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MAXIMAL AND SINGULAR INTEGRAL OPERATORS AND THEIR COMMUTATORS ON GENERALIZED WEIGHTED MORREY SPACES WITH VARIABLE EXPONENT

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Abstract. We consider the generalized weighted Morrey spaces $\mathscr{M}_{\omega}^{p(\cdot),\varphi}(\Omega)$ with variable exponent $p(x)$ and a general function $\varphi(x,r)$ defining the Morrey-type norm. In case of unbounded sets $\Omega \subset \mathbb{R}^n$ we prove the boundedness of the Hardy-Littlewood maximal operator and Calderón-Zygmund singular operators with standard kernel, in such spaces. We also prove the boundedness of the commutators of maximal operator and Calderón-Zygmund singular operators in the generalized weighted Morrey spaces with variable exponent

1. Introduction

The classical Morrey spaces were originally introduced by Morrey in [53] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [22, 23, 25, 53]. Mizuhara [54] and Nakai [57] introduced generalized Morrey spaces. Later, Guliyev [25] defined the generalized Morrey spaces $M^{p,\varphi}$ with normalized norm. Recently, Komori and Shirai [48] considered the weighted Morrey spaces $L_w^{p,\kappa}$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [26] gave a concept of generalized weighted Morrey space $M_w^{p,\varphi}$ which could be viewed as extension of both generalized Morrey space $M^{p,\varphi}$ and weighted Morrey space $L_w^{p,\kappa}$. In [26] the boundedness of the classical operators and its commutators in spaces $M_w^{p,\varphi}$ also was studied, see also [33, 42].

As it is known, last two decades there is an increasing interest to the study of variable exponent spaces and operators with variable parameters in such spaces, we refer for instance to the surveying papers [20, 44, 47, 60], on the progress in this field, including topics of Harmonic Analysis and Operator Theory, see also references therein. For mapping properties of maximal functions and singular integrals on Lebesgue spaces with variable exponent we refer to [13, 14, 15, 17, 18, 19, 46, 50].

Variable exponent Morrey spaces $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$, were introduced and studied in [3] and [55] in the Euclidean setting and in [45] in the setting of metric measure spaces,

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in case of bounded sets. The boundedness of the maximal operator in variable exponent Morrey spaces $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$, $\lambda(\cdot)$ was proved in [3]. In [56] the maximal operator was considered in a somewhat more general space, but under more restrictive conditions on $p(x)$. P. Hästö in [35] used his new "local-to-global" approach to extend the result of [3] on the maximal operator to the case of the whole space $\mathbb{R}^n$. The boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathscr{L}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces was proved in [45].

Generalized Morrey spaces of such a kind in the case of constant p were studied in [6], [51], [54], [57]. In the case of bounded sets the boundedness of the maximal operator, singular integral operators and the potential operator in generalized variable exponent Morrey type spaces was proved in [29], [30], [31] and in the case of unbounded sets in [32], see also [39, 40, 58, 62]. Also, in the case of bounded sets the boundedness of these operators in generalized variable exponent weighted Morrey spaces for the power weights was proved in [37] and [38].

In the case of constant p and λ, the results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [59], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [16]; for further results in the case of constant p and λ (see, for instance, [5]–[7]).

We introduce the generalized variable exponent weighted Morrey spaces $\mathscr{M}_{\omega}^{p(\cdot),\varphi}(\Omega)$ over an open set $\Omega \subseteq \mathbb{R}^n$. Within the frameworks of the spaces $\mathscr{M}_{\omega}^{p(\cdot),\varphi}(\Omega)$, over unbounded sets $\Omega \subseteq \mathbb{R}^n$ we consider the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y)|dy$$

and Calderón-Zygmund type singular operator

$$Tf(x) = \int_{\Omega} K(x,y)f(y)dy,$$

where $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \Omega\times\Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \leqslant C|x-y|^{-n} \quad \text{for all } \ x \neq y,$$

$$|K(x,y) - K(x,z)| \leqslant C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \ \text{ if } |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leqslant C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \ \text{ if } |x-y| > 2|x-\xi|.$$

Let

$$T^* f(x) = \sup_{\varepsilon>0} |T_{\varepsilon} f(x)|$$

be the maximal singular operator, where $T_{\varepsilon} f(x)$ is the usual truncation

$$T_{\varepsilon} f(x) = \int_{\{y\in\Omega:|x-y|\geqslant\varepsilon\}} K(x,y)f(y)dy.$$

We find the condition on the Morrey function $\varphi(x,r)$ for the boundedness of the maximal operator M and the singular integral operators T in generalized weighted Morrey space $\mathscr{M}_{\omega}^{p(\cdot),\varphi}(\Omega)$ with variable $p(x)$ under the log-condition on $p(\cdot)$.

The paper is organized as follows. In Section 2 we provide necessary preliminaries on variable exponent weighted Lebesgue and generalized weighted Morrey spaces. In Section 3 we deal with the maximal operator and its commutator. In Section 4 we treat Calderón-Zygmund singular operators and its commutators.

The main results are given in Theorems 3.4, 3.5, 3.8, 3.9, 4.2, 4.3, 4.5, 4.6. We emphasize that the results we obtain for generalized weighted Morrey spaces are new even in the case when $p(x)$ is constant, because we do not impose any monotonicity type condition on $\varphi(x,r)$.

We use the following notation: $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\Omega \subset \mathbb{R}^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$, by c, C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line. By $A \lesssim B$ we mean that $A \leqslant CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries on variable exponent weighted Lebesgue and generalized weighted Morrey spaces

We refer to the book [18] for variable exponent Lebesgue spaces but give some basic definitions and facts. Let $p(\cdot)$ be a measurable function on Ω with values in $(1,\infty)$. An open set Ω which may be unbounded throughout the whole paper. We mainly suppose that

$$1 < p_- \leqslant p(x) \leqslant p_+ < \infty, \tag{2.1}$$

where $p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x)$, $p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : \, I_{p(\cdot)}\left(\frac{f}{\eta}\right) \leqslant 1\right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent.

The space $L^{p(\cdot)}(\Omega)$ coincides with the space

$$\left\{ f(x) : \left| \int_{\Omega} f(y) g(y) dy \right| < \infty \ \text{ for all } \ g \in L^{p'(\cdot)}(\Omega) \right\} \tag{2.2}$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}(\Omega)} \approx \sup_{\|g\|_{L^{p'(\cdot)}}\leqslant 1} \left|\int_\Omega f(y)g(y)dy\right| \tag{2.3}$$

see [41, Proposition 2.2], see also [49, Theorem 2.3], or [61, Theorem 3.5].

For the basics on variable exponent Lebesgue spaces we refer to [63], [49].
$\mathscr{P}(\Omega)$ is the set of bounded measurable functions $p:\Omega\to[1,\infty)$;
$\mathscr{P}^{log}(\Omega)$ is the set of exponents $p\in\mathscr{P}(\Omega)$ satisfying the local log-condition

$$|p(x)-p(y)| \leqslant \frac{A}{-\ln|x-y|}, \quad |x-y|\leqslant\frac{1}{2} \;\; x,y\in\Omega, \tag{2.4}$$

where $A=A(p)>0$ does not depend on x,y;
$\mathscr{A}^{log}(\Omega)$ is the set of bounded exponents $p:\Omega\to\mathbb{R}^n$ satisfying the condition (2.4);
$\mathbb{P}^{log}(\Omega)$ is the set of exponents $p\in\mathscr{P}^{log}(\Omega)$ with $1<p_-\leqslant p_+<\infty$;
for Ω which may be unbounded, by $\mathscr{P}_\infty(\Omega)$, $\mathscr{P}^{log}_\infty(\Omega)$, $\mathbb{P}^{log}_\infty(\Omega)$, $\mathscr{A}^{log}_\infty(\Omega)$ we denote the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x)-p(\infty)| \leqslant \frac{A_\infty}{\ln(2+|x|)}, \quad x\in\mathbb{R}^n. \tag{2.5}$$

where $p_\infty=\lim\limits_{x\to\infty}p(x)>1$.

We will also make use of the estimate provided by the following lemma (see [18], Corollary 4.5.9).

$$\|\chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \leqslant Cr^{\theta_p(x,r)}, \qquad x\in\Omega,\; p\in\mathbb{P}^{log}_\infty(\Omega), \tag{2.6}$$

where $\theta_p(x,r)=\begin{cases}\frac{n}{p(x)}, & r\leqslant 1,\\ \frac{n}{p(\infty)}, & r\geqslant 1\end{cases}$.

By ω we always denote a weight, i.e. a positive, locally integrable function with domain Ω. The weighted Lebesgue space $L^{p(\cdot)}_\omega(\Omega)$ is defined as the set of all measurable functions for which

$$\|f\|_{L^{p(\cdot)}_\omega(\Omega)} = \|f\omega\|_{L^{p(\cdot)}(\Omega)}.$$

Let us define the class $A_{p(\cdot)}(\Omega)$ (see [20], [50]) to consist of those weights ω for which

$$\sup_B |B|^{-1}\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}\|\omega^{-1}\|_{L^{p'(\cdot)}(\widetilde{B}(x,r))}<\infty.$$

THEOREM 2.1. ([36, Theorem 1.1]) *Let $\Omega\subset\mathbb{R}^n$ be an open unbounded set and $p\in\mathbb{P}^{log}_\infty(\Omega)$. Then $M:L^{p(\cdot)}_\omega(\Omega)\to L^{p(\cdot)}_\omega(\Omega)$ if and only if $\omega\in A_{p(\cdot)}(\Omega)$.*

Singular operators within the framework of the spaces with variable exponents were studied in [19]. From Theorem 4.8 and Remark 4.6 of [19] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω, but valid for an arbitrary open set Ω under the corresponding condition in $p(x)$ at infinity.

THEOREM 2.2. ([19, Theorem 4.8]) *Let* $\Omega \subset \mathbb{R}^n$ *be a unbounded open set and* $p \in \mathbb{P}^{log}(\Omega)$*. Then the singular integral operator* T *is bounded in* $L^{p(\cdot)}(\Omega)$*.*

Let $\lambda(x)$ be a measurable function on Ω with values in $[0,n]$. The variable Morrey space $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ and variable weighted Morrey space $\mathscr{L}^{p(\cdot),\lambda(\cdot)}_{\omega}(\Omega)$ are defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{-\frac{\lambda(x)}{p(x)}} \|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

$$\|f\|_{\mathscr{L}^{p(\cdot),\lambda(\cdot)}_{\omega}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{-\frac{\lambda(x)}{p(x)}} \|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}_{\omega}(\Omega)},$$

respectively.

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}| dy,$$

where $f_{\widetilde{B}(x,t)}(x) = |\widetilde{B}(x,t)|^{-1} \int_{\widetilde{B}(x,t)} f(y)dy$.

DEFINITION 2.1. We define the $BMO(\Omega)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO} = \sup_{x\in\Omega} M^\sharp f(x) = \sup_{x\in\Omega,\, r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}| dy.$$

DEFINITION 2.2. We define the $BMO_{p(\cdot),\omega}(\Omega)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO_{p(\cdot),\omega}} = \sup_{x\in\Omega,\, r>0} \frac{\|(f(\cdot) - f_{\widetilde{B}(x,r)})\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_{\omega}(\Omega)}}{\|\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}_{\omega}(\Omega)}}.$$

THEOREM 2.3. ([41, Theorem 4.4]) *Let* $\Omega \subset \mathbb{R}^n$ *be an open unbounded set,* $p \in \mathbb{P}^{log}_{\infty}(\Omega)$ *and* ω *be a Lebesgue measurable function. If* $\omega \in A_{p(\cdot)}(\Omega)$*, then the norms* $\|\cdot\|_{BMO_{p(\cdot),\omega}}$ *and* $\|\cdot\|_{BMO}$ *are mutually equivalent.*

Everywhere in the sequel the functions $\varphi(x,r)$, $\varphi_1(x,r)$ and $\varphi_2(x,r)$ used in the body of the paper, are non-negative measurable functions on $\Omega \times (0,\infty)$. We find it convenient to define the generalized weighted Morrey spaces in the form as follows.

DEFINITION 2.3. Let $1 \leqslant p(x) < \infty$, $x \in \Omega$. The variable exponent generalized Morrey space $\mathscr{M}^{p(\cdot),\varphi}(\Omega)$ and variable exponent generalized weighted Morrey space $\mathscr{M}^{p(\cdot),\varphi(\cdot)}_{\omega}(\Omega)$ are defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{\mathscr{M}^{p(\cdot),\varphi}} = \sup_{x\in\Omega, r>0} \frac{1}{\varphi(x,r) r^{\theta_p(x,r)}} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))},$$

$$\|f\|_{\mathscr{M}_\omega^{p(\cdot),\varphi}} = \sup_{x\in\Omega, r>0} \frac{1}{\varphi(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}} \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,r))},$$

respectively.

According to this definition, we recover the space $\mathscr{L}_\omega^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the choice $\varphi(x,r) = r^{\theta_p(x,r)-\frac{\lambda(x)}{p(x)}}$:

$$\mathscr{L}_\omega^{p(\cdot),\lambda(\cdot)}(\Omega) = \mathscr{M}_\omega^{p(\cdot),\varphi(\cdot)}(\Omega)\Big|_{\varphi(x,r)=r^{\theta_p(x,r)-\frac{\lambda(x)}{p(x)}}}.$$

3. The maximal operator and its commutators in $\mathscr{M}_\omega^{p(\cdot),\varphi}(\Omega)$

Let $L_v^\infty(\mathbb{R}_+)$ be the weighted L^∞-space with the norm

$$\|g\|_{L_v^\infty(\mathbb{R}_+)} = \operatorname*{ess\,sup}_{t>0} v(t)g(t).$$

In the sequel $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$ stand for the set of Lebesgue-measurable functions on $\mathbb{R}_+$, and its subspaces of nonnegative and nonnegative non-decreasing functions, respectively. We also denote

$$\mathbb{A} = \left\{\varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0\right\}.$$

Let u be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by

$$(\overline{S}_u g)(t) := \|u\,g\|_{L_1(0,t)}, \quad t \in (0,\infty).$$

The following theorem was proved in [5].

THEOREM 3.1. *Suppose that v_1 and v_2 are nonnegative measurable functions such that $0 < \|v_1\|_{L_\infty(0,t)} < \infty$ for every $t > 0$. Let u be a continuous nonnegative function on $\mathbb{R}$. Then the operator $\overline{S}_u$ is bounded from $L_{v_1}^\infty(\mathbb{R}_+)$ to $L_{v_2}^\infty(\mathbb{R}_+)$ on the cone $\mathbb{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|_{L_\infty(0,\cdot)}^{-1} \right) \right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

We will use the following results on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_0^t g(s)w(s)ds, \; H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [27, 28].

THEOREM 3.2. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \leqslant C \sup_{t>0} v_1(t) g(t)$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname{ess\,sup}\limits_{s<\tau<\infty} v_1(\tau)} < \infty.$$

THEOREM 3.3. *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w g(t) \leqslant C \sup_{t>0} v_1(t) g(t) \tag{3.1}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_0^t \frac{w(s)ds}{\sup_{0<\tau<s} v_1(\tau)} < \infty.$$

Moreover, the value $C=B$ is the best constant for (3.1).

The following weighted local estimates are valid.

THEOREM 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$. Then*

$$\|Mf\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leqslant C \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \sup_{r\geqslant t} \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,r))} \|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))}, \tag{3.2}$$

for every $f \in L_\omega^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{\widetilde{B}(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\Omega\setminus\widetilde{B}(x,2t)}(y), \quad t>0, \tag{3.3}$$

and have

$$\|Mf\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leqslant \|Mf_1\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} + \|Mf_2\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By Theorem 2.1 we obtain

$$\|Mf_1\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leqslant \|Mf_1\|_{L_\omega^{p(\cdot)}(\Omega)} \leqslant C\|f_1\|_{L_\omega^{p(\cdot)}(\Omega)} = C\|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,2t))}, \tag{3.4}$$

where C does not depend on f. From (3.4) we obtain

$$\|Mf_1\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} \leqslant C \|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \sup_{r\geqslant t} \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,r))} \|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))}. \tag{3.5}$$

Let z be an arbitrary point in $B(x,r)$. If $B(z,r)\cap {}^{\complement}B(x,2t)\neq\emptyset$, then $r>t$. Indeed, if $y\in B(z,r)\cap {}^{\complement}B(x,2t)$, then $r>|y-z|\geqslant|x-z|-|x-y|>2t-t=t$.

On the other hand, $B(z,r)\cap {}^{\complement}B(x,2t)\subset B(x,2r)$. Indeed, for $y\in B(z,r)\cap {}^{\complement}B(x,2t)$, then we get $|x-y|\leqslant|y-z|+|x-z|<t+r<2r$.

$$
\begin{aligned}
Mf_2(z)&=\sup_{r>0}|B(z,r)|^{-1}\int_{\widetilde{B}(z,r)}|f_2(y)|dy\\
&\leqslant C\sup_{r\geqslant 2t}|B(x,2r)|^{-1}\int_{{}^{\complement}\widetilde{B}(x,2t)\cap\widetilde{B}(z,r)}|f(y)|dy\\
&\leqslant C\sup_{r\geqslant t}|B(x,r)|^{-1}\int_{\widetilde{B}(x,r)}|f(y)|dy\\
&\leqslant C\sup_{r\geqslant t}|B(x,r)|^{-1}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega^{-1}\|_{L^{p'(\cdot)}(\widetilde{B}(x,r))}\\
&\leqslant C\sup_{r\geqslant t}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))}.
\end{aligned}
$$

Thus, the function $Mf_2(z)$, with fixed x and t, is dominated by the expression not depending on z. Then

$$
\|Mf_2\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))}\leqslant C\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\sup_{r\geqslant t}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))}. \tag{3.6}
$$

We then obtain (3.2) from (3.5) and (3.6). □

THEOREM 3.5. *Let $\Omega\subset\mathbb{R}^n$ be an open unbounded set, $p\in\mathbb{P}^{log}_{\infty}(\Omega)$, $\omega\in A_{p(\cdot)}(\Omega)$ and the function $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the condition*

$$
\sup_{t>r}\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\leqslant C\varphi_2(x,r), \tag{3.7}
$$

where C does not depend on $x\in\Omega$ and r. Then the operator M is bounded from the space $\mathcal{M}^{p(\cdot),\varphi_1}_{\omega}(\Omega)$ to the space $\mathcal{M}^{p(\cdot),\varphi_2}_{\omega}(\Omega)$.

Proof. Let $f\in\mathcal{M}^{p(\cdot),\varphi_1}_{\omega}(\Omega)$. By Theorems 3.1 and 3.4 we obtain

$$
\begin{aligned}
&\|Mf\|_{\mathcal{M}^{p(\cdot),\varphi_2}_{\omega}(\Omega)}\\
&\leqslant C\sup_{x\in\Omega,\,t>0}\frac{1}{\varphi_2(x,t)}\sup_{r\geqslant t}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))}\\
&\leqslant C\sup_{x\in\Omega,\,t>0}\frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))}t^{-n}\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_t^{\infty}s^{n-1}ds\\
&=C\|f\|_{\mathcal{M}^{p(\cdot),\varphi_1}_{\omega}(\Omega)}
\end{aligned}
$$

by (3.7), which completes the proof. □

In the case $\omega=1$ we get

COROLLARY 3.1. ([32, Theorem 3.5]) *Let* $\Omega \subset \mathbb{R}^n$ *be an open unbounded set,* $p \in \mathbb{P}_\infty^{log}(\Omega)$ *and the functions* $\varphi_1(x,r)$ *and* $\varphi_2(x,r)$ *satisfy the condition*

$$\sup_{t>r} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\theta_p(x,s)}}{t^{\theta_p(x,t)}} \leqslant C\varphi_2(x,r), \tag{3.8}$$

where C *does not depend on* $x \in \Omega$ *and* r. *Then the operator* M *is bounded from the space* $\mathscr{M}^{p(\cdot),\varphi_1}(\Omega)$ *to the space* $\mathscr{M}^{p(\cdot),\varphi_2}(\Omega)$.

The commutator generated by M and a suitable function b is formally defined by

$$[M,b]f = M(bf) - bM(f).$$

Given a measurable function b the maximal commutator is defined by

$$M_b(f)(x) := \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x)-b(y)||f(y)|dy$$

for all $x \in \mathbb{R}^n$.

This operator plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for instance [24], [52]). The maximal operator M_b has been studied intensively and there exist plenty of results about it. Pu Zhang and Jianglong Wu [64] proved the following statement.

THEOREM 3.6. [64, Theorem 3.1] *Let* $b \in L_1^{loc}(\mathbb{R}^n)$ *and* $p \in \mathbb{P}_\infty^{log}(\mathbb{R}^n)$, *then the operator* M_b *is bounded from* $L^{p(\cdot)}(\mathbb{R}^n)$ *to itself if and only if* $b \in BMO(\mathbb{R}^n)$.

Operators M_b and $[M,b]$ essentially differ from each other. For example, M_b is a positive and sublinear operator, but $[M,b]$ is neither positive nor sublinear. However, if b satisfies some additional conditions, then operator M_b controls $[M,b]$.

LEMMA 3.1. ([2, Lemma 3.1]) *Let* b *be any non-negative locally integrable function. Then*

$$|[M,b]f(x)| \leqslant M_b(f)(x), \; x \in \mathbb{R}^n$$

holds for all $f \in L_1^{loc}(\mathbb{R}^n)$.

THEOREM 3.7. ([2, Theorem 1.13]) *Let* $b \in BMO(\mathbb{R}^n)$. *Suppose that* X *is a Banach space of measurable functions defined on* $\mathbb{R}^n$. *Assume that* M *is bounded on* X. *Then the operator* M_b *is bounded on* X, *and the inequality*

$$\|M_b f\|_X \leqslant C\|b\|_* \|f\|_X$$

holds with constant C *independent of* f.

COROLLARY 3.2. *Let* $b \in BMO(\Omega)$, $p \in \mathbb{P}_\infty^{log}(\Omega)$ *and* $\omega \in A_{p(\cdot)}(\Omega)$, *then the operator* M_b *is bounded on* $L_\omega^{p(\cdot)}(\mathbb{R}^n)$.

Before proving the main theorems, we need the following lemma.

LEMMA 3.2. ([34, Lemma 2]) *Let $b \in BMO(\Omega)$. Then there is a constant $C > 0$ such that*

$$\left| b_{\widetilde{B}(x,r)} - b_{\widetilde{B}(x,t)} \right| \leqslant C\|b\|_* \ln\frac{t}{r} \quad \text{for} \quad 0 < 2r < t,$$

where C is independent of b, x, r, and t.

The following weighted local estimates are valid.

THEOREM 3.8. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_{\infty}(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$, $b \in BMO(\Omega)$. Then*

$$\|M_b f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))} \leqslant C\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \sup_{r\geqslant t}\left(1+\ln\frac{r}{t}\right)\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))} \tag{3.9}$$

for every $f \in L^{p(\cdot)}_{\omega}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t.

Proof. We represent function f as in (3.3) and have

$$\|M_b f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))} \leqslant \|M_b f_1\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))} + \|M_b f_2\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))}.$$

By Corollary 3.2 we obtain

$$\|M_b f_1\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))} \leqslant \|M_b f_1\|_{L^{p(\cdot)}_{\omega}(\Omega)} \leqslant C\|b\|_*\|f_1\|_{L^{p(\cdot)}_{\omega}(\Omega)} = C\|b\|_*\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,2t))}, \tag{3.10}$$

where C does not depend on f. From (3.10) we obtain (see also (3.4))

$$\|M_b f_1\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))} \leqslant C\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \sup_{r\geqslant t}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))}, \tag{3.11}$$

easily obtained from the fact that $\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,2t))}$ is non-decreasing in t, so that $\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,2t))}$ on the right-hand side of (3.10) is dominated by the right-hand side of (3.11).

For $z \in \widetilde{B}(x,t)$ we get

$$\begin{aligned}
M_b f_2(z) &= \sup_{r>0}|B(z,r)|^{-1}\int_{\widetilde{B}(z,r)}|b(z)-b(y)||f_2(y)|dy \\
&\leqslant C\sup_{r\geqslant 2t}|B(x,2r)|^{-1}\int_{\complement_{\widetilde{B}(x,2t)}\cap\widetilde{B}(z,r)}|b(z)-b(y)||f(y)|dy \\
&\leqslant C\sup_{r\geqslant t}|B(x,r)|^{-1}\int_{\widetilde{B}(x,r)}|b(z)-b(y)||f(y)|dy \\
&\leqslant \sup_{r\geqslant t}|B(x,r)|^{-1}\int_{\widetilde{B}(x,r)}|b(y)-b_{\widetilde{B}(x,r)}||f(y)|dy \\
&\quad + \sup_{r\geqslant t}|B(x,r)|^{-1}\int_{\widetilde{B}(x,r)}|b(z)-b_{\widetilde{B}(x,r)}||f(y)|dy = I_1 + I_2.
\end{aligned}$$

By Hölder inequality and Theorem 2.3 we obtain

$$\begin{aligned}
I_1 &= \sup_{r\geqslant t}|B(x,r)|^{-1}\int_{\widetilde{B}(x,r)}|b(y)-b_{\widetilde{B}(x,r)}||f(y)|dy\\
&\leqslant C\sup_{r\geqslant t}|B(x,r)|^{-1}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\big\|b(\cdot)-b_{\widetilde{B}(x,r)}\big\|_{L^{p'(\cdot)}_{\omega^{-1}}(\widetilde{B}(x,r))}\\
&\leqslant C\|b\|_*\sup_{r\geqslant t}|B(x,r)|^{-1}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega^{-1}\|_{L^{p'(\cdot)}(\widetilde{B}(x,r))}\\
&\leqslant C\|b\|_*\sup_{r\geqslant t}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))}.
\end{aligned}$$

To estimate I_2, by Lemma 3.2 we get

$$\begin{aligned}
I_2 &= \sup_{r\geqslant t}|B(x,r)|^{-1}|b(z)-b_{\widetilde{B}(x,r)}|\int_{\widetilde{B}(x,r)}|f(y)|dy\\
&\leqslant \sup_{r\geqslant t}|B(x,r)|^{-1}|b(z)-b_{\widetilde{B}(x,t)}|\int_{\widetilde{B}(x,r)}|f(y)|dy\\
&\quad+\sup_{r\geqslant t}|B(x,r)|^{-1}|b_{\widetilde{B}(x,t)}-b_{\widetilde{B}(x,r)}|\int_{\widetilde{B}(x,r)}|f(y)|dy\\
&\leqslant CM_b\chi_{B(x,t)}(z)\sup_{r\geqslant t}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))}\\
&\quad+C\|b\|_*\sup_{r\geqslant t}\ln\frac{r}{t}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))}.
\end{aligned}$$

Then Corollary 3.2 we have

$$\begin{aligned}
\|M_bf_2\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))} &\leqslant \|I_1\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))}+\|I_2\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))}\\
&\leqslant C\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\sup_{r\geqslant t}\left(1+\ln\frac{r}{t}\right)\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))}\\
&\quad+C\|M_b\chi_{B(x,t)}\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))}\sup_{r\geqslant t}\left(1+\ln\frac{r}{t}\right)\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))}\\
&\leqslant C\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\sup_{r\geqslant t}\left(1+\ln\frac{r}{t}\right)\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,r))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,r))}.
\end{aligned} \tag{3.12}$$

Then from (3.11) and (3.12) we obtain (3.9). □

THEOREM 3.9. *Let $\Omega\subset\mathbb{R}^n$ be an open unbounded set, $p\in\mathbb{P}^{log}_{\infty}(\Omega)$, $\omega\in A_{p(\cdot)}(\Omega)$, $b\in BMO(\Omega)$ and the function $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the condition*

$$\sup_{t>r}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\leqslant C\varphi_2(x,r), \tag{3.13}$$

where C does not depend on $x\in\Omega$ and t. Then the operator M_b is bounded from the space $\mathscr{M}^{p(\cdot),\varphi_1}_{\omega}(\Omega)$ to the space $\mathscr{M}^{p(\cdot),\varphi_2}_{\omega}(\Omega)$.

Proof. Let $f \in \mathscr{M}_{\omega}^{p(\cdot),\varphi_1}(\Omega)$. By (3.13), Theorems 3.2 and 3.8 we obtain

$$
\begin{aligned}
&\|M_b f\|_{\mathscr{M}_{\omega}^{p(\cdot),\varphi_2}(\Omega)} \\
&\leqslant C\|b\|_* \sup_{x\in\Omega,\, t>0} \frac{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}} \sup_{t>r}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_{\omega}^{p(\cdot)}(\widetilde{B}(x,t))}\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}^{-1} \\
&\leqslant C\|b\|_* \sup_{x\in\Omega,\, t>0} \frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}\|f\|_{L_{\omega}^{p(\cdot)}(\widetilde{B}(x,t))} = C\|b\|_*\|f\|_{\mathscr{M}_{\omega}^{p(\cdot),\varphi_1}(\Omega)}
\end{aligned}
$$

which completes the proof. □

4. Singular integral operators and its commutators in $\mathscr{M}_{\omega}^{p(\cdot),\varphi}(\Omega)$

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderón [8, 9] studied a kind of commutators, appearing in Cauchy integral problems of Lipschitz curve. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [10] states that the commutator operator $[b,K]f = K(bf) - bKf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [11], [12], [21], [22], [23]).

The following statement holds (see [19, Lemma 3.5]):

PROPOSITION A. *Let $\Omega \subset \mathbb{R}^n$ be unbounded and $p \in \mathbb{P}_{\infty}^{log}(\Omega)$. Then for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$ there holds*

$$
\left|\int_{\Omega} f(y)g(y)dy\right| \leqslant C\int_{\Omega} M^{\sharp}f(y)Mg(y)dy
$$

with a constant $C > 0$ not depending on f.

LEMMA 4.3. *Let $\Omega \subset \mathbb{R}^n$ be unbounded and $p \in \mathbb{P}_{\infty}^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$. Then*

$$
\|f\omega\|_{L^{p(\cdot)}(\Omega)} \leqslant C\|\omega M^{\sharp}f\|_{L^{p(\cdot)}(\Omega)}
$$

with a constant $C > 0$ not depending on f.

Proof. By (2.3) we have

$$
\|f\omega\|_{L^{p(\cdot)}(\Omega)} \leqslant C \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)}\leqslant 1}\left|\int_{\Omega} f(y)g(y)\omega(y)dy\right|.
$$

According to Proposition A,

$$
\|f\omega\|_{L^{p(\cdot)}(\Omega)} \leqslant C \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)}\leqslant 1}\int_{\Omega} M^{\sharp}f(y)M(g\omega)(y)dy.
$$

By the Hölder inequality and Theorem 2.1, we derive

$$\begin{aligned}\|f\omega\|_{L^{p(\cdot)}(\Omega)} &\leqslant C \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)}\leqslant 1} \|\omega M^{\sharp} f\|_{L^{p(\cdot)}(\Omega)} \|\omega^{-1} M(g\omega)\|_{L^{p'(\cdot)}(\Omega)} \\ &\leqslant C \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)}\leqslant 1} \|\omega M^{\sharp} f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)} \leqslant C\|\omega M^{\sharp} f\|_{L^{p(\cdot)}(\Omega)}. \qquad \square\end{aligned}$$

PROPOSITION B. ([4, Theorem 2.1]) *Let T be a Calderón-Zygmund operator. Then for arbitrary s, $0<s<1$, there exists a constant $C_s>0$ such that*

$$[(|Tf|^{s})^{\sharp}]^{\frac{1}{s}}(x) \leqslant C_s Mf(x)$$

for all $f \in C_0^{\infty}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_{\infty}^{log}(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$. Then the operators T and T^* are bounded in the space $L_{\omega}^{p(\cdot)}(\Omega)$.*

Proof. In [46, Theorem 4.2.] was proved that infinitely differentiable functions is dense in $L_{\omega}^{p(\cdot)}(\Omega)$ with any positive, locally integrable function ω. Then by the Proposition B, Lemma 4.3 and Theorem 2.1, we derive the operator T is bounded in the space $L_{\omega}^{p(\cdot)}(\Omega)$.

The boundedness of the operator T^* on $L_{\omega}^{p(\cdot)}(\Omega)$ follows from the known estimate

$$T^* f(x) \lesssim M(Tf)(x) + Mf(x),$$

from Theorem 2.1 and the boundedness of the operator T on $L_{\omega}^{p(\cdot)}(\Omega)$. $\square$

The following weighted local estimates are valid.

THEOREM 4.2. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_{\infty}^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and $f \in L_{\omega}^{p(\cdot)}(\Omega)$. Then*

$$\|Tf\|_{L_{\omega}^{p(\cdot)}(\widetilde{B}(x,t))} \leqslant C\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \int_t^{\infty} \|f\|_{L_{\omega}^{p(\cdot)}(\widetilde{B}(x,s))} \|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}, \tag{4.1}$$

where C does not depend on f, $x \in \Omega$ and t.

Proof. We represent function f as in (3.3) and have

$$\|Tf\|_{L_{\omega}^{p(\cdot)}(\widetilde{B}(x,t))} \leqslant \|Tf_1\|_{L_{\omega}^{p(\cdot)}(\widetilde{B}(x,t))} + \|Tf_2\|_{L_{\omega}^{p(\cdot)}(\widetilde{B}(x,t))}.$$

By the Theorem 4.1 we obtain

$$\|Tf_1\|_{L_{\omega}^{p(\cdot)}(\widetilde{B}(x,t))} \leqslant \|Tf_1\|_{L_{\omega}^{p(\cdot)}(\Omega)} \leqslant C\|f_1\|_{L_{\omega}^{p(\cdot)}(\Omega)},$$

so that

$$\|Tf_1\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))} \leqslant C\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,2t))}.$$

Taking into account the inequality

$$\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))} \leqslant C\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \int_t^\infty \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))} \|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s},$$

we get

$$\|Tf_1\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))} \leqslant C\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \int_t^\infty \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))} \|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}. \tag{4.2}$$

To estimate $\|Tf_2\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))}$, we observe that

$$|Tf_2(z)| \leqslant C \int\limits_{\Omega\setminus B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \leqslant t$, $|z-y| \geqslant 2t$ imply $\frac{1}{2}|z-y| \leqslant |x-y| \leqslant \frac{3}{2}|z-y|$, and therefore

$$|Tf_2(z)| \leqslant C\int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{-n}|f(y)|dy,$$

To estimate Tf_2, we first prove the following auxiliary inequality

$$\begin{aligned}&\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy\\ &\leqslant Ct^{-n}\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} \int_t^\infty s^{n-1}\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))} \|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s}.\end{aligned} \tag{4.3}$$

To this end, we choose $\delta > 0$ and proceed as follows

$$\begin{aligned}\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy &\leqslant \delta\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n+\delta}|f(y)|dy\int_{|x-y|}^\infty s^{-\delta-1}ds\\ &\leqslant Ct^{-n}\int_t^\infty \frac{ds}{s}\int_{\{y\in\Omega:2t\leqslant|x-y|\leqslant s\}}|f(y)|dy\\ &\leqslant Ct^{-n}\int_t^\infty \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega^{-1}\|_{L^{p'(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}\\ &\leqslant Ct^{-n}\int_t^\infty s^{n-1}\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}ds.\end{aligned} \tag{4.4}$$

Hence by inequality (4.4), we get

$$\begin{aligned}\|Tf_2\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))} &\leqslant C\|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}_\omega(\Omega)}\int_t^\infty \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}\\ &= C\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_t^\infty \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}.\end{aligned} \tag{4.5}$$

From (4.2) and (4.5) we arrive at (4.1). □

THEOREM 4.3. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and $\varphi_1(x,t)$ and $\varphi_2(x,r)$ fulfill condition*

$$\int_t^\infty \frac{\operatorname*{ess\,inf}_{s<r<\infty} \varphi_1(x,r)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}}{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,s))}} \frac{ds}{s} \leqslant C\varphi_2(x,t), \tag{4.6}$$

where C does not depend on $x \in \Omega$ and t. Then the singular integral operators T and T^* are bounded from the space $\mathscr{M}_\omega^{p(\cdot),\varphi_1}(\Omega)$ to the space $\mathscr{M}_\omega^{p(\cdot),\varphi_2}(\Omega)$.

Proof. Let $f \in \mathscr{M}_\omega^{p(\cdot),\varphi_1}(\Omega)$. As usual, when estimating the norm

$$\|Tf\|_{\mathscr{M}_\omega^{p(\cdot),\varphi_2}(\Omega)} = \sup_{x\in\Omega,\, t>0} \frac{1}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}} \|Tf\chi_{\widetilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)}. \tag{4.7}$$

We estimate $\|Tf\chi_{\widetilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)}$ in (4.7) by means of Theorem 4.2 and obtain

$$\begin{aligned}
\|Tf\|_{\mathscr{M}_\omega^{p(\cdot),\varphi_2}(\Omega)} &\leqslant C \sup_{x\in\Omega,\, t>0} \frac{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}} \int_t^\infty s^{-1}\|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))} ds \\
&\leqslant C \sup_{x\in\Omega,\, t>0} \frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}} \|f\|_{L_\omega^{p(\cdot)}(\widetilde{B}(x,t))} = C\|f\|_{\mathscr{M}_\omega^{p(\cdot),\varphi_1}(\Omega)}.
\end{aligned}$$

It remains to make use of condition (4.6). □

LEMMA 4.4. ([21, Lemma 1]) *Let $1 < s < \infty$, $b \in BMO(\mathbb{R}^n)$, then there exists $C > 0$ such that for all $x \in \mathbb{R}^n$, the following inequality holds*

$$M^\sharp([b,T]f)(x) \leqslant C\|b\|_* \left((M|Tf|^s)^{\frac{1}{s}}(x) + (M|f|^s)^{\frac{1}{s}}(x) \right).$$

THEOREM 4.4. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $b \in BMO(\Omega)$, $p \in \mathbb{P}_\infty^{log}(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$. Then the commutator operator $[b,T]$ is bounded on the space $L_\omega^{p(\cdot)}(\Omega)$.*

Proof. By Lemma 4.4, Lemma 4.3, Theorem 2.1 and Theorem 4.1, we derive the operator $[b,T]$ is bounded in the space $L_\omega^{p(\cdot)}(\Omega)$. □

In the case $\omega \equiv 1$, we have the following corollary, which proved in [43].

COROLLARY 4.3. ([43, Theorem 1.1]) *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $b \in BMO(\Omega)$ and $p \in \mathbb{P}_\infty^{log}(\Omega)$. Then the operator $[b,T]$ is bounded on the space $L^{p(\cdot)}(\Omega)$.*

The following weighted local estimates are valid.

THEOREM 4.5. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$, $b \in BMO(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$. Then*

$$\|[b,T]f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))} \leqslant C\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_t^\infty \left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s} \tag{4.8}$$

for every $f \in L^{p(\cdot)}_\omega(\Omega)$, where C does not depend on $f, x \in \Omega$ and t.

Proof. We represent function f as in (3.3) and have

$$\|[b,T]f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))} \leqslant \|[b,T]f_1\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))} + \|[b,T]f_2\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))}.$$

By Theorem 4.4 we obtain

$$\begin{aligned}\|[b,T]f_1\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))} \leqslant \|[b,T]f_1\|_{L^{p(\cdot)}_\omega(\Omega)} &\leqslant C\|b\|_*\|f_1\|_{L^{p(\cdot)}_\omega(\Omega)} \\ &= C\|b\|_*\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,2t))},\end{aligned} \tag{4.9}$$

where C does not depend on f. From (4.9) we obtain

$$\|[b,T]f_1\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))} \leqslant C\|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_t^\infty \left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s} \tag{4.10}$$

easily obtained from the fact that $\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,2t))}$ is non-decreasing in t, so that $\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,2t))}$ on the right-hand side of (4.9) is dominated by the right-hand side of (4.10). To estimate $\|[b,T]f_2\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))}$, we observe that

$$|[b,T]f_2(z)| \leqslant C \int\limits_{\Omega\setminus B(x,2t)} |b(z)-b(y)|\frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \leqslant t$, $|z-y| \geqslant 2t$ imply $\frac{1}{2}|z-y| \leqslant |x-y| \leqslant \frac{3}{2}|z-y|$, and therefore

$$|[b,T]f_2(z)| \leqslant C\int_{\Omega\setminus\widetilde{B}(x,2t)} |x-y|^{-n}|b(z)-b(y)|\,|f(y)|dy.$$

To estimate $[b,T]f_2$, we first prove the following auxiliary inequality

$$\int_{\Omega\setminus\widetilde{B}(x,t)} |x-y|^{-n}|b(z)-b(y)||f(y)|dy \leqslant C\|b\|_*\int_t^\infty \left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}. \tag{4.11}$$

To estimate $[b,T]f_2(z)$, we observe that for $z \in \widetilde{B}(x,t)$ we have

$$\begin{aligned}
&\int_{\Omega\backslash\widetilde{B}(x,t)} |x-y|^{-n}|b(z)-b(y)||f(y)|dy \\
&\leqslant \int_{\Omega\backslash\widetilde{B}(x,t)} |x-y|^{-n}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dy \\
&\quad + \int_{\Omega\backslash\widetilde{B}(x,t)} |x-y|^{-n}|b(z)-b_{\widetilde{B}(x,t)}||f(y)|dy = J_1 + J_2.
\end{aligned}$$

To this end, we choose $\delta > 0$, by Theorem 2.3 and Lemma 3.2 we obtain

$$\begin{aligned}
J_1 &= \int_{\Omega\backslash\widetilde{B}(x,t)} |x-y|^{-n}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dy \\
&\leqslant \delta \int_{\Omega\backslash\widetilde{B}(x,t)} |x-y|^{-n+\delta}|b(y)-b_{\widetilde{B}(x,t)}||f(y)|dy \int_{|x-y|}^{\infty} s^{-\delta-1}ds \\
&\leqslant C\int_t^{\infty} s^{-n-1}\int_{\{y\in\Omega:2t\leqslant|x-y|\leqslant s\}} |b(y)-b_{\widetilde{B}(x,t)}||f(y)|dyds \\
&\leqslant C\int_t^{\infty} s^{-n-1}\|b(\cdot)-b_{\widetilde{B}(x,s)}\|_{L^{p'(\cdot)}_{\omega^{-1}}(\widetilde{B}(x,s))}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,s))}ds \\
&\quad + C\int_t^{\infty} s^{-n-1}|b_{\widetilde{B}(x,t)}-b_{\widetilde{B}(x,s)}|\int_{\widetilde{B}(x,s)} |f(y)|dyds \\
&\leqslant C\|b\|_*\int_t^{\infty} s^{-n-1}\|\omega^{-1}\|_{L^{p'(\cdot)}(\widetilde{B}(x,s))}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,s))}ds \\
&\quad + C\|b\|_*\int_t^{\infty} s^{-n-1}\ln\frac{s}{t}\|\omega^{-1}\|_{L^{p'(\cdot)}(\widetilde{B}(x,s))}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,s))}ds \\
&\leqslant C\|b\|_*\int_t^{\infty}\left(1+\ln\frac{s}{t}\right)\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,s))}\frac{ds}{s}.
\end{aligned}$$

To estimate J_2, by (4.3), we have

$$\begin{aligned}
J_2 &= |b(z)-b_{\widetilde{B}(x,t)}|\int_{\Omega\backslash\widetilde{B}(x,t)} |x-y|^{-n}|f(y)|dy \\
&\leqslant C|B(x,t)|^{-1}\int_{\widetilde{B}(x,t)} |b(z)-b(y)|dy\int_t^{\infty}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s} \\
&\leqslant CM_b\chi_{B(x,t)}(z)\int_t^{\infty}\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s},
\end{aligned}$$

where C does not depend on x,t.

Hence by inequality (4.11), we get

$$\begin{aligned}
&\|[b,T]f_2\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,t))} \\
&\lesssim \|\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}_{\omega}(\Omega)}\|b\|_*\int_t^{\infty}\left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s} \\
&= \|b\|_*\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}\int_t^{\infty}\left(1+\ln\frac{s}{t}\right)\|f\|_{L^{p(\cdot)}_{\omega}(\widetilde{B}(x,s))}\|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))}\frac{ds}{s}. \qquad (4.12)
\end{aligned}$$

From (4.10) and (4.12) we arrive at (4.8). □

THEOREM 4.6. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$, $b \in BMO(\Omega)$ and the functions $\varphi_1(x,r)$ and $\varphi_2(x,r)$ satisfy the condition* (3.13)*. Then the operator $[b,T]$ is bounded from the space $\mathscr{M}^{p(\cdot),\varphi_1}_\omega(\Omega)$ to the space $\mathscr{M}^{p(\cdot),\varphi_2}_\omega(\Omega)$.*

Proof. Let $f \in \mathscr{M}^{p(\cdot),\varphi_1}_\omega(\Omega)$. We have

$$\|[b,T]f\|_{\mathscr{M}^{p(\cdot),\varphi_2}_\omega(\Omega)} = \sup_{x\in\Omega,\, t>0} \frac{1}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}} \|[b,T]f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))}.$$

By (3.13), Theorems 3.2 and 4.5 we obtain

$$\begin{aligned}
&\|[b,T]f\|_{\mathscr{M}^{p(\cdot),\varphi_2}_\omega(\Omega)} \\
&\leqslant C\|b\|_* \sup_{x\in\Omega,\, t>0} \frac{\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}}{\varphi_2(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}} \int_t^\infty \left(1+\ln\frac{s}{t}\right) \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,s))} \|\omega\|^{-1}_{L^{p(\cdot)}(\widetilde{B}(x,s))} \frac{ds}{s} \\
&\leqslant C\|b\|_* \sup_{x\in\Omega,\, t>0} \frac{1}{\varphi_1(x,t)\|\omega\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}} \|f\|_{L^{p(\cdot)}_\omega(\widetilde{B}(x,t))} = C\|b\|_*\|f\|_{\mathscr{M}^{p(\cdot),\varphi_1}_\omega(\Omega)}
\end{aligned}$$

which completes the proof. □

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