Operators and Matrices
Volume 11, Number 2 (2017), 337–346 doi:10.7153/oam-11-21

SPECTRAL RADIUS ALGEBRAS OF WCE OPERATORS

Y. ESTAREMI AND M. R. JABBARZADEH

(*Communicated by T. S. S. R. K. Rao*)

Abstract. In this paper, we investigate the spectral radius algebras related to the weighted conditional expectation operators on the Hilbert spaces $L^2(\mathscr{F})$. We give a large classes of operators on $L^2(\mathscr{F})$ that have the same spectral radius algebra. As a consequence we get that the spectral radius algebras of a weighted conditional expectation operator and its Aluthge transformation are equal. Also, we obtain an ideal of the spectral radius algebra related to the rank one operators on the Hilbert space $\mathscr{H}$. Finally we get that the operator T majorizes all closed range elements of the spectral radius algebra of T, when T is a weighted conditional expectation operator on $L^2(\mathscr{F})$ or a rank one operator on the arbitrary Hilbert space $\mathscr{H}$.

1. Introduction

Let $(X,\mathscr{F},\mu)$ be a complete σ-finite measure space. All sets and functions statements are to be interpreted as holding up to sets of measure zero. For a σ-subalgebra $\mathscr{A}$ of $\mathscr{F}$, the conditional expectation operator associated with $\mathscr{A}$ is the mapping $f\rightarrow E^{\mathscr{A}}f$, defined for all non-negative f as well as for all $f\in L^2(\mathscr{F})=L^2(X,\mathscr{F},\mu)$, where $E^{\mathscr{A}}f$ is the unique $\mathscr{A}$-measurable function satisfying $\int_A(E^{\mathscr{A}}f)d\mu=\int_A f d\mu$, for all $A\in\mathscr{A}$. We will often write E for $E^{\mathscr{A}}$. The mapping E is a linear orthogonal projection from $L^2(\mathscr{F})$ onto $L^2(\mathscr{A})$. For more details on the properties of E see [20].

We continue our investigation about the class of bounded linear operators on the L^p-spaces having the form M_wEM_u, where E is the conditional expectation operator, M_w and M_u are (possibly unbounded) multiplication operators and it is called weighted conditional expectation operator. Our interest in operators of the form M_wEM_u stems from the fact that such forms tend to appear often in the study of those operators related to conditional expectation. Weighted conditional expectation operators appeared in [4], where it is shown that every contractive projection on certain L^1-spaces can be decomposed into an operator of the form M_wEM_u and a nilpotent operator. For stronger results about weighted conditional expectation operators one can see [5, 9, 11, 13]. In these papers one can see that a large classes of operators are of the form of weighted conditional expectation operators.

Let $\mathscr{H}$ be a Hilbert space and $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. $\mathscr{N}(T)$ and $\mathscr{R}(T)$ denote the null-space and range of an operator T, respectively. A closed subspace $\mathscr{M}$ of $\mathscr{H}$ is said to be invariant for an operator

Mathematics subject classification (2010): 47A65, 47L30.
Keywords and phrases: Conditional expectation operator, spectral algebras, invariant subspace, rank one operator.

Paper OaM-11-21

$T \in \mathscr{B}(\mathscr{H})$ if $T\mathscr{M} \subseteq \mathscr{M}$. The collection of all invariant subspaces of T is a lattice and it is denoted by $Lat(T)$. If $\mathscr{M}$ is invariant for all operators commute with T, then it is called a hyperinvariant subspace for T. The description $Lat(T)$ is an open problem. Some authors describe $Lat(T)$ in the special case of T. In [12], Lambert and Petrovic introduced a modified version of a class of operator algebras that is called spectral radius algebras. Since a spectral radius algebra related to an operator $T \in \mathscr{B}(\mathscr{H})$ ($\mathscr{B}_T$) contains all operators that commute with T ($\{T\}'$), then the invariant subspaces of $\mathscr{B}_T$ are hyperinvariant subspaces of T. In [12], the authors established several sufficient conditions for $\mathscr{B}_T$ to have a nontrivial invariant subspace. When T is compact the results of [12] generalizes the Lomonosov's theorem. In [2], the authors demonstrated that for a subclasses of normal operators $\mathscr{B}_T$ has a nontrivial invariant subspace. Spectral radius algebras for complex symmetric operators are discussed in [10]. Moreover, spectral radius algebras were studied recently by S. Petrovic in [14, 15, 16, 17, 18, 19].

In this paper we investigate the spectral radius algebras related to the weighted conditional expectation operators on the Hilbert spaces $L^2(\mathscr{F})$. We will show that there are lots of operators on $L^2(\mathscr{F})$ such as T with $\mathscr{B}_T \neq \{T\}'$. In addition, we obtain an ideal of the spectral radius algebra related to the rank one operators on the Hilbert space $\mathscr{H}$. Finally we get that the operator T majorizes all closed range elements of the spectral radius algebra of T, when T is a weighted conditional expectation operator on $L^2(\mathscr{F})$ or a rank one operator on the arbitrary Hilbert space $\mathscr{H}$.

2. Spectral radius algebras

For notation and basic terminology concerning spectral radius algebras, we refer the reader to [3, 12].

Let $\mathscr{H}$ be a Hilbert space, $T \in \mathscr{B}(\mathscr{H})$ and let $r(T)$ be the spectral radius of T. For $m \geqslant 1$ we define

$$R_m(T) = R_m := \left(\sum_{n=0}^{\infty} d_m^{2n} T^{*^n} T^n \right)^{\frac{1}{2}}, \tag{2.1}$$

where $d_m = \frac{1}{1/m + r(T)}$. Since $d_m \uparrow 1/r(T)$, the sum in (2.1) is norm convergent and for each m, R_m is well defined, positive and invertible. The spectral radius algebra $\mathscr{B}_T$ of T consists of all operators $S \in \mathscr{B}(\mathscr{H})$ such that

$$\sup_{m \in \mathbb{N}} \|R_m S R_m^{-1}\| < \infty.$$

$\mathscr{B}_T$ is an algebra and it contains all operators commute with T. Throughout this section we assume that $w, u \in \mathscr{D}(E) := \{f \in L^0(\mathscr{F}) : E(|f|) \in L^0(\mathscr{A})\}$, where $L^0(\mathscr{F})$ is the space of almost every where finite valued $\mathscr{F}$- measurable functions on X. Now we recall the definition of weighted conditional expectation operators on $L^2(\mathscr{F})$.

DEFINITION 2.1. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and let $\mathscr{A}$ be a σ-subalgebra of $\mathscr{F}$ such that $(X, \mathscr{A}, \mu_{\mathscr{A}})$ is also σ-finite. Let E be the conditional expectation operator relative to $\mathscr{A}$. If $u,w \in L^0(\mathscr{F})$, such that uf is conditionable and $wE(uf) \in L^2(\mathscr{F})$ for all $f \in \mathscr{D} \subseteq L^2(\mathscr{F})$, where $\mathscr{D}$ is a linear subspace, then the corresponding weighted conditional expectation (or briefly WCE) operator is the linear transformation $M_wEM_u : \mathscr{D} \to L^2(\mathscr{F})$ defined by $f \to wE(uf)$.

As was proved in [8] we have an equivalent condition for boundedness of the weighted conditional expectation operators M_wEM_u on $L^2(\mathscr{F})$ as the next theorem.

THEOREM 2.2. *The operator $T = M_wEM_u : L^2(\mathscr{F}) \to L^2(\mathscr{F})$ is bounded if and only if $(E(|w|^2)^{\frac{1}{2}})(E(|u|^2)^{\frac{1}{2}}) \in L^\infty(\mathscr{A})$, in this case $\|T\| = \|(E(|w|^2)^{\frac{1}{2}})(E(|u|^2)^{\frac{1}{2}})\|_\infty$.*

Let $T = M_wEM_u$ be a bounded operator on $L^2(\mathscr{F})$. Direct computations shows that for every $n \in \mathbb{N}$ (natural numbers) we have

$$T^n f = (E(uw))^{n-1} wE(uf);$$
$$T^{*^n} f = (\overline{E(uw)})^{n-1} \bar{u}E(\bar{w}f).$$

Since $R_m = R_m(M_wEM_u)$ is positive and invertible operator, we obtain

$$R_m = \left(I + M_{(E(|w|^2)\sum_{n=1}^{\infty} d_m^{2n}|E(uw)|^{2(n-1)})} M_{\bar{u}}EM_u\right)^{\frac{1}{2}}.$$

It is easy to see that the following equality holds almost every where on X.

$$\sum_{n=1}^{\infty} d_m^{2n}|E(uw)|^{2(n-1)} = \frac{d_m^2}{1 - d_m^2|E(uw)|^2}.$$

If we set

$$v_m = \frac{d_m^2 E(|w|^2)}{1 - d_m^2|E(uw)|^2},$$

then we have

$$R_m = \left(I + M_{v_m\bar{u}}EM_u\right)^{\frac{1}{2}}.$$

By an elementary technical method we can compute the inverse of R_m as follow:

$$R_m^{-1} = \left(I + M_{\frac{v_m\bar{u}}{v_mE(|u|^2)-1}} EM_u\right)^{\frac{1}{2}}.$$

Here we recall a fundamental lemma in operator theory.

LEMMA 2.3. *Let T be a bounded operator on the Hilbert space $\mathscr{H}$ and $\lambda \geqslant 0$. Then we have*

$$\|\lambda I + T^*T\| = \lambda + \|T^*T\| = \lambda + \|T\|^2.$$

Specially, if T is a positive operator, then $\|\lambda I + T\| = \lambda + \|T\|$.

Proof. It is an easy exercise. □

From now on, we assume that $E(|u|^2) \in L^\infty(\mathscr{A})$. Now we characterize the spectral radius algebra corresponding to the WCE operator M_wEM_u in the next theorem.

THEOREM 2.4. *Let $S \in \mathscr{B}(L^2(\mathscr{F}))$. Then $S \in \mathscr{B}_{M_wEM_u}$ if and only if $\mathscr{N}(EM_u)$ is invariant under S.*

Proof. Since R_m and R_m^{-1} are positive operators and $(EM_u)^* = M_{\bar{u}}E$, then by Lemma 2.3 and Theorem 2.2 we have

$$\|R_m\|^2 = \|R_m^2\| = 1 + \|E(|u|^2)v_m\|_\infty$$

and

$$\|R_m^{-1}\|^2 = \|R_m^{-2}\| = 1 + \left\|\frac{E(|u|^2)v_m}{v_mE(|u|^2) - 1}\right\|_\infty.$$

If we decompose $L^2(\mathscr{F})$ as a direct sum $\mathscr{H}_1 \oplus \mathscr{H}_2$, in which

$$\mathscr{H}_2 = \mathscr{N}(EM_u) = \{f \in L^2(\mathscr{F}) : E(uf) = 0\}$$

and

$$\mathscr{H}_1 = \mathscr{H}_2^\perp = \overline{\bar{u}L^2(\mathscr{A})},$$

then the corresponding block matrix of R_m is

$$R_m = \begin{pmatrix} M_{(q_m)^{\frac{1}{2}}} & 0 \\ 0 & I \end{pmatrix} \text{ and } R_m^{-1} = \begin{pmatrix} M_{(q_m)^{\frac{-1}{2}}} & 0 \\ 0 & I \end{pmatrix},$$

where $q_m = 1 + v_mE(|u|^2)$. Notice that for $m > m'$ we have $q_m \geqslant q_{m'}$ and $\|q_m\|_\infty \to \infty$ as $m \to \infty$. If $S \in \mathscr{B}(L^2(\mathscr{F}))$ say $S = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, the block matrix with respect to the decomposition $\mathscr{H}_1 \oplus \mathscr{H}_2$, then

$$\begin{aligned} R_mSR_m^{-1} &= \begin{pmatrix} M_{(q_m)^{\frac{1}{2}}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} M_{(q_m)^{\frac{-1}{2}}} & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} M_{(q_m)^{\frac{1}{2}}}XM_{(q_m)^{\frac{-1}{2}}} & M_{(q_m)^{\frac{1}{2}}}Y \\ ZM_{(q_m)^{\frac{-1}{2}}} & W \end{pmatrix}. \end{aligned}$$

Since $\|M_{(q_m)^{\frac{1}{2}}}XM_{(q_m)^{\frac{-1}{2}}}\| \leqslant \|X\|$, then we get that $\sup_m \|R_mSR_m^{-1}\| < \infty$ if and only if $\sup_m \|M_{(q_m)^{\frac{1}{2}}}Y\| < \infty$. Direct computations shows that $\sup_m \|M_{(q_m)^{\frac{1}{2}}}Y\| < \infty$ if and only if $Y = 0$. This means that $\mathscr{H}_2$ is an invariant subspace for S. □

Therefore by Theorem 2.4 we get that there are many different operators that have the same spectral radius algebra.

COROLLARY 2.5. *Let $w, w', u \in \mathscr{D}(E)$. If M_wEM_u and $M_{w'}EM_u$ are bounded operator on the Hilbert space $L^2(\mathscr{F})$, then $\mathscr{B}_{M_{w'}EM_u} = \mathscr{B}_{M_wEM_u}$.*

Also in the next corollary we have a sufficient condition for $\mathscr{B}_{M_wEM_u}$ to be equal to $\mathscr{B}(L^2(\mathscr{F}))$.

COROLLARY 2.6. *If $\mathscr{N}(EM_u) = \{0\}$, then $\mathscr{B}_{M_wEM_u} = \mathscr{B}(L^2(\mathscr{F}))$.*

In the next Proposition we find some special elements of $\mathscr{B}_{M_wEM_u}$.

PROPOSITION 2.7. *If $a \in L^0(\mathscr{A})$ such that $a \geqslant 0$ and $M_{a\bar{u}}EM_u \in \mathscr{B}(L^2(\mathscr{F}))$, then $M_{a\bar{u}}EM_u \in \mathscr{B}_{M_wEM_u}$.*

Proof. Since $R_m = \left(I + M_{v_m\bar{u}}EM_u\right)^{\frac{1}{2}}$ and $v_m = \frac{d_m^2 E(|w|^2)}{1-d_m^2|E(uw)|^2}$ is an $\mathscr{A}$-measurable function, it holds that $R_mM_{a\bar{u}}EM_u = M_{a\bar{u}}EM_uR_m$. Therefore we have $\|R_mM_{a\bar{u}}EM_uR_m^{-1}\| = \|M_{a\bar{u}}EM_u\|$, and so we get that $M_{a\bar{u}}EM_u \in \mathscr{B}_{M_wEM_u}$. □

Every operator T on a Hilbert space $\mathscr{H}$ can be decomposed into $T = U|T|$ with a partial isometry U, where $|T| = (T^*T)^{\frac{1}{2}}$. U is determined uniquely by the kernel condition $\mathscr{N}(U) = \mathscr{N}(|T|)$. Then this decomposition is called the polar decomposition. The Aluthge transformation $\widetilde{T}$ of the operator T is defined by $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Here we recall that the Aluthge transformation of $T = M_wEM_u$ is

$$\widetilde{T}(f) = \frac{\chi_{z_1}E(uw)}{E(|u|^2)}\bar{u}E(uf), \qquad f \in L^2(\mathscr{F}),$$

in which $z_1 = z(E(|u|^2))$ (see [8]). Thus $\widetilde{T} = M_{w'}EM_{u'}$ where $w' = \frac{E(uw)\bar{u}\chi_{z_1}}{E(|u|^2)}$ and $u' = u$. We recall that $r(M_wEM_u) = \|E(uw)\|_\infty$ (see [6]). Direct computations shows that $E(u'w') = E(uw)$. Hence $r(T) = r(\widetilde{T})$. Hence by using Proposition 2.7 we have the next corollary.

COROLLARY 2.8. *If w and u are positive measurable functions, then $\widetilde{T} \in \mathscr{B}_T$ where $T = M_wEM_u$.*

By the proof of Proposition 2.7 we get that the commutant of M_wEM_u (in symbol $\{M_wEM_u\}'$) is a proper subset of $\mathscr{B}_{M_wEM_u}$ when w, u are positive and $w \neq u$. In the next theorem we get that $\mathscr{B}_T = \mathscr{B}_{\widetilde{T}}$ when $T = M_wEM_u$ and $w, u \geqslant 0$.

COROLLARY 2.9. *If $T = M_wEM_u$ and $w, u \geqslant 0$, then $\mathscr{B}_T = \mathscr{B}_{\widetilde{T}}$.*

Recall that for $f, g \in L^2(\mathscr{F})$ we can define a rank one operator $f \otimes g$ on $L^2(\mathscr{F})$ by the action $(f \otimes g)(h) = \langle h, g\rangle f$ for every $h \in L^2(\mathscr{F})$, in which $\langle\ ,\ \rangle$ is the inner product of the Hilbert space $L^2(\mathscr{F})$. In the next proposition we give some conditions under which a rank one operator belongs to the $\mathscr{B}_{M_wEM_u}$.

PROPOSITION 2.10. *If $T = M_wEM_u$ and $f,g \in L^2(\mathscr{F})$, then $f \otimes g \in \mathscr{B}_T$ if and only if*

$$\sup_m \|\alpha_m^{\frac{1}{2}} E(ug)\|^2 \|f\|^2 + \|v_m^{\frac{1}{2}} E(uf)\|^2 (\|g\|^2 + \|\alpha_m^{\frac{1}{2}} E(ug)\|^2) < \infty,$$

where $\alpha_m = \frac{v_m}{v_m E(|u|^2)-1}$.

Proof. By using the properties of inner product we have

$$\|R_m f\|^2 = \|f\|^2 + \|v_m^{\frac{1}{2}} E(uf)\|^2$$

and

$$\|R_m^{-1} f\|^2 = \|g\|^2 + \|\alpha_m^{\frac{1}{2}} E(ug)\|^2.$$

Now, the desired conclusion follows by [12, Lemma 3.9]. □

By using some results of [12] we get that the conditional expectation corresponding to σ- subalgebra $\mathscr{A} \subseteq \mathscr{B}$ is in $\mathscr{B}_{E^{\mathscr{A}} M_u}$ as we mentioned in the next remark.

REMARK 2.11. Let $T = EM_u \in \mathscr{B}(L^2(\mathscr{F}))$, $u \in L^\infty(\mathscr{A})$ and let $\mathscr{A}, \mathscr{B}$ be σ-subalgebras of $\mathscr{F}$ such that $\mathscr{A} \subseteq \mathscr{B}$. If $E = E^{\mathscr{A}}$ and S is an operator for which $TS = E^{\mathscr{B}} ST$, then $S \in \mathscr{B}_T$.

Proof. It is not hard to see that $EM_u E^{\mathscr{B}} = E^{\mathscr{B}} EM_u$. Since $E^{\mathscr{B}}$ is a projection on $L^2(\mathscr{F})$, then it is power bounded. Therefore, by [12, Proposition 2.3] we get that $S \in \mathscr{B}_T$. □

COROLLARY 2.12. *If $T = M_wEM_u \in \mathscr{B}(L^2(\mathscr{F}))$ and $a \in L^\infty(\mathscr{A})$, then $M_a \in \mathscr{B}_T$.*

Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Here we recall the definition of Q_T, that is defined in [12], as follows:

$$Q_T = \{S \in \mathscr{B}(\mathscr{H}) : \|R_m S R_m^{-1}\| \to 0\}.$$

In the next theorem we illustrate Q_T when $T = M_wEM_u \in \mathscr{B}(L^2(\mathscr{F}))$.

THEOREM 2.13. *Let $T = M_wEM_u$ and $S \in \mathscr{B}(L^2(\mathscr{F}))$. Then $S \in Q_T$ if and only if $\mathscr{N}(EM_u)$ is invariant under S and $\mathscr{N}(EM_u) \subseteq \mathscr{N}(S)$.*

Proof. Let $S = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, the block matrix with respect to the decomposition $\mathscr{H}_1 \oplus \mathscr{H}_2$, in which $\mathscr{H}_2 = \mathscr{N}(EM_u)$ and $\mathscr{H}_1 = \mathscr{H}_2^{\perp}$. So similar to the proof of Theorem 2.4 we have

$$R_m S R_m^{-1} = \begin{pmatrix} M_{(q_m)^{\frac{1}{2}}} X M_{(q_m)^{\frac{-1}{2}}} & M_{(q_m)^{\frac{1}{2}}} Y \\ Z M_{(q_m)^{\frac{-1}{2}}} & W \end{pmatrix}.$$

Hence $\|W\| \leqslant \|R_m S R_m^{-1}\|$. Since $S \in Q_T$, then $Y = 0$ and $\|W\| = 0$. This means that $\mathscr{H}_2$ is invariant under S and $PSP = 0$ in which $P = P_{\mathscr{H}_2}$. Therefore $SP = PSP = 0$, and so $\mathscr{H}_2 \subseteq \mathscr{N}(S)$. Conversely, if $\mathscr{N}(EM_u)$ is invariant under S and $\mathscr{N}(EM_u) \subseteq \mathscr{N}(S)$, then we get that $W = Y = 0$ and

$$R_m S R_m^{-1} = \begin{pmatrix} M_{(q_m)^{\frac{1}{2}}} X M_{(q_m)^{\frac{-1}{2}}} & 0 \\ Z M_{(q_m)^{\frac{-1}{2}}} & 0 \end{pmatrix}.$$

Hence

$$\|R_m S R_m^{-1}\| \leqslant \|M_{(q_m)^{\frac{1}{2}}} X M_{(q_m)^{\frac{-1}{2}}}\| + \|Z M_{(q_m)^{\frac{-1}{2}}}\|.$$

Since $\|M_{(q_m)^{\frac{-1}{2}}}\| = \|(q_m)^{\frac{-1}{2}}\|_\infty \to 0$, then $\|R_m S R_m^{-1}\| \to 0$ when $m \to \infty$. This completes the proof. $\square$

Now by using [12, Theorem 2.6] and some information about WCE operators we have an equivalent condition for the spectral radius algebra of a WCE operator to be equal to $\mathscr{B}(L^2(\mathscr{F}))$.

PROPOSITION 2.14. *If $T = M_w E M_u$, then $\mathscr{B}_T = \mathscr{B}(L^2(\mathscr{F}))$ if and only if*

$$\sup_m \left(\|E(|u|^2)v_m\|_\infty + \left\|\frac{E(|u|^2)v_m}{v_m E(|u|^2) - 1}\right\|_\infty (1 + \|E(|u|^2)v_m\|_\infty)\right) < \infty,$$

where $v_m = \frac{d_m^2 E(|w|^2)}{1 - d_m^2 |E(uw)|^2}$.

Proof. It is a direct consequence of [12, Theorem 2.6] and some information of the proof of Theorem 2.4. $\square$

By using Proposition 2.14 and some results of [2] we have an equivalent condition for the WCE operator $M_w E M_u$ to be a constant multiple of an isometry.

THEOREM 2.15. *If $T = M_w E M_u$ is a bounded operator on the Hilbert space $L^2(\mathscr{F})$, then T is a constant multiple of an isometry if and only if*

$$\sup_m \left(\|E(|u|^2)v_m\|_\infty + \left\|\frac{E(|u|^2)v_m}{v_m E(|u|^2) - 1}\right\|_\infty (1 + \|E(|u|^2)v_m\|_\infty)\right) < \infty,$$

where $v_m = \frac{d_m^2 E(|w|^2)}{1 - d_m^2 |E(uw)|^2}$.

Proof. It is a direct consequence of [2, Theorem 2.7] and Proposition 2.14. $\square$

Now in the next theorem we obtain some sufficient conditions for $\mathscr{B}_{M_w E M_u}$ to a nontrivial invariant subspace.

THEOREM 2.16. *If the measure space $(X, \mathscr{A}, \mu)$ is not a non-atomic measure space and $E(uw) = 0$, then $\mathscr{B}_{M_w E M_u}$ has a nontrivial invariant subspace.*

Proof. Since $E(uw) = 0$ then M_wEM_u is quasinilpotent. Also since the σ-algebra $\mathscr{A}$ has at least one atom, then we have a compact multiplication operator M_a for some $a \in L^\infty(\mathscr{A})$. Hence by Corollary 2.12 we have $M_a \in \mathscr{B}_{M_wEM_u}$. Moreover by using [12, Lemma 3.1] we get that $M_wEM_u \in Q_{M_wEM_u}$. Therefore by [12, Theorem 3.4] we get the proof. □

Here we give a remark on [12, Proposition 2.8] as follows:

REMARK 2.17. For the unit vectors u, v, w of the Hilbert space $\mathscr{H}$ we have $\mathscr{B}_{u\otimes w} = \mathscr{B}_{v\otimes w}$.

In the next theorem we describe $Q_{u\otimes v}$ for a rank one operator $u \otimes v$ in which u, v are in the Hilbert space $\mathscr{H}$.

THEOREM 2.18. *Let $\mathscr{H}$ be a Hilbert space and $S \in \mathscr{B}(\mathscr{H})$. If $u, v \in \mathscr{H}$, then $S \in Q_{u\otimes v}$ if and only if $S = (I-P)SP$, where $P = P_{\mathscr{H}_1}$ and $\mathscr{H}_1$ is the one-dimensional space spanned by v.*

Proof. As was computed in [12, Proposition 2.8] we have

$$R_m^2 = I + \frac{d_m^2}{1-d_m^2r^2} v\otimes v,$$

in which $r = r(u\otimes v) = |\langle u, v\rangle|$. Let $\lambda_m = \sqrt{1+\frac{d_m^2}{1-d_m^2r^2}}$. If $\mathscr{H}_1$ is the one-dimensional space spanned by v and $\mathscr{H}_2 = \mathscr{H}_1^\perp$. For $S \in \mathscr{B}(\mathscr{H})$, we have the corresponding block matrix of R_m, R_m^{-1} and S with respect to the decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ as follows:

$$R_m = \begin{pmatrix} M_{\lambda_m} & 0 \\ 0 & I \end{pmatrix}, \quad R_m^{-1} = \begin{pmatrix} M_{\frac{1}{\lambda_m}} & 0 \\ 0 & I \end{pmatrix}$$

and

$$S = \begin{pmatrix} PSP & PS(I-P) \\ (I-P)SP & (I-P)S(I-P) \end{pmatrix}.$$

Therefore, we have

$$R_mSR_m^{-1} = \begin{pmatrix} PSP & M_{\lambda_m}PS(I-P) \\ M_{\frac{1}{\lambda_m}}(I-P)SP & (I-P)S(I-P) \end{pmatrix}.$$

If $S \in Q_{u\otimes v}$, then $S \in \mathscr{B}_{u\otimes v}$. Hence by [12, Proposition 2.8] we obtain $PS(I-P) = 0$. Since $S \in Q_{u\otimes v}$, $\|PSP\| \leqslant \|R_mSR_m^{-1}\|$ and $\|(I-P)S(I-P)\| \leqslant \|R_mSR_m^{-1}\|$, then $\|PSP\| = 0$ and $\|(I-P)S(I-P)\| = 0$. Hence $PSP = 0$ and $(I-P)S(I-P) = 0$. Thus

$$S = \begin{pmatrix} 0 & 0 \\ (I-P)SP & 0 \end{pmatrix} = (I-P)SP.$$

Conversely, If $S=(I-P)SP$, then

$$\|R_mSR_m^{-1}\| = \left\| \begin{pmatrix} 0 & 0 \\ M_{\frac{1}{\lambda_m}}(I-P)SP & 0 \end{pmatrix} \right\| = \|M_{\frac{1}{\lambda_m}}(I-P)SP\|.$$

Since $\|M_{\frac{1}{\lambda_m}}(I-P)SP\| \to 0$, then $\|R_mSR_m^{-1}\| \to 0$ as $m \to \infty$. So $S \in Q_{u\otimes v}$. □

Let X,Y,Z be Banach spaces. Assume that $T \in \mathscr{B}(X,Y)$ and $S \in \mathscr{B}(X,Z)$. Then T majorizes S if there exists $M>0$ such that

$$\|Sx\| \leqslant M\|Tx\|$$

for all $x \in X$ (see [1]). Here we recall a result of [1] that gives us an equivalent condition for a closed range operator to majorize another bounded operator.

REMARK 2.19. [1, Proposition 4] Let X be Banach spaces and $T,S \in \mathscr{B}(X)$ with $\mathscr{R}(T)$ closed. Then T majorizes S if and only if $\mathscr{N}(T) \subseteq \mathscr{N}(S)$.

Now we recall an assertion about closed range weighted conditional expectation operators.

PROPOSITION 2.20. [7, Theorem 2.1] *If $z(E(u)) = z(E(|u|^2))$ and for some $\delta > 0$, $E(u) \geqslant \delta$ on $z(E(|u|^2))$, then the operator EM_u has closed range on $L^2(\mathscr{F})$.*

PROPOSITION 2.21. *Let $T = M_wEM_u$ and $u \geqslant 0$. If $S \in Q_T$ and $E(u) \geqslant \delta$, then EM_u majorizes S.*

Proof. Since $u \geqslant 0$, then $z(E(u)) = z(E(|u|^2))$. Hence by the Remark 2.19, Theorem 2.13 and Proposition 2.20 we get the proof. □

Finally, since the rank one operator $x \otimes y$ has closed range, the we can obtain the next proposition.

PROPOSITION 2.22. *Let $x,y \in \mathscr{H}$. If $T \in Q_{x\otimes y}$, then $x \otimes y$ majorizes T.*

Proof. If $T \in Q_{x\otimes y}$, then by the proof of Theorem 2.18 we have $\mathscr{H}_2 = \mathscr{N}(x\otimes y)$ and $\mathscr{N}(x \otimes y) \subseteq \mathscr{N}(T)$. Since $x \otimes y$ has closed range, then by the Remark 2.19 we conclude that $x \otimes y$ majorizes T. □

REFERENCES

[1] B. A. BARNES, *Majorization, Range inclusion and factorization for bounded linear operators*, Proc. Amer. Math. Soc. **133** (2004), 155–162.

[2] A. BISWAS, A. LAMBERT AND S. PETROVIC, *On spectral radius algebras and normal operators*, In. Univ. Math. J. **4** (2007), 1661–1674.

[3] A. BISWAS, A. LAMBERT, S. PETROVIC AND B. WEINSTOCK, *On spectral radius algebras*, Oper. Matrices. **2** (2008), 167–176.

[4] R. G. DOUGLAS, *Contractive projections on an L_1 space*, Pacific J. Math. **15** (1965), 443–462.
[5] P. G. DODDS, C. B. HUIJSMANS AND B. DE PAGTER, *Characterizations of conditional expectation-type operators*, Pacific J. Math. **141** (1990), 55–77.
[6] Y. ESTAREMI, *Some classes of weighted conditional type operators and their spectra*, Positivity **19** (2015), 83–93.
[7] Y. ESTAREMI, *On properties of Multiplication conditional type operators between L^p -space*, Filomat, preprint.
[8] Y. ESTAREMI AND M. R. JABBARZADEH, *Weighted lambert type operators on L^p -spaces*, Oper. Matrices **1** (2013), 101–116.
[9] J. J. GROBLER AND B. DE PAGTER, *Operators representable as multiplication-conditional expectation operators*, J. Operator Theory **48** (2002), 15–40.
[10] S. JUNG, E. KO AND J. E. LEE, *Remarks on complex symetric operators*, Mediterr. J. Math. (2015), 1–10.
[11] A. LAMBERT, *L^p multipliers and nested sigma-algebras*, Oper. Theory Adv. Appl. **104** (1998), 147–153.
[12] A. LAMBERTA AND S. PETROVIC, *Beyond hyperinvariance for compact operators*, J. Funct. Anal. **219** (2005) 93–108.
[13] SHU-TEH CHEN, MOY, *Characterizations of conditional expectation as a transformation on function spaces*, Pacific J. Math. **4** (1954), 47–63.
[14] S. PETROVIC, *Spectral Radius Algebras and C Contractions*, Proceedings of the American Mathematical Society **12** (2008), 4283–4288.
[15] S. PETROVIC, *Spectral radius algebras, Deddens algebras, and weighted shifts*, Bull. London Math. Soc. **43** (2011) 513–522.
[16] S. PETROVIC, *On the structure of the spectral radius algebras*, J. Operator Theory. **60** (2008), 137–148.
[17] S. PETROVIC, *Spectral radius algebras and C_0 contractions*, Proc, Amer, Math. Soc. **12** (2008), 4283–4288.
[18] S. PETROVIC, *Spectral radius algebras and C_0 contractions II*, J. Aust. Math. Soc. **89** (2010), 91–104.
[19] S. PETROVIC, *Spectral radius algebras and shift*, Proc, Amer, Math. Soc. **5** (2010), 1639–1644.
[20] M. M. RAO, *Conditional measure and applications*, Marcel Dekker, New York, 1993.

(Received May 20, 2016)

Y. Estaremi and M. R. Jabbarzadeh
Department of Mathematics
Payame Noor University
P. O. Box: 19395-3697, Tehran, Iran
e-mail: yestaremi@pnu.ac.ir

M. R. Jabbarzadeh
Faculty of Mathematical Sciences
University of Tabriz
P. O. Box: 5166615648, Tabriz, Iran
e-mail: mjabbar@tabrizu.ac.ir