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A GENERALIZATION OF YOUNG–TYPE INEQUALITIES

DAESHIK CHOI

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Abstract. In this paper, we prove a simple but useful result and apply it to give a generalization of Young-type inequalities developed by many researchers. Applications to positive definite matrices will be also provided.

1. Introduction

The classical Young's inequality states that if $a,b>0$ and $0\leqslant v\leqslant 1$, then

$$(1-v)a+vb\geqslant a^{1-v}b^{v}. \tag{1}$$

There are many refinements of Young's inequality and its reverse. Kittaneh and Manasrah [6, 7] improved (1) as follows:

$$(1-v)a+vb\geqslant a^{1-v}b^{v}+r_0(\sqrt{a}-\sqrt{b})^2, \tag{2}$$
$$(1-v)a+vb\leqslant a^{1-v}b^{v}+R_0(\sqrt{a}-\sqrt{b})^2, \tag{3}$$

where $r_0=\min\{v,1-v\}$ and $R_0=\max\{v,1-v\}$ whose notations will be used throughout this paper. Hirzallah and Kittaneh [5] and He and Zou [4], respectively, obtained other refinements:

$$((1-v)a+vb)^2\geqslant\left(a^{1-v}b^{v}\right)^2+r_0^2(a-b)^2, \tag{4}$$
$$((1-v)a+vb)^2\leqslant\left(a^{1-v}b^{v}\right)^2+R_0^2(a-b)^2. \tag{5}$$

Recently, Manasrah and Kittaneh refined Young's inequality as

$$((1-v)a+vb)^m\geqslant\left(a^{1-v}b^{v}\right)^m+r_0^m(a^{m/2}-b^{m/2})^2 \tag{6}$$

for any positive integer m. We remark that (2) and (4) are the cases $m=1,2$ of (6), respectively.

This paper is motivated by the following observation:

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- (2) and (4) hold equality when $v=\frac{1}{2}$, but (6) which gives
$$(a+b)^m \geqslant a^m+b^m+(2^m-2)(ab)^{m/2}$$
when $v=\frac{1}{2}$ does not satisfy such a property for $m>2$.
- Considering the form of (3) and (5), we may expect that
$$((1-v)a+vb)^m \leqslant \left(a^{1-v}b^v\right)^m + R_0^m(a^{m/2}-b^{m/2})^2$$
holds for any positive integer m. However, as numerical examples show, the above does not hold for $m>2$.

In this paper, we will present a generalization of (2)–(5), which solves the problems stated above. As applications, we will give some inequalities involving positive definite matrices. In the last section, we will generalize other Young-type inequalities. For readers interested in recent papers closely connected to this paper, we refer to [1, 3].

2. Generalized Young's inequalities

Our main result is the following theorem which will be proved later in this section.

THEOREM 1. *If $a,b>0$ and $0\leqslant v\leqslant 1$, then for any positive integer m, we have*
$$((1-v)a+vb)^m \geqslant (a^{1-v}b^v)^m+(2r_0)^m\left(\left(\frac{a+b}{2}\right)^m-\sqrt{ab}^m\right), \tag{7}$$
$$((1-v)a+vb)^m \leqslant (a^{1-v}b^v)^m+(2R_0)^m\left(\left(\frac{a+b}{2}\right)^m-\sqrt{ab}^m\right), \tag{8}$$
where $r_0=\min\{v,1-v\}$ and $R_0=\max\{v,1-v\}$.

Before we give a proof of the theorem, we observe the following obvious properties of the result:

1. (2)–(5) are the cases $m=1,2$ of the theorem.
2. Equality holds in (7) and (8) when $v=\frac{1}{2}$.

We start with a simple, but useful result in studying Young-type inequalities.

PROPOSITION 1. *Let a,b,c,d be positive numbers with $a>c$ and $b>d$, and let $\alpha_m=\left(\frac{a^m-c^m}{b^m-d^m}\right)^{1/m}$ for $m=1,2,\ldots$ Then,*

1. *$\alpha_m\to\frac{a}{b}$ as $m\to\infty$.*
2. *$\{\alpha_m\}$ is decreasing if $ad\geqslant bc$ and increasing if $ad\leqslant bc$.*

Proof. Since $\alpha_m = \frac{a}{b}\left(\frac{1-(c/a)^m}{1-(d/b)^m}\right)^{1/m}$, $c/a<1$, and $d/b<1$, it is obvious that $\alpha_m \to \frac{a}{b}$ as $m\to\infty$. Next, we show monotonicity of $\{\alpha_m\}$. Since $\alpha_m = \frac{c}{d}\left(\frac{(a/c)^m-1}{(b/d)^m-1}\right)^{1/m}$, we may assume $c=d=1$ and show that $\alpha_m = \left(\frac{a^m-1}{b^m-1}\right)^{1/m}$ is decreasing if $a \geqslant b > 1$ and increasing if $b \geqslant a > 1$. Moreover, it suffices only to show the decreasing case, since

$$\left(\frac{a^m-1}{b^m-1}\right)^{1/m} = 1/\left(\frac{b^m-1}{a^m-1}\right)^{1/m}.$$

Simple algebra shows that $\alpha_m \geqslant \alpha_{m+1}$ is equivalent to

$$\frac{(b^{m+1}-1)^m}{(b^m-1)^{m+1}} \geqslant \frac{(a^{m+1}-1)^m}{(a^m-1)^{m+1}}. \tag{9}$$

For a fixed positive integer m, let $f(t) = \frac{(t^{m+1}-1)^m}{(t^m-1)^{m+1}}$ for $t>1$. Since

$$f'(t) = \frac{-m(m+1)(t-1)t^{m-1}(t^{m+1}-1)^{m-1}}{(t^m-1)^{m+2}} < 0,$$

f is decreasing on $(1,\infty)$. Thus if $a \geqslant b > 1$, then (9) holds. □

COROLLARY 1. *For $0<v<1$ and $a,b>0$ with $a\neq b$, let $\mu_v = (1-v)a+vb$ and $\delta_v = a^{1-v}b^v$. For convenience, we denote $\mu_{1/2}$ and $\delta_{1/2}$ by μ and δ, respectively. Then for all positive integers m, if $\frac{\mu_v}{\mu} \geqslant \frac{\delta_v}{\delta}$, then*

$$\frac{\mu_v}{\mu} \leqslant \left(\frac{\mu_v^m-\delta_v^m}{\mu^m-\delta^m}\right)^{1/m} \leqslant \frac{\mu_v-\delta_v}{\mu-\delta}$$

and if $\frac{\mu_v}{\mu} \leqslant \frac{\delta_v}{\delta}$, then

$$\frac{\mu_v-\delta_v}{\mu-\delta} \leqslant \left(\frac{\mu_v^m-\delta_v^m}{\mu^m-\delta^m}\right)^{1/m} \leqslant \frac{\mu_v}{\mu}.$$

Equivalently, the following holds for all positive integers m:

$$\min\left\{\frac{\mu_v}{\mu}, \frac{\mu_v-\delta_v}{\mu-\delta}\right\} \leqslant \left(\frac{\mu_v^m-\delta_v^m}{\mu^m-\delta^m}\right)^{1/m} \leqslant \max\left\{\frac{\mu_v}{\mu}, \frac{\mu_v-\delta_v}{\mu-\delta}\right\}.$$

Proof. Let $\alpha_m = \left(\frac{\mu_v^m-\delta_v^m}{\mu^m-\delta^m}\right)^{1/m}$. Since $\mu_v > \delta_v$ for all $0<v<1$, the desired result follows from Proposition 1. □

Now we prove Theorem 1 as follows.

Proof of Theorem 1. Under the same notation of Corollary 1, the inequalities in Theorem 1 can be written as

$$\mu_v^m \geqslant \delta_v^m + (2r_0)^m(\mu^m-\delta^m),$$
$$\mu_v^m \leqslant \delta_v^m + (2R_0)^m(\mu^m-\delta^m).$$

Since they obviously hold when $v=0$, $v=1$, or $a=b$, we assume $0<v<1$ and $a\neq b$. Then the above inequalities are equivalent to

$$2r_0 \leqslant \left(\frac{\mu_v^m-\delta_v^m}{\mu^m-\delta^m}\right)^{1/m} \leqslant 2R_0. \tag{10}$$

Since $\left(\frac{\mu_v^m-\delta_v^m}{\mu^m-\delta^m}\right)^{1/m}$ is between $\frac{\mu_v}{\mu}$ and $\frac{\mu_v-\delta_v}{\mu-\delta}$ by Corollary 1, it suffices to show that $\frac{\mu_v}{\mu}$ and $\frac{\mu_v-\delta_v}{\mu-\delta}$ are between $2r_0$ and $2R_0$. Simple algebra shows that

$$\begin{aligned}
&2r_0 \leqslant \frac{\mu_v}{\mu} \leqslant 2R_0\\
\Longleftrightarrow\ &(a+b)r_0 \leqslant (1-v)a+vb \leqslant (a+b)R_0\\
\Longleftrightarrow\ &(1-v-r_0)a+(v-r_0)b \geqslant 0,\\
&\text{and } (R_0-1+v)a+(R_0-v)b \geqslant 0.
\end{aligned}$$

Since $r_0 \leqslant v$ and $1-v \leqslant R_0$, the above holds for all v. Meanwhile, (10) holds for $m=1$ by (2) and (3). □

REMARK 1. By direct computation, we can show that (7) improves (6). Since

$$\begin{aligned}
&(a^{1-v}b^v)^m+(2r_0)^m\left(\left(\frac{a+b}{2}\right)^m-\sqrt{ab}^m\right) \geqslant \left(a^{1-v}b^v\right)^m+r_0^m(a^{m/2}-b^{m/2})^2\\
\Longleftrightarrow\ &(a+b)^m-(2\sqrt{ab})^m \geqslant a^m+b^m-2\sqrt{ab}^m,
\end{aligned}$$

we will show $f_m(x)\geqslant 0$ for $x>0$ and $m=1,2,\ldots$, where

$$f_m(x)=(1+x^2)^m-2^mx^m-1-x^{2m}+2x^m.$$

Moreover, since $x^{2m}f_m(x^{-1})=f_m(x)$, we may assume $x\geqslant 1$. We use induction on m. Clearly, $f_1\equiv 0$. Suppose $f_m(x)\geqslant 0$ for all $x\geqslant 1$. Then, since

$$(1+x^2)^m \geqslant 2^mx^m+1+x^{2m}-2x^m,$$

we have

$$\begin{aligned}
f'_{m+1}(x) &= 2(m+1)\left\{x(1+x^2)^m-2^mx^m-x^{2m+1}+x^m\right\}\\
&\geqslant 2(m+1)\left\{x\left(2^mx^m+1+x^{2m}-2x^m\right)-2^mx^m-x^{2m+1}+x^m\right\}\\
&= 2(m+1)x\left\{2^mx^m+1-2x^m-2^mx^{m-1}+x^{m-1}\right\}.
\end{aligned}$$

Let $g_m(x)=2^mx^m+1-2x^m-2^mx^{m-1}+x^{m-1}$. Since $g_m(1)=0$ and

$$\begin{aligned}
g'_m(x) &= x^{m-2}\left\{m(2^m-2)x-(m-1)(2^m-1)\right\}\\
&\geqslant x^{m-2}\left\{m(2^m-2)-(m-1)(2^m-1)\right\}\\
&= x^{m-2}(2^m-m-1)\\
&\geqslant 0,
\end{aligned}$$

$f'_{m+1}(x)\geqslant 0$ for $x\geqslant 1$. Thus $f_{m+1}\geqslant 0$ follows from $f_{m+1}(1)=0$.

3. Applications

In this section, we show some applications of Theorem 1 to matrix inequalities. Let M_n be the space of $n \times n$ complex matrices. For $A \in M_n$, $A > 0$ denotes that A is positive definite, *i.e.*, $x^*Ax > 0$ for all nonzero $x \in \mathbb{C}^n$. For $A, B > 0$ and $0 \leqslant v \leqslant 1$, the v-arithmetic mean of A and B is defined by $A\nabla_v B = (1-v)A + vB$. Similarly, we define the t-weighted geometric mean of two positive definite matrices $A, B \in M_n$ by $A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ for any real number t. For convenience, we denote $A\nabla_{\frac{1}{2}}B$ by $A\nabla B$.

Using (6), Manasrah and Kittaneh [8] improved the inequalities

$$[\mathrm{tr}(A\nabla_v B)]^m \geqslant \left(\mathrm{tr}|A^{1-v}B^v|\right)^m,$$

$$[\det(A\nabla_v B)]^m \geqslant \det(A^{1-v}B^v)^m,$$

$$[(1-v)|||AX||| + v|||XB|||]^m \geqslant |||A^{1-v}XB^v|||^m$$

for $A, B, X \in M_n$ with $A, B > 0$ and $0 \leqslant v \leqslant 1$, where $|X| = (X^*X)^{1/2}$ and $|||\cdot|||$ stands for any unitarily invariant norm on M_n. Since (7) is stronger than (6), their results can be further improved. We state the following without proof. See [8] for detailed proofs.

THEOREM 2. *Let $A, B \in M_n$ be positive definite. Then for $0 \leqslant v \leqslant 1$ and $m = 1, 2, \ldots$, we have*

1. $[tr(A\nabla_v B)]^m \geqslant \left(tr|A^{1-v}B^v|\right)^m + (2r_0)^m \left((trA\nabla B)^m - \sqrt{trAtrB}^m\right)$,
2. $[\det(A\nabla_v B)]^m \geqslant \det(A^{1-v}B^v)^m + (2r_0)^{mn}\left(\left(\frac{\det A^{1/n} + \det B^{1/n}}{2}\right)^{mn} - \sqrt{\det(AB)}^m\right)$,
3. $[(1-v)|||AX||| + v|||XB|||]^m \geqslant |||A^{1-v}XB^v|||^m + (2r_0)^m \left(\left(\frac{|||AX||| + |||XB|||}{2}\right)^m - \sqrt{|||AX|||\,|||XB|||}^m\right)$

   for all $X \in M_n$.

To show matrix (or operator) inequalities corresponding to their scalar versions, we can use the operator monotonicity of continuous functions; that is, if f is a real valued continuous function defined on the spectrum of a self-adjoint operator A, then $f(t) \geqslant 0$ for every t in the spectrum of A implies $f(A)$ is a positive operator. The following follows from the property and Theorem 1.

THEOREM 3. *Let $A, B \in M_n$ be positive definite. Then for $0 \leqslant v \leqslant 1$ and $m = 1, 2, \ldots$, we have*

$$A\#_m(A\nabla_v B) \geqslant A\#_{vm}B + (2r_0)^m \left(A\#_m(A\nabla B) - A\#_{m/2}B\right), \tag{11}$$

$$A\#_m(A\nabla_v B) \leqslant A\#_{vm}B + (2R_0)^m \left(A\#_m(A\nabla B) - A\#_{m/2}B\right). \tag{12}$$

Proof. For any positive definite matrix $X \in M_n$, it follows from Theorem 1 with $a = 1$ that

$$((1-v)I + vX)^m \geqslant X^{vm} + (2r_0)^m \left(\left(\frac{I+X}{2}\right)^m - X^{m/2}\right),$$

where I is the identity matrix in M_n. Letting $X = A^{-1/2}BA^{-1/2}$ and multiplying both sides of the above inequality by $A^{1/2}$ on their left and right hand sides, we get (11). We can also derive (12) in the same way. □

4. Other generalized inequalities

In this section, we will show more examples which can be generalized by Proposition 1. For convenience of notation, we have used μ_v and δ_v, respectively, for the v-weighted arithmetic and geometric means of a and b in the second section. As shown in [2] and other references, however, we will now use the following well-known notation:

$$\begin{aligned} a\nabla_v b &= (1-v)a + vb, \\ a\#_v b &= a^{1-v}b^v, \\ a!_v b &= ((1-v)a^{-1} + vb^{-1})^{-1} \end{aligned}$$

for $a,b > 0$ and $0 \leqslant v \leqslant 1$. When $v = \frac{1}{2}$, we write $a\nabla b$, $a\#b$, and $a!b$ for brevity, respectively.

There are many inequalities which can be generalized by Proposition 1, but we will apply the proposition to the following ones shown in [2] and its references.

THEOREM 4. *For $a,b > 0$ and $0 \leqslant v \leqslant 1$, we have*

1. $a!_v b + 2r_0(a\nabla b - a!b) \leqslant a\nabla_v b \leqslant a!_v b + 2R_0(a\nabla b - a!b)$,
2. $K(h,2)^{r_0} a\#_v b \leqslant a\nabla_v b \leqslant K(h,2)^{R_0} a\#_v b$, *where* $h = \frac{b}{a}$ *and* $K(t,2)$ *is the Kantorovich constant defined by* $K(t,2) = \frac{(t+1)^2}{4t}$.

We generalize the first one in the above theorem as follows.

THEOREM 5. *For $a,b > 0$, $0 \leqslant v \leqslant 1$, and $m = 1,2,\ldots$, we have*

$$\begin{aligned} (a\nabla_v b)^m &\geqslant (a!_v b)^m + (2r_0)^m\left((a\nabla b)^m - (a!b)^m\right), \\ (a\nabla_v b)^m &\leqslant (a!_v b)^m + (2R_0)^m\left((a\nabla b)^m - (a!b)^m\right). \end{aligned}$$

Proof. Since equality holds in them when either $a = b$ or $v = 0,1$, we may assume $a \neq b$ and $0 < v < 1$. Then we have

$$2r_0 \leqslant \frac{a\nabla_v b - a!_v b}{a\nabla b - a!b} \leqslant 2R_0$$

by the first part of Theorem 4. Thus it suffices to show

$$2r_0 \leqslant \frac{a\nabla_v b}{a\nabla b} \leqslant 2R_0$$

by Proposition 1, and the above is already shown in the proof of Theorem 1. □

Note that the second part of Theorem 4 can be written as

$$\left(\frac{a\nabla b}{a\# b}\right)^{2r_0} a\#_v b \leqslant a\nabla_v b \leqslant \left(\frac{a\nabla b}{a\# b}\right)^{2R_0} a\#_v b. \tag{13}$$

or equivalently, using natural logarithm,

$$\begin{aligned} \ln(a\nabla_v b) &\geqslant \ln(a\#_v b) + 2r_0\left(\ln(a\nabla b) - \ln(a\# b)\right), \\ \ln(a\nabla_v b) &\leqslant \ln(a\#_v b) + 2R_0\left(\ln(a\nabla b) - \ln(a\# b)\right). \end{aligned} \tag{14}$$

THEOREM 6. *For* $a,b > 1$, $0 \leqslant v \leqslant 1$, *and* $m = 1,2,\ldots$, *we have*

$$\begin{aligned} [\ln(a\nabla_v b)]^m &\geqslant [\ln(a\#_v b)]^m + (2r_0)^m\left([\ln(a\nabla b)]^m - [\ln(a\# b)]^m\right), \\ [\ln(a\nabla_v b)]^m &\leqslant [\ln(a\#_v b)]^m + (2R_0)^m\left([\ln(a\nabla b)]^m - [\ln(a\# b)]^m\right). \end{aligned}$$

Proof. Without loss of generality, we may assume $a \neq b$ and $0 < v < 1$. Since

$$2r_0 \leqslant \frac{\ln(a\nabla_v b) - \ln(a\#_v b)}{\ln(a\nabla b) - \ln(a\# b)} \leqslant 2R_0$$

by (14), it suffices to show

$$2r_0 \leqslant \frac{\ln(a\nabla_v b)}{\ln(a\nabla b)} \leqslant 2R_0$$

by Proposition 1, which is equivalent to

$$(a\nabla b)^{2r_0} \leqslant a\nabla_v b \leqslant (a\nabla b)^{2R_0}. \tag{15}$$

Note that (13) can be expressed as

$$(a\nabla b)^{2r_0} a^{1-v-r_0} b^{v-r_0} \leqslant a\nabla_v b \leqslant (a\nabla b)^{2R_0} a^{1-v-R_0} b^{v-R_0}. \tag{16}$$

Since $a,b > 1$ and $r_0 \leqslant v, 1-v \leqslant R_0$, we have $a^{1-v-r_0}b^{v-r_0} \geqslant 1$ and $a^{1-v-R_0}b^{v-R_0} \leqslant 1$. Thus (15) follows from (16). □

Replacing a,b with a^{-1},b^{-1}, respectively, in the above theorem, we get the following.

COROLLARY 2. *For* $a,b \in (0,1)$, $0 \leqslant v \leqslant 1$, *and* $m = 1,2,\ldots$, *we have*

$$\begin{aligned} |\ln(a!_v b)|^m &\geqslant |\ln(a\#_v b)|^m + (2r_0)^m\left(|\ln(a!b)|^m - |\ln(a\# b)|^m\right), \\ |\ln(a!_v b)|^m &\leqslant |\ln(a\#_v b)|^m + (2R_0)^m\left(|\ln(a!b)|^m - |\ln(a\# b)|^m\right). \end{aligned}$$

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Daeshik Choi
Department of Mathematics and Statistics
Southern Illinois University Edwardsville
Box 1653, Edwardsville, IL 62026
e-mail: dchoi@siue.edu