb{R}^d)} \rightarrow 0 \quad \text{when } j \rightarrow \infty.$$

It suffices to show that

$$\|D_l \mathfrak{M}_{\alpha}(\vec{f}_j) - D_l \mathfrak{M}_{\alpha}(\vec{f})\|_{L^q(\mathbb{R}^d)} \rightarrow 0 \quad \text{when } j \rightarrow \infty \quad (14)$$

for any $l = 1, 2, \dots, d$. We only prove (14) for $l = d$ (since other cases are analogous). We may assume without loss of generality that all $f_i \geq 0$ and $f_{i,j} \geq 0$.

For any $i = 1, 2, \dots, m$, let $\vec{f}^i = (f_1, \dots, f_{i-1}, D_d f_i, f_{i+1}, \dots, f_m)$. Given $\varepsilon > 0$, there exists $R > 0$ such that $\sum_{i=1}^m \|\mathfrak{M}_{\alpha}(\vec{f}^i)\|_{q, G_1} < \varepsilon$ with $G_1 = \mathbb{R}^d \setminus B_R$. By absolute continuity, there exists $\eta > 0$ such that $\sum_{i=1}^m \|\mathfrak{M}_{\alpha}(\vec{f}^i)\|_{q, A} < \varepsilon$ whenever $|A| < \eta$ and A is a measurable subset of B_R . As we already observed, for a.e. $x \in \mathbb{R}^d$ and any $1 \leq i \leq m$, the function u_{x, \vec{f}^i} is uniformly continuous on $[0, \infty)$. Thus, for a.e. $x \in \mathbb{R}^d$, the function $\sum_{i=1}^m u_{x, \vec{f}^i}$ is uniformly continuous on $[0, \infty)$ and we can find $\delta_x > 0$ such that

$$\left| \sum_{i=1}^m u_{x, \vec{f}^i}(r_1) - \sum_{i=1}^m u_{x, \vec{f}^i}(r_2) \right| < |B_R|^{-1/q} \varepsilon \quad \text{whenever } |r_1 - r_2| < \delta_x.$$

It follows that there exists $\delta > 0$ such that

$$\begin{aligned} & \left| \left\{ x \in B_R : \left| \sum_{i=1}^m u_{x, \vec{f}_i}(r_1) - \sum_{i=1}^m u_{x, \vec{f}_i}(r_2) \right| \right. \right. \\ & \left. \left. \geq |B_R|^{-1/q} \varepsilon \text{ for some } r_1, r_2 \text{ with } |r_1 - r_2| < \delta \right\} \right| \\ & =: |G_2| < \frac{\eta}{2}. \end{aligned}$$

Applying Lemma 1, there exists $j_1 \in \mathbb{N}$ such that

$$|\{x \in B_R; \mathcal{R}(\vec{f}_j)(x) \not\subseteq \mathcal{R}(\vec{f})(x)_{(\delta)}\}| =: |B^j| < \frac{\eta}{2} \text{ when } j \geq j_1.$$

Let $\vec{f}_{i,j} = (f_{1,j}, \dots, f_{i-1,j}, D_d f_{i,j}, f_{i+1,j}, \dots, f_{m,j})$ and $r \in \mathcal{R}(\vec{f}_j)(x)$. We consider the following two cases: (i) $r > 0$. We can write

$$\begin{aligned} & |u_{x, \vec{f}_{i,j}}(r) - u_{x, \vec{f}_i}(r)| \\ & = \left| |B(x,r)|^{\alpha/d-m} \left(\prod_{1 \leq \mu \neq i \leq m} \int_{B(x,r)} f_{\mu,j}(y) dy \right) \int_{B(x,r)} D_d f_{i,j}(y) dy \right. \\ & \quad \left. - |B(x,r)|^{\alpha/d-m} \left(\prod_{1 \leq \mu \neq i \leq m} \int_{B(x,r)} f_{\mu}(y) dy \right) \int_{B(x,r)} D_d f_i(y) dy \right| \\ & \leq \sum_{\mu=1}^{i-1} \mathfrak{M}_{\alpha}(F_{\mu,j}^{\vec{r}})(x) + \sum_{v=i+1}^m \mathfrak{M}_{\alpha}(G_{v,j}^{\vec{r}})(x) + \mathfrak{M}_{\alpha}(\vec{H}_{i,j}^{\vec{r}})(x) =: \mathcal{G}_{i,j}(x), \end{aligned}$$

where

$$F_{\mu,j}^{\vec{r}} = (f_1, \dots, f_{\mu-1}, f_{\mu,j} - f_{\mu}, f_{\mu+1,j}, \dots, f_{i-1,j}, D_d f_{i,j}, f_{i+1,j}, \dots, f_{m,j}),$$

$$G_{v,j}^{\vec{r}} = (f_1, \dots, f_{i-1}, D_d f_i, f_{i+1}, \dots, f_{v-1}, f_{v,j} - f_v, f_{v+1,j}, \dots, f_{m,j}),$$

$$\vec{H}_{i,j}^{\vec{r}} = (f_1, \dots, f_{i-1}, D_d f_{i,j} - D_d f_i, f_{i+1,j}, \dots, f_{m,j}).$$

(ii) $r = 0$. If $0 < \alpha < md$, then $|u_{x, \vec{f}_{i,j}}(r) - u_{x, \vec{f}_i}(r)| = 0$. If $\alpha = 0$, we have

$$\begin{aligned} & |u_{x, \vec{f}_{i,j}}(r) - u_{x, \vec{f}_i}(r)| \\ & \leq \sum_{\mu=1}^{i-1} \left(\prod_{l_1=1}^{\mu-1} f_{l_1}(x) \right) (f_{\mu,j}(x) - f_{\mu}(x)) \left(\prod_{l_2=\mu+1}^{i-1} f_{l_2,j}(x) \right) |D_d f_{i,j}(x)| \left(\prod_{l_3=i+1}^m f_{l_3,j}(x) \right) \\ & \quad + \sum_{v=i+1}^m \left(\prod_{l_1=1}^{i-1} f_{l_1}(x) \right) |D_d f_i(x)| \left(\prod_{l_2=i+1}^{v-1} f_{l_2}(x) \right) |f_{v,j}(x) - f_v(x)| \left(\prod_{l_3=v+1}^m f_{l_3,j}(x) \right) \\ & \quad + \left(\prod_{l_1=1}^{i-1} f_{l_1}(x) \right) |D_d f_{i,j}(x) - D_d f_i(x)| \left(\prod_{l_2=i+1}^m f_{l_2,j}(x) \right). \end{aligned}$$

From the above, the Lebesgue differential theorem and Lemma 2, we have that for a.e.

$x \in \mathbb{R}^d$,

$$\begin{aligned}
& |D_d \mathfrak{M}_\alpha(\vec{f}_j)(x) - D_d \mathfrak{M}_\alpha(\vec{f})(x)| \\
&= \left| \sum_{i=1}^m u_{x, \vec{f}_i}(\vec{f}_j)(r_1) - \sum_{i=1}^m u_{x, \vec{f}_i}(r_2) \right| \\
&\leq \left| \sum_{i=1}^m u_{x, \vec{f}_i}(\vec{f}_j)(r_1) - \sum_{i=1}^m u_{x, \vec{f}_i}(r_1) \right| + \left| \sum_{i=1}^m u_{x, \vec{f}_i}(r_1) - \sum_{i=1}^m u_{x, \vec{f}_i}(r_2) \right| \\
&\leq \sum_{i=1}^m \mathcal{G}_{i,j}(x) + \left| \sum_{i=1}^m u_{x, \vec{f}_i}(r_1) - \sum_{i=1}^m u_{x, \vec{f}_i}(r_2) \right|
\end{aligned} \tag{15}$$

for any $r_1 \in \mathcal{R}(\vec{f}_j)(x)$ and $r_2 \in \mathcal{R}(\vec{f})(x)$. One can easily check that

$$\lim_{j \rightarrow \infty} \|\mathcal{G}_{i,j}\|_{L^q(\mathbb{R}^d)} = 0$$

for any $i = 1, 2, \dots, m$. It follows that there exists $j_2 \in \mathbb{N}$ such that $\|\mathcal{G}_{i,j}\|_{L^q(\mathbb{R}^d)} < \varepsilon$ for any $i = 1, 2, \dots, m$ and $j \geq j_2$.

If $x \notin G_1 \cup G_2 \cup B^j$ we can choose $r_1 \in \mathcal{R}(\vec{f}_j)(x)$ and $r_2 \in \mathcal{R}(\vec{f})(x)$ such that $|r_1 - r_2| < \delta$ and

$$\left| \sum_{i=1}^m u_{x, \vec{f}_i}(r_1) - \sum_{i=1}^m u_{x, \vec{f}_i}(r_2) \right| < |B_R|^{-1/q} \varepsilon.$$

On the other hand, we have that for any $i = 1, 2, \dots, m$, $r_1 \in \mathcal{R}(\vec{f}_j)(x)$ and $r_2 \in \mathcal{R}(\vec{f})(x)$,

$$\left| \sum_{i=1}^m u_{x, \vec{f}_i}(r_1) - \sum_{i=1}^m u_{x, \vec{f}_i}(r_2) \right| \leq \sum_{i=1}^m |u_{x, \vec{f}_i}(r_1) - u_{x, \vec{f}_i}(r_2)| \leq 2 \sum_{i=1}^m \mathfrak{M}_\alpha(\vec{f}_i)(x).$$

Note that $|G_2 \cup B^j| < \eta$ for $j \geq j_1$. Thus we get from (15) that

$$\begin{aligned}
& \|D_d \mathfrak{M}_\alpha(\vec{f}_j) - D_d \mathfrak{M}_\alpha(\vec{f})\|_{L^q(\mathbb{R}^d)} \\
&\leq \left\| \sum_{i=1}^m \mathcal{G}_{i,j} \right\|_{L^q(\mathbb{R}^d)} + \| |B_R|^{-1/q} \varepsilon \|_{q, B_R} + 2 \left\| \sum_{i=1}^m \mathfrak{M}_\alpha(\vec{f}_i) \right\|_{q, G_1 \cup G_2 \cup B^j} \leq C\varepsilon,
\end{aligned}$$

for any $j \geq \max\{j_1, j_2\}$, which gives

$$\lim_{j \rightarrow \infty} \|D_d \mathfrak{M}_\alpha(\vec{f}_j) - D_d \mathfrak{M}_\alpha(\vec{f})\|_{L^q(\mathbb{R}^d)} = 0.$$

This completes the proof of Theorem 1. \square

3. Proof of Theorem 2

This section is devoted to proving Theorem 2. We now recall the definition of approximate differentiability. Let f be a real-valued function defined on a set $E \subset \mathbb{R}^d$.

\mathbb{R}^d . We say that f is approximately differentiable at $x_0 \in E$ if there is a vector $L = (L_1, L_2, \dots, L_d) \in \mathbb{R}^d$ such that for any $\varepsilon > 0$ the set

$$A_\varepsilon = \left\{ x \in \mathbb{R}^d : \frac{|f(x) - f(x_0) - L(x - x_0)|}{|x - x_0|} < \varepsilon \right\}$$

has x_0 as a density point. If this is the case, then x_0 is a density point of E and L is uniquely determined. The vector L is called the approximate differential of f at x_0 and is denoted by $\nabla f(x_0)$. Note that every function $f \in W^{1,1}(\mathbb{R}^d)$ is approximately differentiable a.e. It follows from Theorem A that Mf is approximately differentiable a.e. under the assumption that $f \in W^{1,1}(\mathbb{R}^d)$. However, it is unknown that whether $f \in W^{1,1}(\mathbb{R}^d)$ implies the weak differentiability of Mf when $d \geq 2$. For now, the relationship of approximate differentiability and weak differentiability is also not clear.

To prove Theorem 2, we need the following lemma followed from [35], which provides several characterizations of a.e. approximate differentiability of a function.

LEMMA 3. ([35]) *Let $f : E \rightarrow \mathbb{R}$ be measurable, $E \subset \mathbb{R}^d$. Then the following conditions are equivalent:*

- (i) *f is approximately differentiable a.e.*
- (ii) *For any $\varepsilon > 0$, there is a closed set $F \subset E$ and a locally Lipschitz function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $f = g$ on $x \in F$ and $|E \setminus F| < \varepsilon$.*
- (iii) *For any $\varepsilon > 0$, there is a closed set $F \subset E$ and a function $g \in \mathcal{C}^1(\mathbb{R}^d)$ such that $f = g$ on $x \in F$ and $|E \setminus F| < \varepsilon$.*

LEMMA 4. *Let $0 \leq \alpha \leq md - 1$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^1(\mathbb{R}^d)$. For $\varepsilon > 0$, we define the truncated maximal operator $\mathfrak{M}_\alpha^\varepsilon$ by*

$$\mathfrak{M}_\alpha^\varepsilon(\vec{f})(x) = \sup_{r \geq \varepsilon} |B(x, r)|^{\alpha/d-m} \prod_{i=1}^m \int_{B(x, r)} |f_i(y)| dy.$$

Then $\mathfrak{M}_\alpha^\varepsilon(\vec{f})$ is Lipschitz continuous for every $\varepsilon > 0$.

Proof. For any $r \geq \varepsilon$ and $\delta \geq 1$, we have

$$\left(\frac{r}{r + |x - y|} \right)^\delta \geq \left(\frac{\varepsilon}{\varepsilon + |x - y|} \right)^\delta \geq 1 - \delta \frac{|x - y|/\varepsilon}{1 + |x - y|/\varepsilon} \geq 1 - \frac{\delta}{\varepsilon} |x - y|. \quad (16)$$

Fix $x, y \in \mathbb{R}^d$ and $r \geq \varepsilon$ we have $B(y, r) \subset B(x, r + |x - y|)$. This together with (16) yields that

$$\begin{aligned} \mathfrak{M}_\alpha^\varepsilon(\vec{f})(x) &\geq |B(x, r + |x - y|)|^{\alpha/d-m} \prod_{i=1}^m \int_{B(x, r + |x - y|)} |f_i(y)| dy \\ &\geq \left(\frac{r}{r + |x - y|} \right)^{md-\alpha} |B(y, r)|^{\alpha/d-m} \prod_{i=1}^m \int_{B(y, r)} |f_i(y)| dy \\ &\geq \left(1 - \frac{md-\alpha}{\varepsilon} |x - y| \right) |B(y, r)|^{\alpha/d-m} \prod_{i=1}^m \int_{B(y, r)} |f_i(y)| dy. \end{aligned}$$

whenever $\alpha \leq md - 1$. It follows that

$$\mathfrak{M}_\alpha^\varepsilon(\vec{f})(y) - \mathfrak{M}_\alpha^\varepsilon(\vec{f})(x) \leq \frac{md - \alpha}{\varepsilon} |x - y| \mathfrak{M}_\alpha^\varepsilon(\vec{f})(y).$$

Similarly,

$$\mathfrak{M}_\alpha^\varepsilon(\vec{f})(x) - \mathfrak{M}_\alpha^\varepsilon(\vec{f})(y) \leq \frac{md - \alpha}{\varepsilon} |x - y| \mathfrak{M}_\alpha^\varepsilon(\vec{f})(x).$$

Thus we have

$$\begin{aligned} |\mathfrak{M}_\alpha^\varepsilon(\vec{f})(x) - \mathfrak{M}_\alpha^\varepsilon(\vec{f})(y)| &\leq \frac{md - \alpha}{\varepsilon} |x - y| (\mathfrak{M}_\alpha^\varepsilon(\vec{f})(x) + \mathfrak{M}_\alpha^\varepsilon(\vec{f})(y)) \\ &\leq 2 \frac{md - \alpha}{\varepsilon} |x - y| \left(\frac{1}{\frac{2\pi^{d/2}}{d\Gamma(d/2)} \varepsilon^d} \right)^{m-\alpha/d} \prod_{j=1}^m \|f_j\|_{L^1(\mathbb{R}^d)} \\ &\leq 2 \frac{md - \alpha}{\varepsilon^{md-\alpha+1}} \left(\frac{d\Gamma(d/2)}{2\pi^{d/2}} \right)^{m-\alpha/d} \prod_{j=1}^m \|f_j\|_{L^1(\mathbb{R}^d)} |x - y|. \end{aligned}$$

Then Lemma 4 is proved. \square

We now turn to prove Theorem 2.

Proof of Theorem 2. Let Z_j be the set of all Lebesgue points of f_j and $u_{x,\vec{f}}(r)$ as in Section 2. We set $F = \mathbb{R}^d \setminus (\bigcap_{j=1}^m Z_j)$. Let $x \in \bigcap_{j=1}^m Z_j$ such that $\mathfrak{M}_\alpha(\vec{f})(x) > u_{x,\vec{f}}(0)$. Since $f_j \in L^1(\mathbb{R}^d)$ and $\mathfrak{M}_\alpha(\vec{f})(x) > 0$, there exists a sequence of positive bounded numbers $\{r_k\}_{k \geq 1}$ such that

$$u_{x,\vec{f}}(r_k) \rightarrow \mathfrak{M}_\alpha(\vec{f})(x) \text{ when } k \rightarrow \infty.$$

Hence there exists a subsequence $\{s_k\}_{k \geq 1} \subset \{r_k\}_{k \geq 1}$ such that $\lim_{k \rightarrow \infty} s_k = r > 0$. It follows that

$$\mathfrak{M}_\alpha(\vec{f})(x) = u_{x,\vec{f}}(r).$$

This yields that

$$\mathbb{R}^d = F \cup \{x \in \mathbb{R}^d : \mathfrak{M}_\alpha(\vec{f})(x) = u_{x,\vec{f}}(0)\} \cup E,$$

where $E = \bigcup_{k=1}^{\infty} E_k$ and $E_k = \{x \in \mathbb{R}^d : \mathfrak{M}_\alpha(\vec{f})(x) = \mathfrak{M}_\alpha^{1/k}(\vec{f})(x)\}$. Obviously, $E_k \subset E_{k+1}$. By Lemma 3, we know that $\prod_{j=1}^m |f_j|$ is approximately differentiable a.e. Then $\mathfrak{M}_\alpha(\vec{f})$ is approximately differentiable a.e. in the set $\{x \in \mathbb{R}^d : \mathfrak{M}_\alpha(\vec{f})(x) = u_{x,\vec{f}}(0)\}$.

By Lemma 4 we have that $\mathfrak{M}_\alpha^{1/k}(\vec{f})$ is Lipschitz continuous for any $k \geq 1$. It follows that $\mathfrak{M}_\alpha(\vec{f})\chi_{E_{k+1} \setminus E_k}$ is approximately differentiable a.e. for all $k \geq 1$. Using Lemma 3 again we have that $\mathfrak{M}_\alpha(\vec{f})\chi_E = \mathfrak{M}_\alpha(\vec{f})\chi_{E_k} + \sum_{k=1}^{\infty} \mathfrak{M}_\alpha(\vec{f})\chi_{E_{k+1} \setminus E_k}$ is approximately differentiable a.e. Note that $|F| = 0$. Therefore, $\mathfrak{M}_\alpha(\vec{f})$ is approximately differentiable a.e. \square

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Feng Liu
College of Mathematics and Systems Science
Shandong University of Science and Technology
Qingdao, Shandong 266590, P. R. China
e-mail: liufeng860314@163.com

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