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UPPER AND LOWER BOUNDS, AND OPERATOR MONOTONICITY OF AN EXTENSION OF THE PETZ–HASEGAWA FUNCTION

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Abstract. The Petz-Hasegawa function

$$f_p(x) = p(1-p)\frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)}$$

for $p \in [-1,2]$ is a well-known operator monotone function on $x > 0$. In this paper, we discuss some properties of the following extension of the Petz-Hasegawa function

$$f_{\mathbf{p}}(x) = x^{\gamma}\prod_{i=1}^{n} p_i \frac{x-1}{x^{p_i}-1},$$

where $\mathbf{p} = (p_1, \dots, p_n)$ by only using an elementary technique. Firstly, we get its upper and lower bounds. Secondly, we obtain a result on operator monotonicity.

1. Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $\mathscr{H}$. An operator A is positive semi-definite if and only if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$, and we write it $A \geqslant 0$. If an operator A is positive semi-definite and invertible, A is called positive definite. In this case, we write it $A > 0$. For self-adjoint operators A and B, $B \leqslant A$ is defined by $0 \leqslant A - B$. A real valued function f defined on an interval $I \subset \mathbb{R}$ is called an operator monotone function if

$$B \leqslant A \quad \text{implies} \quad f(B) \leqslant f(A)$$

for all self-adjoint operators A and B whose spectra are contained in I. Typical examples of operator monotone functions are $f(x) = x^{\lambda}$ and $f(x) = (1-\lambda+\lambda x^q)^{1/q}$ on $x > 0$ for $\lambda \in [0,1]$ and $q \in [-1,1] \setminus \{0\}$. Petz and Hasegawa have proven that the function $f_p(x)$ on $x > 0$ defined by

$$f_p(x) = p(1-p)\frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)} \quad (p \neq 0, 1),$$

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$f_0(x) = \lim_{p\to 0} f_p(x) = \frac{x-1}{\log x}$, $f_1(x) = \lim_{p\to 1} f_p(x) = \frac{x-1}{\log x}$ is operator monotone for $-1 \leqslant p \leqslant 2$ in [8] (see also [1, 4]). We call $f_p(x)$ the Petz-Hasegawa function (we write it PH function, simply). In this paper, we shall consider an extension of the PH function as follows:

$$f_{\mathbf{p}}(x) = x^{\gamma} \prod_{i=1}^{n} p_i \frac{x-1}{x^{p_i}-1},$$

where $\mathbf{p} = (p_1, \ldots, p_n)$, and give the following two properties of $f_{\mathbf{p}}(x)$: (i) Upper and lower bounds of $f_{\mathbf{p}}(x)$; (ii) operator monotonicity of $f_{\mathbf{p}}(x)$.

For the first problem, we get two estimations of $f_{\mathbf{p}}(x)$. J. I. Fujii and M. Fujii have been considered a function $\dfrac{p}{p+1}\dfrac{x^{p+1}-1}{x^p-1}$ in [3]. One of our results leads that this function is an upper bound of the PH function.

For the second problem, Nagisa and Wada have given an equivalent condition of $\alpha_i, \beta_i, \gamma$ and s $(i = 1, 2, \ldots, n)$ to that

$$\left(x^{\gamma} \prod_{i=1}^{n} \frac{x^{\alpha_i}-1}{x^{\beta_i}-1} \right)^{s} \qquad (\alpha_i, \beta_i \in [0,2], \gamma \geqslant 0)$$

is operator monotone in [7]. In this paper, we shall only consider the case $\alpha_i = s = 1$ $(i = 1, \ldots, n)$ of the above function, but we consider the cases $\gamma \in \mathbb{R}$ and $\beta_i \in [-2, 2]$. These cases have not been considered in [7].

This paper is organized as follows: In Section 2, we shall give upper and lower bounds of $f_{\mathbf{p}}(x)$. In Section 3, we shall show the operator monotonicity of $f_{\mathbf{p}}(x)$. These results are proved by using only an elementary technique.

Takayuki Furuta passed away on 26 June, 2016. He had obtained a small result (a part of Corollary 4), however it had not been submitted. The rest of the authors found his unpublished manuscript when we visited his home in order to arrange his notebooks. Then we added some results into Furuta's manuscript to make this paper. Takayuki Furuta made outstanding contributions in Operator Inequalities. We dedicate this short note to his memory. We will miss him.

2. Upper and lower bounds of $f_{\mathbf{p}}(x)$

In this section, we shall give upper and lower bounds of $f_{\mathbf{p}}(x)$. In what follows, we consider $p\frac{x-1}{x^p-1}$ for $p = 0$ as $\frac{x-1}{\log x}$, the limit as $p \to 0$.

THEOREM 1. *Let n be a natural number such that $n \geqslant 2$, and let $p_i \in [0,1]$ for $i = 0, 1, 2, \ldots, n$ such that $\sum_{i=0}^{n} p_i = n$. Then*

$$(1-p_0)x^{\gamma}\frac{x-1}{x^{1-p_0}-1} \leqslant f_{\mathbf{p}}(x) = x^{\gamma} \prod_{i=1}^{n} p_i \frac{x-1}{x^{p_i}-1}$$

$$\leqslant x^{\gamma}\left(\mu \frac{x-1}{x^{\mu}-1}\right)^{n} \leqslant x^{\gamma}\left(\frac{x^{\mu}+1}{2}\right)^{\frac{p_0}{\mu}} \leqslant x^{\gamma}\left(\frac{x+1}{2}\right)^{p_0}$$

holds for $\gamma \in \mathbb{R}$ *and* $x > 0$*, where* $p = (p_1, \ldots, p_n)$ *and* $\mu = \frac{1}{n}\sum_{i=1}^{n} p_i$.

To give a proof of Theorem 1, we shall use the following theorem.

THEOREM A. ([9]) *Let* $p, q \in [-1,1] \setminus \{0\}$. *Then*

$$F_{p,q}(x) = \left[\int_0^1 (1-\lambda+\lambda x^p)^{\frac{q}{p}} d\lambda\right]^{\frac{1}{q}} = \left(\frac{p}{p+q}\frac{x^{p+q}-1}{x^p-1}\right)^{\frac{1}{q}}$$

is a positive operator monotone function on $x > 0$*, and increasing on* $p, q \in [-1,1] \setminus \{0\}$.

In [9], Theorem A is shown by using a technique of complex analysis. But it can be shown by the following facts, easily: (i) $(1-\lambda+\lambda x^p)^{1/p}$ is operator monotone on $x > 0$ for $\lambda \in [0,1]$ and $p \in [-1,1] \setminus \{0\}$, and increasing on $p \in [-1,1] \setminus \{0\}$, and (ii) for operator monotone functions $f_i(x)$ $(i = 1, 2, \ldots, n)$, $(\sum_{i=1}^{n} w_i f_i(x)^q)^{1/q}$ is operator monotone for $q \in [-1,1] \setminus \{0\}$ and $w_i > 0$ such that $\sum_{i=1}^{n} w_i = 1$, and increasing on $q \in [-1,1] \setminus \{0\}$.

Proof of Theorem 1. First of all, if we take $p_i = 0$ for an arbitrary i, then $p_j = 1$ for all $j \neq i$ since the condition $\sum_{i=0}^{n} p_i = n$. If $p_i = 1$ for an arbitrary i, then $p_i \frac{x-1}{x^{p_i}-1} = 1$. Hence we only consider $p_i \in (0,1)$. It is enough to show

$$\begin{aligned}(1-p_0)\frac{x-1}{x^{1-p_0}-1} &\leqslant \prod_{i=1}^{n} p_i \frac{x-1}{x^{p_i}-1} \\ &\leqslant \left(\mu\frac{x-1}{x^{\mu}-1}\right)^n \leqslant \left(\frac{x^{\mu}+1}{2}\right)^{\frac{p_0}{\mu}} \leqslant \left(\frac{x+1}{2}\right)^{p_0} \qquad (1)\end{aligned}$$

for $x > 0$.

Firstly, we shall show the first inequality in (1).

$$\begin{aligned}(1-p_0)\frac{x-1}{x^{1-p_0}-1} &= \prod_{i=0}^{n-1} \frac{\sum_{k=0}^{i}(1-p_k)}{\sum_{j=0}^{i+1}(1-p_j)} \frac{x^{\sum_{j=0}^{i+1}(1-p_j)}-1}{x^{\sum_{k=0}^{i}(1-p_k)}-1} \\ &= \prod_{i=0}^{n-1} \left(\frac{\sum_{k=0}^{i}(1-p_k)}{\sum_{j=0}^{i+1}(1-p_j)} \frac{x^{\sum_{j=0}^{i+1}(1-p_j)}-1}{x^{\sum_{k=0}^{i}(1-p_k)}-1}\right)^{\frac{1-p_{i+1}}{1-p_{i+1}}} \\ &= \prod_{i=0}^{n-1} F_{\sum_{k=0}^{i}(1-p_k),1-p_{i+1}}(x)^{1-p_{i+1}} \\ &\leqslant \prod_{i=0}^{n-1} F_{p_{i+1},1-p_{i+1}}(x)^{1-p_{i+1}} = \prod_{i=1}^{n} p_i \frac{x-1}{x^{p_i}-1},\end{aligned}$$

where the inequality follows from Theorem A and the following fact: Since $\sum_{k=0}^{n}(1-p_k) = 1$ and $p_i \in (0,1)$ for $i = 0, 1, 2, \ldots, n$,

$$\sum_{k=0}^{i}(1-p_k) = 1-(1-p_{i+1})-\cdots-(1-p_n) \leqslant p_{i+1}.$$

Next, we shall prove the second inequality in (1). To prove this inequality, we show that for each $x > 0$,

$$g(t) = \log\left(t\frac{x-1}{x^t-1}\right) \quad \text{is a concave function on } (0,1). \tag{2}$$

For $0 < t_1 < t_2 < 1$,

$$\begin{aligned}
t_1\frac{x-1}{x^{t_1}-1}\cdot t_2\frac{x-1}{x^{t_2}-1} &= \frac{t_1}{\frac{t_1+t_2}{2}}\frac{x^{\frac{t_1+t_2}{2}}-1}{x^{t_1}-1}\cdot\frac{t_1+t_2}{2}\frac{x-1}{x^{\frac{t_1+t_2}{2}}-1}\cdot t_2\frac{x-1}{x^{t_2}-1}\\
&= F_{t_1,\frac{t_2-t_1}{2}}(x)^{\frac{t_2-t_1}{2}}F_{\frac{t_1+t_2}{2},1-\frac{t_1+t_2}{2}}(x)^{1-\frac{t_1+t_2}{2}}F_{t_2,1-t_2}(x)^{1-t_2}\\
&\leqslant F_{\frac{t_1+t_2}{2},\frac{t_2-t_1}{2}}(x)^{\frac{t_2-t_1}{2}}F_{\frac{t_1+t_2}{2},1-\frac{t_1+t_2}{2}}(x)^{1-\frac{t_1+t_2}{2}}F_{t_2,1-t_2}(x)^{1-t_2}\\
&= \frac{\frac{t_1+t_2}{2}}{t_2}\frac{x^{t_2}-1}{x^{\frac{t_1+t_2}{2}}-1}\cdot\frac{t_1+t_2}{2}\frac{x-1}{x^{\frac{t_1+t_2}{2}}-1}\cdot t_2\frac{x-1}{x^{t_2}-1}\\
&= \left(\frac{t_1+t_2}{2}\frac{x-1}{x^{\frac{t_1+t_2}{2}}-1}\right)^2
\end{aligned}$$

holds by Theorem A. Then

$$\begin{aligned}
\frac{1}{2}\{g(t_1)+g(t_2)\} &= \frac{1}{2}\log\left(t_1\frac{x-1}{x^{t_1}-1}\cdot t_2\frac{x-1}{x^{t_2}-1}\right)\\
&\leqslant \log\left(\frac{t_1+t_2}{2}\frac{x-1}{x^{\frac{t_1+t_2}{2}}-1}\right) = g\left(\frac{t_1+t_2}{2}\right),
\end{aligned}$$

that is, $g(t)$ is a concave function on $(0,1)$ since $g(t)$ is continuous. Therefore we get

$$\begin{aligned}
\frac{1}{n}\log\left(\prod_{i=1}^{n}p_i\frac{x-1}{x^{p_i}-1}\right) &= \frac{1}{n}\{g(p_1)+\cdots+g(p_n)\}\\
&\leqslant g\left(\frac{p_1+\cdots+p_n}{n}\right) = g(\mu) = \log\left(\mu\frac{x-1}{x^{\mu}-1}\right),
\end{aligned}$$

that is,

$$\prod_{i=1}^{n}p_i\frac{x-1}{x^{p_i}-1} \leqslant \left(\mu\frac{x-1}{x^{\mu}-1}\right)^n.$$

Next, we shall show the third inequality in (1). Since

$$\mu = \frac{1}{n}\sum_{i=1}^{n}p_i = 1-\frac{p_0}{n} \geqslant 1-\frac{p_0}{2} > \frac{1}{2}, \quad \text{that is,} \quad 1-\mu < \mu$$

and

$$(1-\mu)n = n-\sum_{i=1}^{n}p_i = p_0,$$

Theorem A ensures that

$$\left(\mu\frac{x-1}{x^{\mu}-1}\right)^{n}=F_{\mu,1-\mu}(x)^{(1-\mu)n}\leqslant F_{\mu,\mu}(x)^{(1-\mu)n}=\left(\frac{x^{\mu}+1}{2}\right)^{\frac{p_0}{\mu}}.$$

The last inequality in (1) follows from the fact that $F_{q,q}(x)=(\frac{x^q+1}{2})^{1/q}$ is monotone increasing on $q\in[-1,1]\setminus\{0\}$ by Theorem A.

Therefore the proof is completed. □

We remark that (2) can be shown by differential calculations, but we prove it by using Theorem A here.

COROLLARY 2. *Let n be a natural number such that $n\geqslant 2$, and let $p_i\in[0,1]$ for $i=1,2,\ldots,n$ such that $\sum_{i=1}^{n}p_i=n-1$. Then*

$$\begin{aligned}x^{\gamma}\frac{x-1}{\log x}&\leqslant f_p(x)=x^{\gamma}\prod_{i=1}^{n}p_i\frac{x-1}{x^{p_i}-1}\\&\leqslant x^{\gamma}\left(\mu\frac{x-1}{x^{\mu}-1}\right)^{n}\leqslant x^{\gamma}\left(\frac{x^{\mu}+1}{2}\right)^{\frac{1}{\mu}}\leqslant x^{\gamma}\frac{x+1}{2}\end{aligned}$$

holds for $\gamma\in\mathbb{R}$ and $x>0$, where $p=(p_1,\ldots,p_n)$ and $\mu=\frac{1}{n}\sum_{i=1}^{n}p_i=1-\frac{1}{n}$.

Proof. By taking a limit $p_0\to 1$ in Theorem 1, we have the desired inequality since $\lim\limits_{\alpha\to 0}\dfrac{x^{\alpha}-1}{\alpha}=\log x$ holds for all $x>0$. □

We can obtain another upper bound of $f_p(x)$.

THEOREM 3. *Let n be a natural number, and let $p_i\in[0,1]$ for $i=0,1,2,\ldots,n$ such that $\sum_{i=0}^{n}p_i=n$. Then*

$$\begin{aligned}f_p(x)=x^{\gamma}\prod_{i=1}^{n}p_i\frac{x-1}{x^{p_i}-1}&\leqslant x^{\gamma}\left(\prod_{i=1}^{n-1}p_i\frac{x-1}{x^{p_i}-1}\right)\frac{1}{2-p_n}\frac{x^{2-p_n}-1}{x-1}\\&\leqslant x^{\gamma}\prod_{i=1}^{n}\frac{1}{2-p_i}\frac{x^{2-p_i}-1}{x-1}\leqslant x^{\gamma}\left(\frac{x+1}{2}\right)^{p_0}\end{aligned}$$

holds for $\gamma\in\mathbb{R}$ and $x>0$, where $p=(p_1,\ldots,p_n)$.

Proof. It is enough to show

$$\begin{aligned}\prod_{i=1}^{n}p_i\frac{x-1}{x^{p_i}-1}&\leqslant\left(\prod_{i=1}^{n-1}p_i\frac{x-1}{x^{p_i}-1}\right)\frac{1}{2-p_n}\frac{x^{2-p_n}-1}{x-1}\\&\leqslant\prod_{i=1}^{n}\frac{1}{2-p_i}\frac{x^{2-p_i}-1}{x-1}\leqslant\left(\frac{x+1}{2}\right)^{p_0}\end{aligned}\tag{3}$$

for $x > 0$.

We shall show the first inequality in (3).

$$\begin{aligned}\prod_{i=1}^{n} p_i \frac{x-1}{x^{p_i}-1} &= \prod_{i=1}^{n} F_{p_i,1-p_i}(x)^{1-p_i} \\ &\leqslant \left(\prod_{i=1}^{n-1} F_{p_i,1-p_i}(x)^{1-p_i}\right) F_{1,1-p_n}(x)^{1-p_n} \quad \text{(by Theorem A)} \\ &= \left(\prod_{i=1}^{n-1} p_i \frac{x-1}{x^{p_i}-1}\right) \frac{1}{2-p_n} \frac{x^{2-p_n}-1}{x-1}.\end{aligned}$$

The second and the third inequalities in (3) are obtained by Theorem A as follows:

$$\begin{aligned}\left(\prod_{i=1}^{n-1} p_i \frac{x-1}{x^{p_i}-1}\right) \frac{1}{2-p_n} \frac{x^{2-p_n}-1}{x-1} &= \left(\prod_{i=1}^{n-1} F_{p_i,1-p_i}(x)^{1-p_i}\right) F_{1,1-p_n}(x)^{1-p_n} \\ &\leqslant \prod_{i=1}^{n} F_{1,1-p_i}(x)^{1-p_i} = \prod_{i=1}^{n} \frac{1}{2-p_i} \frac{x^{2-p_i}-1}{x-1} \\ &\leqslant \prod_{i=1}^{n} F_{1,1}(x)^{1-p_i} \\ &= \prod_{i=1}^{n} \left(\frac{x+1}{2}\right)^{1-p_i} = \left(\frac{x+1}{2}\right)^{p_0},\end{aligned}$$

where the last equality holds by $\sum_{i=1}^{n}(1-p_i) = p_0$. □

Especially, we have upper and lower bounds of the PH function by Corollary 2 and Theorem 3 as follows:

COROLLARY 4. *Let* $p \in [0,1]$.

(i) *The inequality*

$$\begin{aligned}f_0(x) = f_1(x) = \frac{x-1}{\log x} \leqslant f_p(x) &= p(1-p)\frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)} \\ &\leqslant \left(\frac{\sqrt{x}+1}{2}\right)^2 \leqslant \frac{x+1}{2}\end{aligned}$$

holds for $x > 0$.

(ii) *The inequality*

$$\begin{aligned}f_0(x) = f_1(x) = \frac{x-1}{\log x} \leqslant f_p(x) &= p(1-p)\frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)} \\ &\leqslant \frac{p}{p+1}\frac{x^{p+1}-1}{x^p-1} \\ &\leqslant \frac{1}{(p+1)(2-p)}\frac{(x^{p+1}-1)(x^{2-p}-1)}{(x-1)^2} \leqslant \frac{x+1}{2}\end{aligned}$$

holds for $x>0$.

$\dfrac{p}{p+1}\dfrac{x^{p+1}-1}{x^p-1}$ in Corollary 4 has been considered in [2, 3].

Proof. (i) and the first inequality in (ii) are obtained by putting $n=2$, $\gamma=0$, $p_1=p$ and $p_2=1-p$ in Corollary 2. The other inequalities in (ii) are obtained by putting $n=2$, $\gamma=0$, $p_0=1$, $p_1=p$ and $p_2=1-p$ in Theorem 3. $\square$

3. Operator monotonicity of $f_p(x)$

First of all, we shall give an elementary proof of the following known result.

THEOREM B. ([5]) *For* $-2\leqslant p\leqslant 2$, $s_p(x)=\left(p\dfrac{x-1}{x^p-1}\right)^{\frac{1}{1-p}}$ *is an operator monotone function on* $x>0$, *where* $s_0(x)$ *and* $s_1(x)$ *are defined by using the limit as follows:*

$$s_0(x)=\lim_{p\to 0}s_p(x)=\frac{x-1}{\log x}\quad \textit{and}\quad s_1(x)=\lim_{p\to 1}s_p(x)=\frac{1}{e}x^{\frac{x}{x-1}}.$$

In [5], Theorem B has been proven by using a technique of complex analysis. This proof is very nice, but it is a little bit difficult. Here, we shall give an alternative proof of Theorem B by using only Theorem A and the following well-known fact:

LEMMA C. (ex. [6]) *Let* $f(x)$ *and* $g(x)$ *be operator monotone functions. Then the following functions are also operator monotone:*

(i) $f(x)^{\alpha}g(x)^{\beta}$ *for* $\alpha,\beta\geqslant 0$ *such that* $\alpha+\beta\leqslant 1$,

(ii) $f(x^{-1})^{-1}$.

Alternative proof of Theorem B. (i) *The case* $0\leqslant p\leqslant 1$. In the case $0<p<1$, $s_p(x)$ is operator monotone since $s_p(x)=F_{p,1-p}(x)$ and Theorem A. If $p=0,1$, it is also operator monotone by taking a limit $p\to 0+$ or $p\to 1-0$. It is still true (see [9]).

(ii) *The case* $1<p\leqslant 2$. $s_p(x)$ is operator monotone since $s_p(x)=\left(\dfrac{1}{p}\dfrac{x^p-1}{x-1}\right)^{\frac{1}{p-1}}$ $=F_{1,p-1}(x)$ and Theorem A.

(iii) *The case* $-1\leqslant p<0$.

$$s_p(x)=\left(-|p|\frac{x-1}{x^{-|p|}-1}\right)^{\frac{1}{1+|p|}}=\left(x^{|p|}|p|\frac{x-1}{x^{|p|}-1}\right)^{\frac{1}{1+|p|}}=x^{\frac{|p|}{1+|p|}}s_{|p|}(x)^{\frac{1-|p|}{1+|p|}}.$$

Since $\frac{|p|}{1+|p|},\frac{1-|p|}{1+|p|}\in[0,1]$ and (i), we have operator monotonicity of $s_p(x)$ for $p\in[-1,0]$ by Lemma C.

(iv) *The case* $-2 \leqslant p \leqslant -1$.

$$\begin{aligned}
s_p(x) &= \left(-|p|\frac{x-1}{x^{-|p|}-1}\right)^{\frac{1}{1+|p|}} \\
&= \left(x|p|\frac{x^{-1}-1}{x^{-|p|}-1}\right)^{\frac{1}{1+|p|}} \\
&= \left\{xs_{|p|}(x^{-1})^{1-|p|}\right\}^{\frac{1}{1+|p|}} \\
&= x^{\frac{1}{1+|p|}}\left\{s_{|p|}(x^{-1})^{-1}\right\}^{\frac{|p|-1}{1+|p|}}.
\end{aligned}$$

Since $\frac{1}{1+|p|}, \frac{|p|-1}{1+|p|} \in [0,1]$ and (ii), we have operator mononicity of $s_p(x)$ by Lemma C.

Therefore $s_p(x)$ is operator monotone for $p \in [-2,2]$. □

By the same way, we can obtain operator monotonicity of $f_{\mathbf{p}}(x)$.

THEOREM 5. *Let* $\gamma \in \mathbb{R}$ *and* $\mathbf{p} = (p_i) = (a_1,\ldots,a_l,b_1,\ldots,b_m,c_1,\ldots,c_u,d_1,\ldots,d_v)$ $(n = l+m+u+v)$ *such that*

$$-2 \leqslant d_1 \leqslant \cdots \leqslant d_v < -1 \leqslant c_1 \leqslant \cdots \leqslant c_u < 0 \leqslant b_1 \leqslant \cdots \leqslant b_m < 1 \leqslant a_1 \leqslant \cdots \leqslant a_l \leqslant 2,$$

$$0 \leqslant \gamma + (l+v) - \sum_{i=1}^{l} a_i - \sum_{i=1}^{u} c_i \leqslant 1 \quad \text{and} \quad 0 \leqslant \gamma + (m+u) - \sum_{i=1}^{m} b_i - \sum_{i=1}^{v} d_i \leqslant 1.$$

Then $f_{\mathbf{p}}(x) = x^{\gamma}\prod_{i=1}^{n} p_i \frac{x-1}{x^{p_i}-1}$ *is operator monotone on* $x > 0$.

In [7], Nagisa and Wada have obtained the following result on operator monotonicity of the function related to $f_{\mathbf{p}}(x)$.

THEOREM D. ([7]) *Let* $\alpha_i, \beta_i \in [0,2]$ *such that* $\beta_i < \alpha_i$ *and* $\beta_i \leqslant 1$ *(i=1,2,...,n). Then the function*

$$\left(x^{\gamma}\prod_{i=1}^{n}\frac{x^{\alpha_i}-1}{x^{\beta_i}-1}\right)^{s}$$

is operator monotone on $x > 0$ *if* $0 \leqslant s \leqslant 1/(\gamma + \sum_{i=1}^{n}(\alpha_i - \beta_i))$ *and is not operator monotone on* $x > 0$ *if* $s > 1/(\gamma + \sum_{i=1}^{n}(\alpha_i - \beta_i))$ *for any* $\gamma \geqslant 0$.

In particular, by putting $s = 1$, $\alpha_i = 1$, $\beta_i = p_i$, we have the following corollary.

COROLLARY E. *Let* $p_i \in (0,1)$ $(i = 1,2,\ldots,n)$. *Then the function*

$$f_{\mathbf{p}}(x) = x^{\gamma}\prod_{i=1}^{n}\frac{x-1}{x^{p_i}-1}$$

is operator monotone on $x > 0$ *if* $0 \leqslant \gamma + \sum_{i=1}^{n}(1-p_i) \leqslant 1$ *and is not operator monotone on* $x > 0$ *if* $\gamma + \sum_{i=1}^{n}(1-p_i) > 1$ *for any* $\gamma \geqslant 0$.

Theorem 5 is a kind of an extension of Corollary E, because we consider the cases $\gamma < 0$ and $p_i \in [-2,0) \cup (1,2]$ in Theorem 5. Moreover, Theorem 5 implies operator monotonicity of the PH function $f_p(x)$ for $p \in [-1,2]$. If $p \in [\frac{1}{2},1]$, we can get it by putting $\gamma = 0$, $n = 2$, $b_1 = 1-p$ and $b_2 = p$. If $p \in (1,2]$, we can get it by putting $\gamma = 0$, $n = 2$, $a_1 = p$ and $c_1 = 1-p$. We have the case $p \in [-1,\frac{1}{2}]$ since $f_{1-p}(x) = f_p(x)$ always holds.

Proof of Theorem 5. Since

$$
\begin{aligned}
a_i\frac{x-1}{x^{a_i}-1} &= x^{1-a_i}\left\{a_i\frac{x^{-1}-1}{(x^{-1})^{a_i}-1}\right\}^{\frac{-1}{1-a_i}\cdot\{-(1-a_i)\}} &&= x^{1-a_i}\left\{s_{a_i}(x^{-1})^{-1}\right\}^{a_i-1},\\
b_i\frac{x-1}{x^{b_i}-1} &= s_{b_i}(x)^{1-b_i},\\
c_i\frac{x-1}{x^{c_i}-1} &= x^{-c_i}\left\{(-c_i)\frac{x-1}{x^{-c_i}-1}\right\}^{\frac{1}{1+c_i}\cdot(1+c_i)} &&= x^{-c_i}s_{-c_i}(x)^{1+c_i} \text{ and}\\
d_i\frac{x-1}{x^{d_i}-1} &= x\left\{(-d_i)\frac{x^{-1}-1}{(x^{-1})^{-d_i}-1}\right\}^{\frac{-1}{1+d_i}\cdot\{-(1+d_i)\}} &&= x\left\{s_{-d_i}(x^{-1})^{-1}\right\}^{-(1+d_i)}
\end{aligned}
$$

hold for each i, we have

$$
\begin{aligned}
f_P(x) &= x^{\gamma}\prod_{i=1}^{l}a_i\frac{x-1}{x^{a_i}-1}\prod_{i=1}^{m}b_i\frac{x-1}{x^{b_i}-1}\prod_{i=1}^{u}c_i\frac{x-1}{x^{c_i}-1}\prod_{i=1}^{v}d_i\frac{x-1}{x^{d_i}-1}\\
&= x^{w}\prod_{i=1}^{l}\left\{s_{a_i}(x^{-1})^{-1}\right\}^{a_i-1}\prod_{i=1}^{m}s_{b_i}(x)^{1-b_i}\prod_{i=1}^{u}s_{-c_i}(x)^{1+c_i}\prod_{i=1}^{v}\left\{s_{-d_i}(x^{-1})^{-1}\right\}^{-(1+d_i)},
\end{aligned}
$$

where $w = \gamma + \sum_{i=1}^{l}(1-a_i) + \sum_{i=1}^{u}(-c_i) + \sum_{i=1}^{v}1 = \gamma + (l+v) - \sum_{i=1}^{l}a_i - \sum_{i=1}^{u}c_i$.

By the assumption, $w, a_i-1, 1-b_i, 1+c_i, -(1+d_i) \in [0,1]$ for every i and

$$
\begin{aligned}
w &+ \sum_{i=1}^{l}(a_i-1) + \sum_{i=1}^{m}(1-b_i) + \sum_{i=1}^{u}(1+c_i) - \sum_{i=1}^{v}(1+d_i)\\
&= \gamma + (m+u) - \sum_{i=1}^{m}b_i - \sum_{i=1}^{v}d_i \in [0,1].
\end{aligned}
$$

Hence $f_P(x)$ is operator monotone by Theorem B and Lemma C. $\square$

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