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OPTIMALITY OF THE REARRANGEMENT INEQUALITY WITH APPLICATIONS TO LORENTZ–TYPE SEQUENCE SPACES

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Abstract. We characterize the sequences $(w_i)_{i=1}^{\infty}$ of non-negative numbers for which

$$\sum_{i=1}^{\infty} a_i w_i \quad \text{is of the same order as} \quad \sup_n \sum_{i=1}^{n} a_i w_{1+n-i}$$

when $(a_i)_{i=1}^{\infty}$ runs over all non-increasing sequences of non-negative numbers. As a by-product of our work we settle a problem raised in [1] and prove that Garling sequences spaces have no symmetric basis.

1. Introduction

The rearrangement inequality states that, for $n \in \mathbb{N}$, if $(a_i)_{i=1}^{n}$ and $(b_i)_{i=1}^{n}$ are a pair of non-increasing n-tuples of non-negative scalars then we have

$$\sum_{i=1}^{n} a_i b_{1+n-i} \leqslant \sum_{i=1}^{n} a_i b_{\sigma(i)} \leqslant \sum_{i=1}^{n} a_i b_i$$

for every permutation σ of the set $\{1,\dots,n\}$ (see Theorem 368 of [3]). Consequently, if $(a_i)_{i=1}^{\infty}$ and $(w_i)_{i=1}^{\infty}$ are non-increasing sequences of non-negative scalars,

$$\sup_{n\in\mathbb{N}} \sum_{i=1}^{n} a_i w_{1+n-i} \leqslant \sum_{i=1}^{\infty} a_i w_i.$$

In this note we wonder about which are the non-increasing sequences $(w_i)_{i=1}^{\infty}$ of non-negative scalars that verify a reverse inequality, i.e., in which cases there is a constant $C < \infty$ such that

$$\sum_{i=1}^{\infty} a_i w_i \leqslant C \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} a_i w_{1+n-i} \tag{1}$$

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for every sequence $(a_i)_{i=1}^\infty$ of non-negative scalars. For the time being some simple answers can be given. Indeed, on the one hand, if $w_\infty := \inf_i w_i > 0$ then

$$\sum_{i=1}^\infty a_i w_i \leqslant w_1 \sum_{i=1}^\infty a_i = w_1 \sup_n \sum_{i=1}^n a_i \leqslant \frac{w_1}{w_\infty} \sup_n \sum_{i=1}^n a_i w_{1+n-i}.$$

On the other hand, if we consider $W := \sum_{i=1}^\infty w_i < \infty$ and let $w_1 > 0$ (the case $w_1 = 0$ is trivial) then

$$\sum_{i=1}^\infty a_i w_i \leqslant a_1 \sum_{i=1}^\infty w_i = \frac{W}{w_1} a_1 w_1 \leqslant \frac{W}{w_1} \sup_n \sum_{i=1}^n a_i w_{1+n-i}.$$

In fact, as we will show below, these two cases are the only ones for which (1) holds. This will be our main result as far as inequalities is concerned:

THEOREM 1. (Main Theorem) *Let $(w_i)_{i=1}^\infty$ be a non-increasing sequence consisting of non-negative scalars. The following are equivalent:*

(i) There is a constant $C < \infty$ such that

$$\sum_{i=1}^\infty a_i w_i \leqslant C \sup_{n\in\mathbb{N}} \sum_{i=1}^n a_i w_{1+n-i}$$

for every sequence $(a_i)_{i=1}^\infty$ of non-negative scalars.

(ii) Either $\sum_{i=1}^\infty w_i < \infty$ or $\inf_{i\in\mathbb{N}} w_i > 0$.

Section 2 is devoted to proving Theorem 1. In Section 3 we use Theorem 1 to give some functional analytic properties of a recently introduced class of Lorentz-type spaces, called Garling sequence spaces. In particular, Theorem 1 is applied to show that Garling sequence spaces have no symmetric basis, answering thus a problem that was recently posed in [1].

Throughout this note we use standard terminology and notation in Banach space theory. As is customary, we denote by ℓ_q, $1 \leqslant q \leqslant \infty$, the Banach space consisting of all q-summable sequences (bounded sequences in the case $q = \infty$) and by c_0 the subspace of ℓ_∞ consisting of all sequences converging to zero. For background on bases in Banach spaces we refer the reader to [2].

2. Proof of the Main Theorem

Proof of Theorem 1. As explained in the Introduction, we must only prove that (i) implies (ii).

Assume that (ii) does not hold, that is, $\mathbf{w} = (w_i)_{i=1}^\infty \in c_0 \setminus \ell_1$. Let us denote by $\mathscr{D}$ the set of (nonzero) non-increasing sequences of non-negative integers. For $f = (a_i)_{i=1}^\infty \in \mathscr{D}$ and $n \in \mathbb{N}$ we put

$$A(f,\mathbf{w}) = \sum_{i=1}^\infty a_i w_i, \text{ and}$$

$$B(f,\mathbf{w}) = \sup_{n\in\mathbb{N}} B(f,\mathbf{w},n),$$

where, for $n \in \mathbb{N}$,

$$B(f,\mathbf{w},n) = \sum_{i=1}^{n} a_i w_{1+n-i}.$$

With this notation we must prove that

$$S(\mathbf{w}) := \sup_{f\in\mathscr{D}} \frac{A(f,\mathbf{w})}{B(f,\mathbf{w})} = \infty.$$

We will use the convention that $\sum_{i=1}^{0} c_i = 0$ for all sequences of scalars $(c_i)_{i=1}^{\infty}$.

For $n \in \mathbb{N}$ put $W(n) = \sum_{i=1}^{n} w_i$. Since $\mathbf{w} \notin \ell_1$ we have

$$\lim_n W(n) = \infty.$$

Moreover, since $\mathbf{w} \in c_0$,

$$\lim_{n\in\mathbb{N}} (W(s+n) - W(n)) = 0$$

for any non-negative integer s. We use these properties to recursively construct an increasing sequence $(d_k)_{k=0}^{\infty}$ of non-negative integers with $d_0 = 0$ verifying

(i) $W(\sum_{j=1}^{k-1} d_j) \leqslant 2^{-1} W(d_k)$, and

(ii) $W(d_{k-1} + d_k) - W(d_k) \leqslant 2^{1-k} W(d_{k-1})$

for $k = 1, 2, \dots$.

For every integer $k \geqslant 0$ put $n_k = \sum_{j=1}^{k} d_j$. For each $r \in \mathbb{N}$ we define a sequence $f^{(r)} = (a_{i,r})_{i=1}^{\infty}$ by

$$a_{i,r} = \begin{cases} 1/W(d_k) & \text{if, for some } 1 \leqslant k \leqslant r,\ n_{k-1} < i \leqslant n_k \\ 0 & \text{if } i > n_r. \end{cases}$$

It is clear that $f^{(r)} \in \mathscr{D}$ for all $r \in \mathbb{N}$. Taking into account the inequality in (i) we obtain

$$\begin{aligned} A(f^{(r)},\mathbf{w}) &= \sum_{k=1}^{r} \frac{1}{W(d_k)} \sum_{i=1+n_{k-1}}^{n_k} w_i = \sum_{k=1}^{r} \frac{W(n_k) - W(n_{k-1})}{W(d_k)} \\ &\geqslant \sum_{k=1}^{r} \frac{W(d_k) - 2^{-1} W(d_k)}{W(d_k)} = \sum_{k=1}^{r} \frac{1}{2} = \frac{r}{2}. \end{aligned}$$

Let $n \in \mathbb{N}$. In case that $n > n_r$ we have

$$B(f^{(r)},\mathbf{w},n) = \sum_{i=1}^{n_r} a_{i,r} w_{1+n-i} \leqslant \sum_{i=1}^{n_r} a_{i,r} w_{1+n_r-i} = B(f^{(r)},\mathbf{w},n_r).$$

In case that $n \leqslant n_r$, pick $1 \leqslant k \leqslant r$ with $n_{k-1} < n \leqslant n_k$. We have

$$\begin{aligned} B(f^{(r)}, \mathbf{w}, n) &= \frac{W(n-n_{k-1})}{W(d_k)} + \sum_{j=1}^{k-1} \frac{W(n-n_{j-1}) - W(n-n_j)}{W(d_j)} \\ &\leqslant \frac{W(n_k-n_{k-1})}{W(d_k)} + \sum_{j=1}^{k-1} \frac{W(n-n_{j-1}) - W(n-n_j)}{W(d_j)} \\ &= 1 + \sum_{j=1}^{k-1} \frac{W(n-n_{j-1}) - W(n-n_j)}{W(d_j)}. \end{aligned}$$

If $k=1$ we get $B(f^{(r)}, \mathbf{w}, n) \leqslant 1$. Assume that $k \geqslant 2$. Taking into account inequality (ii) and that, since $\mathbf{w}$ is non-increasing, the sequence $(W(n+t)-W(n+s))_{n=1}^\infty$ is non-increasing for any $s \leqslant t$, we obtain

$$\begin{aligned} B(f^{(r)}, \mathbf{w}, n) &\leqslant 1 + \frac{W(n-n_{k-2}) - W(n-n_{k-1})}{W(d_{k-1})} + \sum_{j=1}^{k-2} \frac{W(n-n_{j-1}) - W(n-n_j)}{W(d_j)} \\ &\leqslant 1 + \frac{W(n_{k-1}-n_{k-2}) - W(n_{k-1}-n_{k-1})}{W(d_{k-1})} \\ &\quad + \sum_{j=1}^{k-2} \frac{W(n_{j+1}-n_{j-1}) - W(n_{j+1}-n_j)}{W(d_j)} \\ &= 2 + \sum_{j=1}^{k-2} \frac{W(d_{j+1}+d_j) - W(d_{j+1})}{W(d_j)} \leqslant 2 + \sum_{j=1}^{k-2} 2^{-j} = 3 - 2^{2-k}. \end{aligned}$$

Therefore $B(f^{(r)}, \mathbf{w}) \leqslant 3$. Thus

$$S(\mathbf{w}) \geqslant \sup_{r\in\mathbb{N}} \frac{A(f^{(r)}, \mathbf{w})}{B(f^{(r)}, \mathbf{w})} \geqslant \sup_{r\in\mathbb{N}} \frac{r}{6} = \infty,$$

and the proof is over. □

3. Applications to Garling squence spaces

Let $1 \leqslant p < \infty$ and let $\mathbf{w} = (w_n)_{n=1}^\infty$ be a non-increasing sequence of positive scalars. Given a sequence of (real or comnplex) scalars $f = (b_k)_{k=1}^\infty$ we put

$$\|f\|_{g(\mathbf{w},p)} = \sup_{\phi\in\mathscr{O}} \left(\sum_{i=1}^\infty |b_{\phi(i)}|^p w_i \right)^{1/p}$$

where $\mathscr{O}$ denotes the set of increasing functions from $\mathbb{N}$ to $\mathbb{N}$. The Garling sequence space $g(\mathbf{w}, p)$ is the Banach space consisting of all sequences f with $\|f\|_{g(\mathbf{w},p)} < \infty$.

Notice that if in (3) we replace "$\phi \in \mathscr{O}$" with "ϕ is a permutation of $\mathbb{N}$" we obtain the norm that defines the weighted Lorentz sequence space

$$d(\mathbf{w},p) := \left\{ (b_k)_{k=1}^{\infty} \in c_0 \colon \left(\sum_{i=1}^{\infty} (b_i^*)^p w_i \right)^{1/p} < \infty \right\},$$

where $(b_i^*)_{i=1}^{\infty}$ denotes the decreasing rearrangement of $(b_k)_{k=1}^{\infty}$. So, the Garling sequence space $g(\mathbf{w},p)$ can be regarded as a variation of the weighted Lorentz sequence space $d(\mathbf{w},p)$.

Imposing the further conditions $\mathbf{w} \in c_0$ and $\mathbf{w} \notin \ell_1$ will prevent us, respectively, from having $g(\mathbf{w},p) = \ell_p$ or $g(\mathbf{w},p) = \ell_\infty$. We will assume as well that $\mathbf{w}$ is normalized, i.e., $w_1 = 1$. Thus, we put

$$\mathscr{W} := \{(w_i)_{i=1}^{\infty} \in c_0 \setminus \ell_1 : 1 = w_1 \geqslant w_2 \geqslant \cdots \geqslant w_i \geqslant w_{i+1} \geqslant \cdots > 0\}$$

and we restrict our attention to *weights* $\mathbf{w} \in \mathscr{W}$.

For $n \in \mathbb{N}$, we will denote $\mathbf{e}_n = (\delta_{i,n})_{i=1}^{\infty}$, where $\delta_{i,n} = 1$ if $n = i$ and $\delta_{i,n} = 0$ otherwise. We have (see Theorem 3.1 of [1]) that the canonical sequence $(\mathbf{e}_n)_{n=1}^{\infty}$ is a Schauder basis for $g(\mathbf{w},p)$. A question posed and partially solved in [1] is to determine the weights $\mathbf{w} \in \mathscr{W}$ and the indices $p \in [1,\infty)$ for which $(\mathbf{e}_n)_{n=1}^{\infty}$ is a symmetric basis of $g(\mathbf{w},p)$. Here we provide a complete intrinsic solution to this problem, in the sense that our approach is entirely based on Theorem 1.

LEMMA 2. *The canonical sequence $(\mathbf{e}_n)_{n=1}^{\infty}$ is not a symmetric basis for $g(\mathbf{w},p)$ for any $\mathbf{w} \in \mathscr{W}$ and any $1 \leqslant p < \infty$.*

Proof. Assume that $(\mathbf{e}_n)_{n=1}^{\infty}$ is a symmetric basis for $g(\mathbf{w},p)$. Then, there is a constant C so that

$$\|g\|_{g(\mathbf{w},p)} \leqslant C\|f\|_{g(\mathbf{w},p)}$$

whenever the sequence g is a permutation of the sequence f.

Given $r \in \mathbb{N}$ and $\phi \in \mathscr{O}$ let $n(r,\phi)$ be the largest integer n such that $\phi(n) \leqslant r$. We have $\phi(i) \leqslant i + r - n(r,\phi)$ for $1 \leqslant i \leqslant n(r,\phi)$. Given a non-increasing sequence $(a_i)_{i=1}^{\infty}$ of non-negative numbers we have

$$\begin{aligned}
\sum_{i=1}^{\infty} a_i w_i &= \sup_r \sum_{i=1}^{r} a_i w_i \leqslant \sup_{r\in\mathbb{N}} \left\| \sum_{i=1}^{r} a_i^{1/p} \mathbf{e}_i \right\|_{g(\mathbf{w},p)}^p \\
&\leqslant C \sup_{r\in\mathbb{N}} \left\| \sum_{i=1}^{r} a_{1+r-i}^{1/p} \mathbf{e}_i \right\|_{g(\mathbf{w},p)}^p = C \sup_{r\in\mathbb{N},\,\phi\in\mathscr{O}} \sum_{i=1}^{n(r,\phi)} a_{1+r-\phi(i)} w_i \\
&\leqslant C \sup_{r\in\mathbb{N},\,\phi\in\mathscr{O}} \sum_{i=1}^{n(r,\phi)} a_{1+n(r,\phi)-i} w_i = C \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} a_{1+n-i} w_i \\
&= C \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} a_i w_{1+n-i}.
\end{aligned}$$

Theorem 1 yields the absurdity $\mathbf{w} \in \ell_1$ or $\mathbf{w} \notin c_0$. $\square$

Now we are ready to establish the advertised structural properties of Garling sequence spaces.

THEOREM 3. *Let* $\mathbf{w} \in \mathscr{W}$ *and* $1 \leqslant p < \infty$.

(i) There is no symmetric basis for $g(\mathbf{w}, p)$.

(ii) $d(\mathbf{w}, p) \subsetneq g(\mathbf{w}, p)$.

(iii) No subspace of $d(\mathbf{w}, p)$ *is isomorphic to* $g(\mathbf{w}, p)$.

(iv) Let $I_{d,g} \colon d(\mathbf{w}, p) \to g(\mathbf{w}, p)$ *be the natural inclusion map, and let* $T \colon g(\mathbf{w}, p) \to d(\mathbf{w}, p)$ *be a bounded linear operator. Then (despite the fact that* $I_{d,g}$ *is not a strictly singular operator)* $T \circ I_{d,g}$ *does not preserve a copy of* $d(\mathbf{w}, p)$*, i.e., if* X *is a subspace of* $d(\mathbf{w}, p)$ *isomorphic to* $d(\mathbf{w}, p)$ *then* $T \circ I_{d,g}|_X$ *is not an isomorphism.*

Proof. It follows using Lemma 2 in combiantion with Theorem 5.1 of [1]. $\square$

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