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COVERING UNIT SPHERES AND BALLS OF NORMED SPACES BY SMALLER BALLS

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Abstract. We amend a widely used result given by Doyle, Lagarias, and Randall concerning the side length of equilateral Minkowski m-gons inscribed in the unit circle of a Minkowski plane. Based on this, we obtain the smallest positive number γ such that the unit circle S_X of a Minkowski plane X can be covered by m translates of γB_X, where B_X is the unit ball of X. Moreover, we improve a recent estimation of the smallest positive number γ such that the unit ball B_X of a Minkowski space X can be covered by m translates of γB_X.

1. Introduction

Let $n \geqslant 2$ be a positive integer. We denote by $[n]$ the set $\{m \in \mathbb{Z}^+ :\ 1 \leqslant m \leqslant n\}$ and by $X = (\mathbb{R}^n, \|\cdot\|)$ an n-dimensional (normed or) *Minkowski space* whose *unit ball* and *unit sphere* are denoted by B_X and S_X, respectively. Clearly, B_X is a *convex body* (i.e., a compact convex set having interior points) symmetric with respect to the *origin* o of $\mathbb{R}^n$. For each $m \in \mathbb{Z}^+$, put

$$\Gamma_m(X) = \inf\left\{\gamma > 0 :\ \exists\{x_i :\ i \in [m]\} \subseteq X \text{ s.t. } B_X \subseteq \bigcup_{i\in[m]} (x_i + \gamma B_X)\right\},$$

$$\gamma_m(X) = \inf\left\{\gamma > 0 :\ \exists\{x_i :\ i \in [m]\} \subseteq X \text{ s.t. } S_X \subseteq \bigcup_{i\in[m]} (x_i + \gamma B_X)\right\}.$$

It is clear that

$$\gamma_m(X) \leqslant \Gamma_m(X) \leqslant 1,\ \forall m \in \mathbb{Z}^+,$$

and that "inf"s in the definition of $\Gamma_m(X)$ and $\gamma_m(X)$ can be replaced by "min"s.

As shown in [11], [16], [8], and [12], $\Gamma_m(X)$ and $\gamma_m(X)$ are both closely related to the special case of the famous Hadwiger's covering problem when the convex body under consideration is centrally symmetric. We refer to [4], [13], [2], and [3] for more

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information and more references concerning this open problem. Recently several progresses have been made in this direction by using tools and methods from Banach space theory, see [15], [8], and [12].

In Section 2, we amend a result given by Doyle, Lagarias, and Randall in [5] saying that the side length of equilateral Minkowski m-gons inscribed in the unit circle of a *Minkowski plane* (i.e., a two-dimensional Minkowski space) having a given point x_0 as one of its vertices is uniquely determined by x_0, which is generally not true (see Example 1). Based on this, in Section 3 we compute the precise value of $\gamma_m(X)$ and show that $\gamma_3(X)=\Gamma_3(X)$ when X is a Minkowski plane. Inspired by the proof idea of Theorem 19 in [12], we provide an estimation of $\Gamma_m(X)$ for Minkowski spaces which is better than the one given in [8] in the last section. As in [8] and [12], we are following the philosophy provided in [17]: studying classical problems from Discrete and Convex Geometry by introducing and studying proper functionals defined on the space of convex bodies.

For each pair of points $u,v\in S_X$ satisfying $u\neq -v$, we denote by

$$\operatorname{arc}(u,v):=\left\{\frac{\alpha u+\beta v}{\|\alpha u+\beta v\|}:\ \alpha,\beta\geqslant 0,\ \alpha u+\beta v\neq o\right\}$$

the *minor arc connecting* u *and* v, see Figure 1.

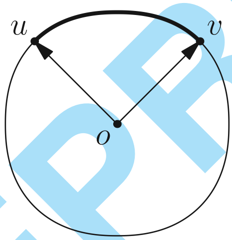

Figure 1: *The definition of a minor arc*

For each bounded subset A of X, put

$$r(A)=\inf\{\gamma\geqslant 0:\ \exists x\in X \text{ s.t. } A\subseteq(x+\gamma B_X)\}.$$

It is not difficult to verify that, when A is compact, then "inf" in the definition of $r(A)$ can be replaced by "min". By the triangle inequality, one can easily verify that

$$r(A)\geqslant\frac{1}{2}\delta(A),$$

where $\delta(A)$ is the *diameter* of A.

2. Side lengths of equilateral Minkowski m-gons

We shall use the following Monotonicity Lemma.

LEMMA 1. (Monotonicity Lemma, cf. Proposition 31 in [14]) *Let X be a Minkowski plane, and $a,b,c \neq o$ be three points with $a \neq c$ such that the ray $[o,b\rangle$ lies between $[o,a\rangle$ and $[o,c\rangle$. If $\|b\| = \|c\|$, then $\|a-b\| \leqslant \|a-c\|$, with equality if and only if either*

1. *b=c;*
2. *or o and b are on opposite sides of $\langle a,c\rangle$, and*

$$\left[\frac{c-a}{\|c-a\|}, \frac{b}{\|b\|}\right] \subseteq S_X;$$

3. *or o and b are on the same side of $\langle a,c\rangle$, and*

$$\left[\frac{c-a}{\|c-a\|}, \frac{-c}{\|-c\|}\right] \subseteq S_X.$$

LEMMA 2. *Let X be a Minkowski plane, $u,v \in S_X$ be two points satisfying $u \neq -v$. Then*

$$r(\mathrm{arc}(u,v)) = \frac{1}{2}\|u-v\|.$$

Proof. Clearly,

$$r(\mathrm{arc}(u,v)) \geqslant \frac{1}{2}\delta(\mathrm{arc}(u,v)) \geqslant \frac{1}{2}\|u-v\|.$$

Hence we only need to prove that

$$r(\mathrm{arc}(u,v)) \leqslant \frac{1}{2}\|u-v\|. \tag{1}$$

Put

$$w = \frac{u+v}{\|u+v\|} \quad \text{and} \quad p = \frac{1}{2}(u+v).$$

Let q be an arbitrary point in $\mathrm{arc}(u,v)$. If $q = u$ or $q = v$, then

$$\|p-q\| = \frac{1}{2}\|u-v\|.$$

If $q = w$, then

$$\|p-q\| = 1 - \|p\| \leqslant 1 - (\|u\| - \|u-p\|) = \frac{1}{2}\|u-v\|.$$

In the following we may assume, without loss of generality, that

$$q \in \operatorname{arc}(u,w) \setminus \{u,w\}.$$

Then the ray $[o,q\rangle$ lies between $[o,u\rangle$ and $[o,p\rangle$, and $\|u\| = \|q\|$. From Lemma 1 it follows that

$$\|q-p\| \leqslant \|u-p\| = \frac{1}{2}\|u-v\|.$$

Therefore, (1) holds. □

LEMMA 3. *Let X be a Minkowski plane, $x \in X$, and $\gamma \in (0,1)$. If*

$$B_X \cap (x+\gamma B_X) \neq \emptyset,$$

then there exist two points $u,v \in S_X$ such that

$$S_X \cap (x+\gamma B_X) = \operatorname{arc}(u,v) \quad \textit{and} \quad \frac{x}{\|x\|} \in \operatorname{arc}(u,v).$$

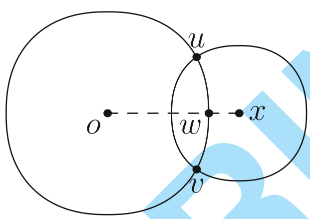

Figure 2: *The intersection of the unit circle and a smaller disc*

Proof. Put $w = x/\|x\|$, see Figure 2. The hypothesis shows that

$$1-\gamma \leqslant \|x\| \leqslant 1+\gamma.$$

Therefore,

$$\|x-w\| = \left\| x - \frac{x}{\|x\|} \right\| = |\,\|x\|-1| \leqslant \gamma,$$

which shows that

$$w \in S_X \cap (x+\gamma B_X).$$

Since

$$\|x-(-w)\| = \|x+w\| = \left\| x + \frac{x}{\|x\|} \right\| = 1+\|x\| > 1 > \gamma,$$

there exist two points p and q in S_X lying on different open half-planes bounded by the line $\langle -x,x\rangle$ such that $\min\{\|p-x\|,\|q-x\|\} \geqslant 1 > \gamma$. Put

$$\lambda_1 = \max\left\{ \lambda \in [0,1] : \left\| x - \frac{\lambda p + (1-\lambda)w}{\|\lambda p + (1-\lambda)w\|} \right\| \leqslant \gamma \right\},$$

$$\lambda_2 = \max\left\{\lambda \in [0,1]: \left\| x - \frac{\lambda q + (1-\lambda)w}{\|\lambda q + (1-\lambda)w\|}\right\| \leqslant \gamma\right\},$$

$$u = \frac{\lambda_1 p + (1-\lambda_1)w}{\|\lambda_1 p + (1-\lambda_1)w\|}, \quad \text{and} \quad v = \frac{\lambda_2 q + (1-\lambda_2)w}{\|\lambda_2 q + (1-\lambda_2)w\|}.$$

By applying Lemma 1, we have

$$\mathrm{arc}(u,w) \cup \mathrm{arc}(v,w) = S_X \cap (x + \gamma B_X).$$

Since $\delta(\mathrm{arc}(u,w) \cup \mathrm{arc}(v,w)) \leqslant 2\gamma < 2$, we have

$$\mathrm{arc}(u,w) \cup \mathrm{arc}(v,w) = \mathrm{arc}(u,v). \quad \square$$

LEMMA 4. *Let X be a Minkowski plane, $u,v \in S_X$ be two distinct points such that $[u,v] \cap \mathrm{int} B_X \neq \emptyset$, H^+ be an open halfplane bounded by $\langle u,v\rangle$ not containing o. If s and t are two distinct points in $S_X \cap H^+$, then*

$$\|s-t\| \leqslant \|u-v\|; \tag{2}$$

equality holds if and only if $\|s-t\| = 2$.

Proof. The case when $u = -v$ is clear. In the following we assume that $u \neq -v$.

The inequality (2) follows directly from the Monotonicity Lemma (Lemma 1). We only need to consider the case when equality holds. Clearly, if $\|s-t\| = 2$, then

$$2 = \|s-t\| \leqslant \|u-v\| \leqslant 2,$$

which implies that $\|s-t\| = \|u-v\|$.

Conversely, suppose that $\|s-t\| = \|u-v\|$.

Case I: $[s,t]$ is parallel to $[u,v]$. Clearly, the parallelogram P having s, t, $-s$, $-t$ as vertices is contained in B_X. Then

$$\langle u,v\rangle \cap P \subseteq \langle u,v\rangle \cap B_X = [u,v].$$

Since $\langle u,v\rangle \cap P$ is a segment parallel to $[s,t]$ whose length is $\|s-t\|$, we have

$$\langle u,v\rangle \cap P = [u,v].$$

Then the line $\langle -s,t\rangle$ contains three distinct points from S_X, which shows that $[-s,t] \subseteq S_X$. Similarly, $[s,-t] \subseteq S_X$. Hence $\|u-v\| = \|s-t\| = 2$.

Case II: $[s,t]$ is not parallel to $[u,v]$. We may assume that the distance from s to the line $\langle u,v\rangle$ is strictly smaller than the distance from t to $\langle u,v\rangle$. Let t' be the point of intersection of S_X and the line passing through s and parallel to $\langle u,v\rangle$. Then Lemma 1 shows that

$$\|s-t\| \leqslant \left\|s-t'\right\| \leqslant \|u-v\|.$$

It follows that

$$\|s-t\| = \left\|s-t'\right\| = \|u-v\|.$$

By Case I, $\|s-t'\| = 2$. Hence $\|s-t\| = 2$. $\square$

The following example shows that Lemma 2.4 in [5] is not true.

EXAMPLE 1. Let $X = (\mathbb{R}^2, \|\cdot\|)$, where the norm is defined by $\|(\alpha,\beta)\| = \max\{|\alpha|,|\beta|\}$. Put $a = (1,1)$, $b = (-1,1)$, $c = (1,-1)$, $b' = (0,1)$, $c' = (1,0)$. Then both the triangles abc and $ab'c'$ are equilateral, but their side lengths are different, see Figure 3.

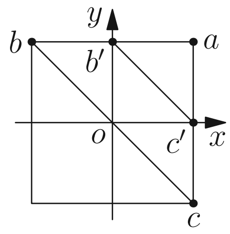

Figure 3: *Two equilateral triangles inscribed in S_X*

PROPOSITION 5. *Let X be a Minkowski plane. If there exist a point $a \in S_X$ and two equilateral triangles T and T' inscribed in S_X having a as one vertex whose side lengths are different, then S_X is a parallelogram.*

Proof. Let $\{a,b,c\}$ and $\{a,b',c'\}$ be the set of vertices of T and T', respectively.

First we show that at least one of T and T' dose not contain o. Otherwise both T and T' contain o. Without loss of generality, we may assume that the side length of T is strictly less than the side length of T', which is at most 2. Then it is clear that T contains o in its interior. Also, $-a \notin [b,c]$, since otherwise we would have

$$\|a-b\| = \|a-c\| = \|-a-a\| = 2,$$

which implies that the triangles T and T' have the same side length.

In this situation we have

$$S_X = \operatorname{arc}(a,b) \cup \operatorname{arc}(a,c) \cup \operatorname{arc}(b,c).$$

Lemma 1 and the assumption that $\|a-c'\| = \|a-b'\| > \|a-b\|$ show that

$$b', c' \in \operatorname{arc}(b,c) \setminus \{b,c\}.$$

Then Lemma 4 shows that $\|b'-c'\| < \|b-c\|$, a contradiction.

Thus at least one of T and T' does not contain o. Assume without loss of generality that $o \notin T$ and that a and o lie on different sides of $\langle b,c \rangle$. Therefore, $\{o,a,b,c\}$ is the set of vertices of a convex quadrilateral having $[o,a]$ and $[b,c]$ as diagonals. We have

$$2(\|o-a\| + \|b-c\|) = \|o-b\| + \|a-c\| + \|o-c\| + \|a-b\|.$$

By Corollary 8 in [14], S_X is a parallelogram. □

The following theorem amends a false statement in Lemma 2.4 in [5] as we have claimed.

THEOREM 6. *Let X be a Minkowski plane. For any integer $m \geqslant 3$ and each $x \in S_X$, there exists a convex m-gon inscribed in B_X and having x as a vertex which is equilateral with respect to $\|\cdot\|$ (called an* equilateral Minkowski m-gon *having x as a vertex). If X is strictly convex, this m-gon is unique. Moreover, if S_X is not a parallelogram or if $m \geqslant 4$, the side length of the m-gon is uniquely determined by x.*

Proof. Note that Lemma 2.4 in [5] correctly proved the existence of such convex m-gons and the uniqueness of the equilateral Minkowski m-gon having $x \in S_X$ as a vertex when X is strictly convex. Proposition 5 shows that when S_X is not a parallelogram, for each $x \in S_X$, the length of equilateral triangles inscribed in S_X having x as a vertex is uniquely determined. In the rest we show that, when $m \geqslant 4$, for each Minkowski plane X and each $x \in S_X$ the length of an equilateral Minkowski m-gon having x as a vertex is uniquely determined.

For two distinct points $x, y \in S_X$, we denote by $\overrightarrow{\mathrm{arc}}(x,y)$ the *directed arc that connects x counter-clockwisely with y*. Hence

$$\overrightarrow{\mathrm{arc}}(x,y) \neq \overrightarrow{\mathrm{arc}}(y,x) \quad \text{and} \quad S_X = \overrightarrow{\mathrm{arc}}(x,y) \cup \overrightarrow{\mathrm{arc}}(y,x).$$

First we show that if $m \geqslant 4$ and if $\{a_1, \cdots, a_m\}$ is the set of vertices of an equilateral Minkowski m-gon P inscribed in S_X, which are ordered counter-clockwisely on S_X, then

$$\overrightarrow{\mathrm{arc}}(a_i, a_{i+1}) = \mathrm{arc}(a_i, a_{i+1}), \ \forall i \text{ with } 1 \leqslant i \leqslant m,$$

where a_{m+1} is set to be a_1. Take $\overrightarrow{\mathrm{arc}}(a_1, a_2)$ for example. If $\overrightarrow{\mathrm{arc}}(a_1, a_2) \neq \mathrm{arc}(a_1, a_2)$, then either $a_2 = -a_1$ or P is contained in the closed halfplane bounded by $\langle a_1, a_2 \rangle$ not containing o. If $a_2 = -a_1$, then the side length of P is 2 and $\overrightarrow{\mathrm{arc}}(a_2, a_1)$ is a semicircle whose length is not smaller than

$$\|a_m - a_1\| + \|a_m - a_{m-1}\| + \|a_{m-1} - a_{m-2}\| = 6.$$

This shows that the circumference of S_X is at least 12. Since the circumference of S_X ranges from 6 to 8 (see, e.g., [14, p. 130]), this is impossible. Now suppose that P is contained in the closed halfplane bounded by $\langle a_1, a_2 \rangle$ not containing o. In this case, $[a_3, a_4]$ is contained in the open halfplane bounded by $\langle a_1, a_2 \rangle$ not containing o. Since $\|a_3 - a_4\| = \|a_1 - a_2\|$, Lemma 4 shows that the side length of P is 2. Again this would show that the circumference of S_X is at least 12.

Note that these arguments also show that o is in the interior of P.

Suppose that there exist two equilateral Minkowski m-gons P and P' having $a_1, \cdots, a_m$ and $b_1, \cdots, b_m$ as vertices, respectively, where

1. $a_1 = b_1 = x$,

2. both $a_1, \cdots, a_m$ and $b_1, \cdots, b_m$ are in counter-clockwise order,

3. $\|a_2 - a_1\| > \|b_2 - b_1\|$.

Since P and P' both contain o in their interiors, a_2 and b_2 are both in $\overrightarrow{\mathrm{arc}}(a_1,-a_1)$. Lemma 1 shows that b_2 is in the relative interior of $\overrightarrow{\mathrm{arc}}(a_1,a_2)$. Now suppose that we have proved that b_i is in the relative interior of $\overrightarrow{\mathrm{arc}}(a_1,a_i)$ for some integer $2 \leqslant i < m$. We distinguish two cases.

Case I: $-b_i \in \overrightarrow{\mathrm{arc}}(-a_1,a_1)$. If $a_{i+1} \in \overrightarrow{\mathrm{arc}}(-b_i,a_1)$, then

$$\overrightarrow{\mathrm{arc}}(a_1,a_{i+1}) = \overrightarrow{\mathrm{arc}}(a_1,b_i) \cup \overrightarrow{\mathrm{arc}}(b_i,-b_i) \cup \overrightarrow{\mathrm{arc}}(-b_i,a_{i+1}),$$

and $b_{i+1} \in \overrightarrow{\mathrm{arc}}(b_i,-b_i) \setminus \{-b_i\}$. Thus b_{i+1} is in the relative interior of $\overrightarrow{\mathrm{arc}}(a_1,a_{i+1})$.

If $a_{i+1} \in \overrightarrow{\mathrm{arc}}(b_i,-b_i)$, then a_i is in the relative interior of $\mathrm{arc}(b_i,a_{i+1})$. If $b_{i+1} \in \mathrm{arc}(a_{i+1},-b_i)$, then Lemma 1 shows that

$$\|b_{i+1}-b_i\| \geqslant \|a_{i+1}-a_i\|,$$

a contradiction. Thus b_{i+1} is in the relative interior of $\overrightarrow{\mathrm{arc}}(b_i,a_{i+1})$ which is contained in the relative interior of $\overrightarrow{\mathrm{arc}}(a_1,a_{i+1})$.

Case II: $-b_i \in \overrightarrow{\mathrm{arc}}(a_1,-a_1) \setminus \{-a_1\}$. In this case we have

$$\mathrm{arc}(a_i,a_{i+1}) \subseteq \mathrm{arc}(b_i,a_1) = \overrightarrow{\mathrm{arc}}(b_i,a_1).$$

Again, Lemma 1 shows that b_{i+1} is in the relative interior of $\mathrm{arc}(b_i,a_{i+1})$ which is contained in the relative interior of $\overrightarrow{\mathrm{arc}}(a_1,a_{i+1})$.

By induction, we know that b_m is in the relative interior of $\overrightarrow{\mathrm{arc}}(a_1,a_m)$. By Lemma 1 again, we have

$$\|b_m-a_1\| = \|b_m-b_1\| \geqslant \|a_m-a_1\|,$$

a contradiction. □

For each integer $m \geqslant 3$, each Minkowski plane X, and each $x \in S_X$, denote by $\alpha_m(x,X)$ the maximal side lengths of equilateral Minkowski m-gons inscribed in S_X having x as a vertex. Theorem 6 shows that if B_X is not a parallelogram or if $m \geqslant 4$, the side length of each equilateral Minkowski m-gon inscribed in S_X having x as a vertex is $\alpha_m(x,X)$. It is not difficult to verify that, for fixed x, $\alpha_m(x,X)$ is non-increasing with respect to m, see [5, p. 178]. Put

$$S(m,X) = \inf\{\alpha_m(x,X) : x \in S_X\}.$$

PROPOSITION 7. *Let X be a Minkowski plane. Then $S(3,X)=2$ if and only if B_X is a parallelogram.*

Proof. It is not difficult to verify that if B_X is a parallelogram then $S(3,X)=2$.

Conversely, suppose that $S(3,X)=2$. Then there exists an equilateral tirangle T having vertices $a,b,c \in S_X$ whose side length is 2.

First suppose that one pair of points from a,b,c, say a and b, are linearly dependent. Then $b=-a$. Then we have

$$\|c+a\| = \|c-a\| = 2.$$

It follows that $[a,c] \subseteq S_X$ and $[b,c] \subseteq S_X$, which implies that B_X is a parallelogram having $a,b,c,-c$ as vertices.

Now we assume that a,b,c are pairwise linearly independent. Since

$$2 = \|a-b\| = \|a\| + \|b\|,$$

we have $[a,-b] \subseteq S_X$. Suppose that $[s,t]$ is the longest segment contained in S_X and containing $[a,-b]$. Let u be the midpoint of $[s,t]$. By the hypothesis, there exist two points $v,w \in S_X$ such that

$$\|u-v\| = \|u-w\| = \|v-w\| = 2.$$

It follows that $[-u,v] \subseteq S_X$. However, $[-s,-t]$ is the unique segment containing $-u$, which implies that $v \in [-s,-t]$. In a similar way, we can show that $w \in [-s,-t]$. Since

$$2 = \|v-w\| \leqslant \|s-t\| \leqslant 2,$$

$[s,t]$ is a segment contained in S_X whose length is 2. Hence, B_X is a parallelogram having $s,t,-s,-t$ as vertices. □

3. $\gamma_m(X)$ of Minkowski planes

For the discussion in the sequel, we shall use the following equivalent representations of the so called *non-square constants* $J(X)$ and $S(X)$, which were provided in [10] (see also [7]):

$$J(X) := \sup\{\|x+y\| : x,y \in S_X, \ \|x+y\| = \|x-y\|\}$$

and

$$S(X) := \inf\{\|x+y\| : x,y \in S_X, \ \|x+y\| = \|x-y\|\}.$$

It is always true that (see, e.g., [6] and [7])

$$1 \leqslant S(X) \leqslant \sqrt{2} \leqslant J(X) \leqslant 2.$$

REMARK 8. One can easily verify that, for each Minkowski plane X, $S(4,X)$ is the Schäffer constant $S(X)$. Therefore

$$1 \leqslant S(4,X) \leqslant \sqrt{2}.$$

THEOREM 9. *Let X be a Minkowski plane and $m \geqslant 3$. Then*

$$\gamma_m(X) = \frac{1}{2} S(m,X).$$

Proof. The case when $m = 3$ and S_X is a parallelogram is clear. In the following we always assume that $m > 3$ or S_X is not a parallelogram. In this case the side length of equilateral Minkowski m-gons inscribed in S_X having $x \in S_X$ as a vertex is uniquely determined by m and x, and each equilateral Minkowski m-gon inscribed in S_X contains o in its interior. By Lemma 2, we only need to show that $\gamma_m(X) \geqslant (1/2)S(m,X)$. Suppose the contrary, namely that

$$\gamma := \gamma_m(X) < \frac{1}{2}S(m,X) \leqslant 1.$$

Then there exists a set $\{p_i :\ i \in [m]\}$ such that

$$S_X \subseteq \bigcup_{i\in[m]} (p_i + \gamma B_X).$$

By Lemma 3, there exist two points $u_1, v_1 \in S_X$ such that

$$(p_1 + \gamma B_X) \cap S_X = \overrightarrow{\mathrm{arc}}(u_1, v_1) = \mathrm{arc}(u_1, v_1).$$

There exists an equilateral Minkowski m-gon $P = x_1 x_2 \cdots x_m$ inscribed in S_X such that $x_1 = u_1$, and that, for each $i \in [m-1]$, the orientation from x_i to x_{i+1} is counter-clockwise. We may also require that

$$\overrightarrow{\mathrm{arc}}(x_i, x_{i+1}) = \mathrm{arc}(x_i, x_{i+1}), \ \forall i \in [m].$$

Moreover, the side length of this equilateral Minkowski m-gon is strictly greater than 2γ. Then, for each $i \in [m]$, $p_i + \gamma B_X$ can only cover at most one (and, consequencely, precisely one) vertex of P. Thus we may assume without loss of generality that, for each $i \in [m]$, there exist two points $u_i, v_i \in S_X$ such that $x_i \in (p_i + \gamma B_X) \cap S_X = \overrightarrow{\mathrm{arc}}(u_i, v_i) = \mathrm{arc}(u_i, v_i)$.

Next we show that, for each $i \in [m]$, $v_i \in \mathrm{arc}(x_i, x_{i+1}) \setminus \{x_{i+1}\}$ (we put $x_{m+1} = x_1$ and $x_0 = x_m$). Since

$$\|u_1 - v_1\| \leqslant 2\gamma < \|x_1 - x_2\| = \|u_1 - x_2\|,$$

Lemma 1 shows that $v_1 \in \mathrm{arc}(x_1, x_2) \setminus \{x_2\}$. Now suppose that $v_k \in \mathrm{arc}(x_k, x_{k+1}) \setminus \{x_{k+1}\}$ for some $k \in [m-1]$. It is clear that there exists a $j \in [m] \setminus \{k\}$ such that

$$v_k \in (p_j + \gamma B_X) \cap S_X = \mathrm{arc}(u_j, v_j).$$

If x_j lies in the open semicircle connecting v_k with $-v_k$ and containing x_k, then, since $\|u_k - v_k\| < \|x_k - x_j\|$, we have $x_j \in \mathrm{arc}(u_k, -v_k)$. It follows that

$$\|p_j - v_k\| \geqslant \|v_k - x_j\| - \|x_j - p_j\| \geqslant \|x_k - x_j\| - \gamma > \gamma,$$

which is impossible. Therefore, x_j has to be in the semicircle connecting v_k with $-v_k$ and containing x_{k+1}. We claim that $j = k+1$, since otherwise, by using the fact that $x_j \notin \mathrm{arc}(x_k, x_{k+1})$, we would have

$$\|p_j - v_k\| \geqslant \|v_k - x_j\| - \|x_j - p_j\| \geqslant \|x_{k+1} - x_j\| - \gamma > \gamma,$$

a contradiction. Now it is clear that $\mathrm{arc}(v_k, x_{k+1}) \subseteq \mathrm{arc}(u_{k+1}, v_{k+1})$. If x_{k+2} lies in the semicircle connecting u_{k+1} and $-u_{k+1}$ and containing v_{k+1}, then

$$\|u_{k+1} - v_{k+1}\| \leqslant 2\gamma < \|x_{k+1} - x_{k+2}\| \leqslant \|u_{k+1} - x_{k+2}\|.$$

Otherwise, v_{k+1} lies in $\mathrm{arc}(x_{k+1}, -u_{k+1})$. In both cases we have

$$v_{k+1} \in \mathrm{arc}(x_{k+1}, x_{k+2}) \setminus \{x_{k+2}\}.$$

It follows by induction that

$$v_m \in \mathrm{arc}(x_m, x_{m+1}) \setminus \{x_{m+1}\} = \mathrm{arc}(x_m, x_1) \setminus \{x_1\}.$$

Thus the relative interior of $\mathrm{arc}(v_m, x_1)$ is not contained in $\cup_{i\in[m]}(p_i + \gamma B_X)$, a contradiction. □

REMARK 10. Proposition 14 in [12] proved a result similar to Theorem 9 for planar convex bodies that are strictly convex and smooth.

LEMMA 11. *Let X be a Minkowski plane, $u \in S_X$, and $[a,b]$ and $[s,t]$ be two chords of S_X parallel to $\langle -u, u\rangle$ such that $\langle s,t\rangle$ lies strictly between $\langle a,b\rangle$ and $\langle -u,u\rangle$. Then*

$$\|s+t\| < \|a+b\|. \tag{3}$$

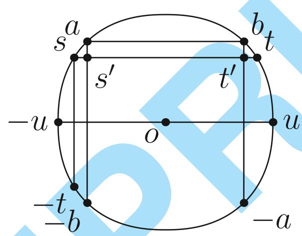

Figure 4: *The proof of Lemma* 11

Proof. It is not difficult to see that we only need to consider the case when $\|a+b\| < 2$. Without loss of generality we may assume that $t-s$ and $b-a$ are both positive scalar multiples of u. Since $\langle s,t\rangle$ lies strictly between $\langle a,b\rangle$ and $\langle -u,u\rangle$, the lines $\langle a,-b\rangle$ and $\langle -a,b\rangle$ intersect $\langle s,t\rangle$ in a point s' and t', respectively; see Figure 4. Clearly, $[s',t'] \subseteq [s,t]$. If one of the two points s' and t' is in S_X (note that these two points lie in the relative interior of $[a,-b]$ and $[-a,b]$), then both $[a,-b]$ and $[-a,b]$ are contained in S_X. It follows that $[s,-t]$ and $[-s,t]$ are contained in the relative interiors of $[a,-b]$ and $[-a,b]$, respectively. Hence, (3) holds. In the following we assume that s' and t' are both interior points of B_X. Then s and $-t$ is contained in the open halfplane bounded by $\langle a,-b\rangle$ not containing o. By Lemma 4 and the fact that $\|a+b\| < 2$, we have (3). □

Let $x, y \in X$. If $\|x+y\| = \|x-y\|$, then we say that x *is isosceles orthogonal to* y, denoted $x \perp_I y$, see [9].

COROLLARY 12. *Let X be a Minkowski plane, $u \in S_X$. Then there exists a unique pair of chords $[a,b]$ and $[-a,-b]$ parallel to $\langle -u,u\rangle$ such that $\|a+b\| = \|a-b\|$.*

Proof. By the uniqueness of isosceles orthogonality (see [1, Theorem 4.35]), there exists a unique $v \in S_X$ (except for the sign) such that $u \perp_I v$. Put

$$a = \frac{-u+v}{\|-u+v\|} \quad \text{and} \quad b = \frac{u+v}{\|u+v\|}.$$

Then

$$a,b \in S_X, \ b-a = \frac{2}{\|u+v\|}u, \ a+b = \frac{2}{\|u+v\|}v, \ \|a-b\| = \frac{2}{\|u+v\|} = \|a+b\|.$$

Thus, $[a,b]$ and $[-a,-b]$ is a pair of chords having the desired properties.

In the following we show the uniqueness. Otherwise, there exists another pair of chords $[s,t]$ and $[-s,-t]$ having the required properties. Without loss of generality we may assume that $\langle s,t\rangle$ lies strictly between $\langle a,b\rangle$ and $\langle -u,u\rangle$, and that $t-s$ is a positive scalar multiple of u. Suppose that $s \in [a,-b]$ or $t \in [b,-a]$. Then $-t \in [a,-b] \subset S_X$, $-s \in [b,-a] \subset S_X$. It follows that

$$\|s+t\| < \|a+b\| = \|a-b\| = \|s-t\|,$$

a contradiction. Similarly, $s \notin [a,-b]$, $-s \notin [b,-a]$. Lemma 11 and Lemma 4 show that

$$\|s+t\| < \|a+b\| = \|a-b\| \leqslant \|s-t\|,$$

a contradiction. $\square$

LEMMA 13. *Let X be a Minkowski plane, and T be an equilateral triangle having a,b,c as vertices which is inscribed in S_X so that $o \in T$. Then*

$$\|a+b\| \leqslant \|a-b\|.$$

Proof. The case $\|a-b\| = 2$ is clear. In the rest of the proof we assume that

$$\|a-b\| = \|a-c\| = \|b-c\| < 2.$$

In this case, o is an interior point of T, which implies that $-c$ is a relatively interior point of $\mathrm{arc}(a,b)$. Put

$$u = \frac{a-b}{\|a-b\|}.$$

Suppose the contrary, that $\|a+b\| > \|a-b\|$. Then there exists a unique pair of points $s,t \in S_X$ such that

1. $\langle s,t\rangle$ lies strictly between $\langle a,b\rangle$ and $\langle -u,u\rangle$,

2. $t-s$ is a positive scalar multiple of $-u$, and

3. $\|s+t\| = \|s-t\|$.

Since o is an interior point of T, c lies in the open halfplane bounded by $\langle -u,u\rangle$ not containing $[a,b]$. If $c \in \operatorname{arc}(-u,-s)$, then T is completely contained in the open halfplane bounded by $\langle -s,s\rangle$ containing $[a,b]$, which is in contradiction to $o \in \operatorname{int} T$. Thus $c \notin \operatorname{arc}(-u,-s)$. Similarly, $c \notin \operatorname{arc}(u,-t)$. It follows that c is a relatively interior point of $\operatorname{arc}(-s,-t)$. By Lemma 1 and Lemma 4, we have

$$\|c-b\| \geqslant \|-s-t\| = \|s+t\| = \|s-t\| > \|a-b\|,$$
$$\|c-a\| \geqslant \|-t-s\| = \|s+t\| = \|s-t\| > \|a-b\|.$$

These are in contradiction to the fact that T is equilateral. □

THEOREM 14. *Let X be a Minkowski plane. Then*

$$\gamma_3(X) = \Gamma_3(X) = \frac{1}{2}S(3,X).$$

Proof. When S_X is a parallelogram, it is clear that

$$\gamma_3(X) = \Gamma_3(X) = \frac{1}{2}S(3,X) = 1.$$

In the following we assume that S_X is not a parallelogram. In this case we have $S(3,X) < 2$, and o is the interior of each equilateral triangle inscribed in S_X.

By Theorem 9, $\gamma_3(X) = (1/2)S(3,X)$. Let $\{a_1,a_2,a_3\}$ be the vertex set of an equilateral triangle inscribed in S_X whose side length is $\gamma = S(3,X)$. Let c_1,c_2,c_3 be the midpoint of $[a_1,a_2]$, $[a_2,a_3]$, and $[a_3,a_1]$, respectively. Then

$$S_X \subseteq \bigcup_{i\in[3]} \left(c_i + \frac{1}{2}S(3,m)\right),$$

and, by Lemma 13,

$$o \in \bigcap_{i\in[3]} \left(c_i + \frac{1}{2}S(3,m)\right).$$

It follows that

$$B_X \subseteq \bigcup_{i\in[3]} \left(c_i + \frac{1}{2}S(3,m)\right).$$

Thus $\Gamma_3(X) \leqslant (1/2)S(3,X)$, which implies that $\Gamma_3(X) = (1/2)S(3,X)$. □

REMARK 15. The situation is much more complicated when $m \geqslant 5$. On the one hand, if S_X is a parallelogram, then we have

$$\frac{1}{2} = \gamma_4(X) \geqslant \gamma_5(X) \geqslant \gamma_6(X) = \frac{1}{2}$$

and

$$\gamma_6(X) \leqslant \Gamma_6(X) \leqslant \Gamma_5(X) \leqslant \Gamma_4(X) = \frac{1}{2}.$$

It follows that

$$\gamma_5(X) = \gamma_6(X) = \Gamma_5(X) = \Gamma_6(X) = \frac{1}{2}.$$

On the other hand, when X is the Euclidean plane we clearly have

$$\Gamma_5(X) > \gamma_5(X).$$

4. A better estimation for $\Gamma_m(X)$

Let X be a Minkowski space, $u \in S_X$ and $\varepsilon \in [0,2]$. The *directional modulus of convexity* $\delta_X(u,\varepsilon)$ is defined by

$$\delta_X(u,\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \ x,y \in S_X,\ \exists\lambda \in \mathbb{R} \text{ s.t. } x-y = \lambda u \notin \operatorname{int} \varepsilon B_X\right\}.$$

For each $u \in S_X$ and each $\lambda > 0$, we put

$$I(u,\lambda) = \{z \in S_X : z + \lambda u \in B_X\},$$

$$\mathscr{U}_m(X) = \left\{\{u_i : \ i \in [m]\} \subset S_X : \ \exists\lambda > 0 \text{ such that } S_X \subseteq \bigcup_{i\in[m]} I(u_i,\lambda)\right\},$$

and, for each $U = \{u_i : \ i \in [m]\} \in \mathscr{U}_m(X)$,

$$\lambda(U) = \sup\left\{\lambda > 0 : \ S_X \subseteq \bigcup_{i\in[m]} I(u_i,\lambda)\right\},$$

and

$$\delta(U) = 1 - \min\{\delta_X(u_i,\lambda(U)) : \ i \in [m]\}.$$

Note that, for a particular choices of $m \in \mathbb{Z}^+$, $\mathscr{U}_m(X)$ might be empty. By standard compactness arguments, one can show that "sup" in the definition of $\lambda(U)$ can be replaced by "max".

Now we improve the estimation of $\Gamma_m(X)$ provided in [8].

THEOREM 16. *Suppose that $m \in \mathbb{Z}^+$ and $\mathscr{U}_m(X) \neq \emptyset$. Then*

$$\Gamma_m(X) \leqslant \inf\left\{\max\left\{\delta(U), \frac{1}{2}\lambda(U)\right\} : \ U \in \mathscr{U}_m(X)\right\}.$$

Proof. Let $U = \{u_i :\ i \in [m]\}$ be an arbitrary element in $\mathscr{U}_m(X)$. We show that

$$\Gamma_m(X) \leqslant \max\left\{\delta(U), \frac{1}{2}\lambda(U)\right\}.$$

The case when $\delta(U) = 1$ or $\lambda(U) = 2$ is clear. In the following we assume that $\delta(U) < 1$ and $\lambda(U) < 2$. In this case we have

$$\delta_X(u_i, \lambda(U)) \geqslant 1 - \delta(U) > 0,\ \forall i \in [m]. \tag{4}$$

Therefore, if $[a,b]$ is a chord of S_X whose length is not smaller than $\lambda(U)$ and which is parallel to $\langle -u_i, u_i\rangle$ for some $i \in [m]$, then $[a,b] \setminus \{a,b\} \subset \operatorname{int} B_X$.

Let i be an arbitrary integer in $[m]$. Put

$$x_i = -\frac{\lambda(U)}{2} u_i.$$

We show that $I(u_i, \lambda(U)) \subseteq B_X(x_i, \delta(U))$, where $B_X(x_i, \delta(U))$ is the ball centered at x_i whose radius is $\delta(U)$.

For each $x \in I(u_i, \lambda(U))$, there exists a point $y \in S_X$ such that

$$[x,y] = B_X \cap (x + \langle -u_i, u_i\rangle).$$

Clearly, $\|x-y\| \geqslant \lambda(U)$. Whether x and y are linearly independent or not, there exist two points $u, v \in S_X$ such that

$$v - u = \lambda(U) u_i,\ x,y \in \operatorname{span}\{u,v\},\ [u,v] = \langle u,v\rangle \cap B_X,$$

and that $\langle x,y\rangle$ lies between $\langle -u_i, u_i\rangle$ and $\langle u,v\rangle$ (cf. the proof of Theorem 12 in [8]). Then

$$\|u - x_i\| = \left\| u + \frac{\lambda(U)}{2} u_i \right\| = \left\| \frac{1}{2}(u+v) \right\| \leqslant 1 - \delta_X(u_i, \lambda(U)) \leqslant \delta(U).$$

Lemma 1 shows that

$$\|x - x_i\| \leqslant \|u - x_i\| \leqslant \delta(U).$$

Therefore $I(u_i, \lambda(U)) \subseteq B_X(x_i, \delta(U))$.

Moreover,

$$o \in \bigcap_{i \in [m]} B_X\left(x_i, \frac{1}{2}\lambda(U)\right).$$

It follows that

$$B_X \subseteq \bigcup_{i \in [m]} B_X\left(x_i, \max\left\{\delta(U), \frac{1}{2}\lambda(U)\right\}\right),$$

which completes the proof. □

REMARK 17. The estimation of $\Gamma_m(X)$ above is better than the estimation given by Theorem 12 in [8]. Take, for example, $l_\infty^2 = \left(\mathbb{R}^2, \|.\|_\infty\right)$. Theorem 12 in [8] gives $\Gamma_4(l_\infty^2) \leqslant 1$. Let $U = \{(1,1), (-1,1), (-1,-1), (1,-1)\}$. It is not difficult to see that $\lambda(U) = 1$ and $\delta(U) = 1/2$. Theorem 16 yields $\Gamma_4(l_\infty^2) \leqslant 1/2 = \Gamma_4(l_\infty^2)$, an estimation which is best possible.

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