Operators and Matrices
Volume 11, Number 4 (2017), 929–939 doi:10.7153/oam-11-59

INVERSE AND MOORE–PENROSE INVERSE OF TOEPLITZ MATRICES WITH CLASSICAL HORADAM NUMBERS

SHOUQIANG SHEN, WEIJUN LIU AND LIHUA FENG

(*Communicated by T. S. S. R. K. Rao*)

Abstract. For integers s,k with $s \leqslant 0$ and $k \geqslant 0$, we define a class of lower triangular Toeplitz matrices $\mathscr{U}_n^{(s,k)}$ of type (s,k), whose non-zero entries are the classical Horadam numbers $U_i^{(a,b)}$. In this paper, we derive a convolution formula containing the Horadam numbers. Using this formula, we obtain several combinatorial identities involving the Horadam numbers and the generalized Fibonacci numbers. In addition, we derive the inverse of the lower triangular Toeplitz matrix $\mathscr{U}_n^{(0,k)}$ and the Moore-Penrose inverse of the strictly lower triangular Toeplitz matrix $\mathscr{U}_n^{(s,k)} (s < 0)$ by utilizing only the Horadam numbers.

1. Introduction

Let $\mathscr{C}^{m\times n}$ be the set of $m \times n$ complex matrices, for every $\mathscr{A} \in \mathscr{C}^{m\times n}$, the Moore-Penrose inverse of matrix $\mathscr{A}$ is the unique $n \times m$ matrix $\mathscr{A}^\dagger$ with the following four properties:

$$\mathscr{A}\mathscr{A}^\dagger\mathscr{A} = \mathscr{A}, \quad \mathscr{A}^\dagger\mathscr{A}\mathscr{A}^\dagger = \mathscr{A}^\dagger, \quad (\mathscr{A}\mathscr{A}^\dagger)^* = \mathscr{A}\mathscr{A}^\dagger, \quad (\mathscr{A}^\dagger\mathscr{A})^* = \mathscr{A}^\dagger\mathscr{A},$$

where $\mathscr{A}^*$ denotes the conjugate transpose of $\mathscr{A}$.

The Moore-Penrose inverse, also called pseudoinverse, is one fundamental concept in matrix theory, and there are several methods for its computation [6, 3]. The most commonly implemented method in programming languages is the Singular Value Decomposition (SVD) method, that is implemented, for example, in the "pinv" function from Matlab. This method although is very accurate, but time consuming when the matrix is large. There also exist several other well-known means including the Greville's algorithm, the full rank QR factorization by Gram-Schmidt orthonormalization (GSO), and iterative methods of various orders [6].

Up to now, many kinds of generalizations of the Fibonacci and Lucas sequences have been described and studied in the literature [4, 5, 15, 13]. In this paper, we aim

Mathematics subject classification (2010): 15A09, 11B39, 05A19.
Keywords and phrases: Inverse, Moore-Penrose inverse, Horadam number, generalized Fibonacci number, Toeplitz matrix.
This research was supported by NSFC (Nos. 11671402, 11371207, 11301302), Hunan Provincial Natural Science Foundation (2016JJ2138) and Mathematics and Interdisciplinary Sciences Project of CSU, the Fundamental Research Funds for the Central Universities of CSU and the Key Research Project of Higher school in Henan Province (No. 15A110040).

to use the classical Horadam sequence $\{U_n^{(a,b)}\}_{n\in N}$ for the generalization, which is defined as [13],

$$U_n^{(a,b)} = AU_{n-1}^{(a,b)} + BU_{n-2}^{(a,b)},\ \ U_0^{(a,b)} = a,\ \ U_1^{(a,b)} = b, \tag{1}$$

where $a,b \in R$ and $A^2+4B>0$. Obviously, if we choose $A=B=1$ in (1), then we obtain the well-known generalized Fibonacci sequence $\{G_n\}_{n\in N}$ [14].

For the Horadam sequence $\{U_n^{(a,b)}\}_{n\in N}$, the following generalization of the Binet's formula for the Fibonacci number holds [13],

$$U_n^{(a,b)} = c_1\alpha^n + c_2\beta^n, \tag{2}$$

where

$$c_1 = \frac{a(A^2+4B)+(2b-aA)\sqrt{A^2+4B}}{2(A^2+4B)},\ \ c_2 = \frac{a(A^2+4B)-(2b-aA)\sqrt{A^2+4B}}{2(A^2+4B)},$$

$$\alpha = \frac{A+\sqrt{A^2+4B}}{2},\ \ \beta = \frac{A-\sqrt{A^2+4B}}{2}.$$

The Toeplitz matrices belong to a family of special matrices of great interests, they are arised in scientific computing and engineering such as image processing, time series analysis and the control theory [2, 7]. In recent years, the investigation of some special Toeplitz matrices attracts much attention. Akbulak and Bozkurt [1] found lower and upper bounds for the spectral norms of Toeplitz matrices with entries the classical Fibonacci and Lucas numbers. Shen [11] generalized the results of [1]. Lee et al. [8, 9] gave the inverse and Cholesky factorization of the $n\times n$ Fibonacci matrix $\mathscr{F}_n = [f_{i,j}]$ $(i,j=1,2,\cdots,n)$ whose entries are defined as

$$f_{i,j} = \begin{cases} F_{i-j+1}, & \text{if} \quad i-j+1 \geqslant 0, \\ 0, & \text{if} \quad i-j+1 < 0. \end{cases} \tag{3}$$

They also studied the relations between the Pascal matrix and the Fibonacci matrix. Analogous to the Fibonacci matrix, the Lucas matrix $\mathscr{L}_n = [l_{i,j}]$ $(i,j=1,2,\cdots,n)$ of the order n is defined as

$$l_{i,j} = \begin{cases} L_{i-j+1}, & \text{if} \quad i-j \geqslant 0, \\ 0, & \text{if} \quad i-j < 0. \end{cases} \tag{4}$$

The authors in [16] investigated the inverse of the matrix $\mathscr{L}_n$ and the relations between the Lucas matrix and the generalized Pascal matrices.

Obviously, the matrices given in expressions (3) and (4) are lower triangular Toeplitz matrices. Further, Stanimirović et al. [13] defined an $n\times n$ Toeplitz matrix $\mathscr{U}_n^{(a,b,s)} = [u_{i,j}^{(a,b,s)}]$ $(i,j=1,2,\cdots,n)$ of type s, where

$$u_{i,j}^{(a,b,s)} = \begin{cases} U_{i-j+1}^{(a,b)}, & \text{if} \quad i-j+s \geqslant 0, \\ 0, & \text{if} \quad i-j+s < 0. \end{cases}$$

When $A = B = 1$, the matrix $\mathscr{U}_n^{(a,b,s)}$ is the generalized Fibonacci matrix $\mathscr{F}_n^{(a,b,s)}$ of type s, they [13] also derived the inverse of the matrix $\mathscr{U}_n^{(a,b,0)}$, and considered correlations between the matrix $\mathscr{U}_n^{(a,b,0)}$ and the generalized Pascal matrices of the first and the second kinds. In addition, Shen and He [12] presented an explicit formula for the Moore-Penrose inverse of the matrix $\mathscr{U}_n^{(a,b,-1)}$, which generalized the main results from [10].

We organize this paper as follows. In Section 2, A convolution formula containing the Horadam numbers $U_i^{(a,b)}$ is given. Using this formula, we get several combinatorial identities involving the Horadam numbers and the generalized Fibonacci numbers. In Section 3, we define a class of lower triangular Toeplitz matrices $\mathscr{U}_n^{(s,k)}$ of type (s,k), whose non-zero elements are the Horadam numbers. Afterwards, we derive the inverse of the lower triangular Toeplitz matrix $\mathscr{U}_n^{(0,k)}$ and the Moore-Penrose inverse of the strictly lower triangular Toeplitz matrix $\mathscr{U}_n^{(s,k)} (s < 0)$, which are only related to the Horadam numbers. As the special cases of our main result, we obtain several results from [12, 13, 14] as corollaries.

2. Combinatorial identities based on the convolution

In this section, we first introduce the notion of the convolution, then obtain a convolution formula containing the Horadam numbers $U_i^{(a,b)}$. Using this convolution formula, we obtain some known combinatorial identities involving the Horadam numbers and the generalized Fibonacci numbers from [13, 14].

Let $u = \{u_1, u_2, \cdots, u_n\}$ and $v = \{v_1, v_2, \cdots, v_n\}$ be two lists with n elements, then the convolution $*$ of u and v is

$$u * v = \sum_{i=1}^{n} u_i v_{n-i+1}.$$

In the next theorem, we present a convolution formula of a list containing the Horadam numbers $U_i^{(a,b)}$ with corresponding powers of $\frac{-BU_m^{(a,b)}}{U_{m+1}^{(a,b)}}$, where $m \geqslant 0$ and $U_{m+1}^{(a,b)} \neq 0$. For convenience, we use the following notation

$$Con(r,m) := \left\{U_{m+1}^{(a,b)}, \cdots, U_{m+r-1}^{(a,b)}\right\} * \left\{1, \frac{-BU_m^{(a,b)}}{U_{m+1}^{(a,b)}}, \cdots, \left(\frac{-BU_m^{(a,b)}}{U_{m+1}^{(a,b)}}\right)^{r-2}\right\}.$$

THEOREM 2.1. *Let m, r be two integers with $m \geqslant 0$ and $r \geqslant 2$, and $\{U_n^{(a,b)}\}_{n\in N}$ be the Horadam sequence. If $B \neq 0$, $U_{m+1}^{(a,b)} \neq 0$ and $\alpha, \beta \neq \frac{-BU_m^{(a,b)}}{U_{m+1}^{(a,b)}}$, then we have*

$$Con(r,m) = U_{m+1}^{(a,b)} \cdot \frac{U_m^{(a,b)} U_{m+r}^{(a,b)} - U_{m+1}^{(a,b)} U_{m+r-1}^{(a,b)}}{U_m^{(a,b)} U_{m+2}^{(a,b)} - [U_{m+1}^{(a,b)}]^2}. \tag{5}$$

Proof. Obviously (5) is right if $U_m^{(a,b)} = 0$. So we consider the case $U_m^{(a,b)} \neq 0$ in the sequel. Since $\alpha,\beta \neq \frac{-BU_m^{(a,b)}}{U_{m+1}^{(a,b)}}$, then applying (2) and simple transformations, we obtain

$$\begin{aligned}
Con(r,m) &= \sum_{l=0}^{r-2} U_{l+m+1}^{(a,b)} \left(\frac{-BU_m^{(a,b)}}{U_{m+1}^{(a,b)}}\right)^{r-l-2} \\
&= \sum_{l=0}^{r-2} (c_1\alpha^{l+m+1} + c_2\beta^{l+m+1}) \left(\frac{-BU_m^{(a,b)}}{U_{m+1}^{(a,b)}}\right)^{r-l-2} \\
&= c_1\alpha^{m+1}\left[\frac{-BU_m^{(a,b)}}{U_{m+1}^{(a,b)}}\right]^{r-2} \sum_{l=0}^{r-2}\left(\frac{-\alpha U_{m+1}^{(a,b)}}{BU_m^{(a,b)}}\right)^{l} + c_2\beta^{m+1}\left[\frac{-BU_m^{(a,b)}}{U_{m+1}^{(a,b)}}\right]^{r-2} \sum_{l=0}^{r-2}\left(\frac{-\beta U_{m+1}^{(a,b)}}{BU_m^{(a,b)}}\right)^{l} \\
&= BU_m^{(a,b)}\left[\frac{-BU_m^{(a,b)}}{U_{m+1}^{(a,b)}}\right]^{r-2}\left(c_1\alpha^{m+1}\frac{1-\left[\frac{-\alpha U_{m+1}^{(a,b)}}{BU_m^{(a,b)}}\right]^{r-1}}{BU_m^{(a,b)}+\alpha U_{m+1}^{(a,b)}} + c_2\beta^{m+1}\frac{1-\left[\frac{-\beta U_{m+1}^{(a,b)}}{BU_m^{(a,b)}}\right]^{r-1}}{BU_m^{(a,b)}+\beta U_{m+1}^{(a,b)}}\right) \\
&= \frac{c_1\left[(-1)^{r-2}\alpha^{m+1}\cdot\frac{(BU_m^{(a,b)})^{r-1}}{(U_{m+1}^{(a,b)})^{r-2}} + \alpha^{m+r}U_{m+1}^{(a,b)}\right](BU_m^{(a,b)}+\beta U_{m+1}^{(a,b)})}{(BU_m^{(a,b)}+\alpha U_{m+1}^{(a,b)})(BU_m^{(a,b)}+\beta U_{m+1}^{(a,b)})} \\
&\quad + \frac{c_2\left[(-1)^{r-2}\beta^{m+1}\cdot\frac{(BU_m^{(a,b)})^{r-1}}{(U_{m+1}^{(a,b)})^{r-2}} + \beta^{m+r}U_{m+1}^{(a,b)}\right](BU_m^{(a,b)}+\alpha U_{m+1}^{(a,b)})}{(BU_m^{(a,b)}+\alpha U_{m+1}^{(a,b)})(BU_m^{(a,b)}+\beta U_{m+1}^{(a,b)})}.
\end{aligned}$$

From $\alpha\beta = -B$, $\alpha+\beta = A$, we get

$$\left(BU_m^{(a,b)} + \alpha U_{m+1}^{(a,b)}\right)\left(BU_m^{(a,b)} + \beta U_{m+1}^{(a,b)}\right) = B\left(U_m^{(a,b)}U_{m+2}^{(a,b)} - [U_{m+1}^{(a,b)}]^2\right).$$

The numerator of $Con(r,m)$ can be transformed after simple algebraic transformations in the form

$$\begin{aligned}
&(-1)^{r-2}c_1\alpha^{m+1}\left(\frac{(BU_m^{(a,b)})^r}{(U_{m+1}^{(a,b)})^{r-2}} + \beta\frac{(BU_m^{(a,b)})^{r-1}}{(U_{m+1}^{(a,b)})^{r-3}}\right) + c_1\alpha^{m+r}\left(BU_{m+1}^{(a,b)}U_m^{(a,b)} + \beta[U_{m+1}^{(a,b)}]^2\right) \\
&+(-1)^{r-2}c_2\beta^{m+1}\left(\frac{(BU_m^{(a,b)})^r}{(U_{m+1}^{(a,b)})^{r-2}} + \alpha\frac{(BU_m^{(a,b)})^{r-1}}{(U_{m+1}^{(a,b)})^{r-3}}\right) + c_2\beta^{m+r}\left(BU_{m+1}^{(a,b)}U_m^{(a,b)} + \alpha[U_{m+1}^{(a,b)}]^2\right).
\end{aligned}$$

Applying Binet's formula (2), then we obtain

$$
\begin{aligned}
Con(r,m) = & \frac{1}{B(U_m^{(a,b)}U_{m+2}^{(a,b)} - [U_{m+1}^{(a,b)}]^2)} \\
& \times \Bigg[(-1)^{r-2} \left(U_{m+1}^{(a,b)} \frac{(BU_m^{(a,b)})^r}{(U_{m+1}^{(a,b)})^{r-2}} - BU_m^{(a,b)} \cdot \frac{(BU_m^{(a,b)})^{r-1}}{(U_{m+1}^{(a,b)})^{r-3}} \right) \\
& + BU_{m+r}^{(a,b)}U_{m+1}^{(a,b)}U_m^{(a,b)} - BU_{m+r-1}^{(a,b)}[U_{m+1}^{(a,b)}]^2 \Bigg] \\
= & \, U_{m+1}^{(a,b)} \cdot \frac{U_m^{(a,b)}U_{m+r}^{(a,b)} - U_{m+1}^{(a,b)}U_{m+r-1}^{(a,b)}}{U_m^{(a,b)}U_{m+2}^{(a,b)} - [U_{m+1}^{(a,b)}]^2}.
\end{aligned}
$$

Thus, the proof is completed. □

If we take $m = 0$ in Theorem 2.1, then we obtain

COROLLARY 2.2. [13] *For the Horadam sequence* $\{U_n^{(a,b)}\}_{n \in N}$ *satisfying* $b \neq 0$, $\alpha, \beta \neq \frac{-aB}{b}$, *and for two arbitrary integers* i, j *satisfying* $i \geqslant j+2$, *we have*

$$
(a^2B + abA - b^2) \sum_{k=j+2}^{i} (-1)^{k-j} \frac{a^{k-j-2}B^{k-j-1}}{b^{k-j+1}} U_{i-k+1}^{(a,b)} = \frac{aB}{b^2} U_{i-j}^{(a,b)} - \frac{B}{b} U_{i-j-1}^{(a,b)}. \quad (6)
$$

Proof. Obviously, (6) is valid for $B = 0$. In the case $B \neq 0$, we select $m = 0$ in Theorem 2.1, then it follows that

$$
\sum_{l=2}^{r} U_{l-1}^{(a,b)} \left(\frac{-aB}{b} \right)^{r-l} = \frac{b\left(aU_r^{(a,b)} - bU_{r-1}^{(a,b)}\right)}{a^2B + abA - b^2}.
$$

If $r = i - j$, then together with some simple transformations, the result can be derived by applying identity

$$
\sum_{l=2}^{r} U_{l-1}^{(a,b)} \left(\frac{-aB}{b} \right)^{r-l} = \sum_{k=j+2}^{i} U_{i-k+1}^{(a,b)} \left(\frac{-aB}{b} \right)^{k-j-2}.
$$

So we complete the proof. □

If we take $A = B = 1$ in Theorem 2.1, then we obtain the following combinatorial identity involving the generalized Fibonacci numbers G_i.

COROLLARY 2.3. [14] *Let* m, r *be two integers with* $m \geqslant 0$ *and* $r \geqslant 2$, *and* $\{G_n\}_{n \in N}$ *be the generalized Fibonacci sequence. If* $G_{m+1} \neq 0$ *and* $\alpha, \beta \neq \frac{-G_m}{G_{m+1}}$, *then we have*

$$
\sum_{l=2}^{r} G_{l+m-1} \left(\frac{-G_m}{G_{m+1}} \right)^{r-l} = G_{m+1} \cdot \frac{G_m G_{m+r} - G_{m+1} G_{m+r-1}}{G_m G_{m+2} - (G_{m+1})^2}.
$$

3. Inverse and Moore-Penrose inverse of matrices $\mathscr{U}_n^{(s,k)}$

In this section, we will study a class of lower triangular Toeplitz matrices $\mathscr{U}_n^{(s,k)}$ of type (s,k), whose non-zero entries are the Horadam numbers satisfying $U_{k+1}^{(a,b)} \neq 0$.

DEFINITION 3.1. Let s, k be two integers with $s \leqslant 0$ and $k \geqslant 0$, and $\{U_n^{(a,b)}\}_{n\in N}$ be the Horadam sequence satisfying $U_{k+1}^{(a,b)} \neq 0$. The $n\times n$ matrix $\mathscr{U}_n^{(s,k)} = [u_{i,j}^{(s,k)}]$ $(i,j = 1,2,\cdots,n)$ of type (s,k) is defined as

$$u_{i,j}^{(s,k)} = \begin{cases} U_{i-j+s+k+1}^{(a,b)}, & \text{if} \quad i-j+s \geqslant 0, \\ 0, & \text{if} \quad i-j+s < 0. \end{cases}$$

Obviously, if $A = B = 1$, the matrix $\mathscr{U}_n^{(s,k)}$ is just the generalized Fibonacci matrix $\mathscr{G}_n^{(s,k)}$ of type (s,k) reported in [14]. If $s = 0$, we use notation $\mathscr{U}_n^{(k)} := \mathscr{U}_n^{(0,k)}$. Bearing this in mind together with the specific structure of this matrix, $\mathscr{U}_n^{(s,k)}$ can be rewritten as the following block form

$$\mathscr{U}_n^{(s,k)} = \begin{pmatrix} \mathscr{O}_{(-s)\times(n+s)} & \mathscr{O}_{(-s)\times(-s)} \\ \mathscr{U}_{n+s}^{(k)} & \mathscr{O}_{(n+s)\times(-s)} \end{pmatrix},$$

where $\mathscr{O}_{p\times q}$ denotes the $p\times q$ zero matrix, and

$$\mathscr{U}_{n+s}^{(k)} = \begin{pmatrix} U_{k+1}^{(a,b)} & 0 & \cdots & 0 \\ U_{k+2}^{(a,b)} & U_{k+1}^{(a,b)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ U_{k+n+s}^{(a,b)} & U_{k+n+s-1}^{(a,b)} & \cdots & U_{k+1}^{(a,b)} \end{pmatrix}. \tag{7}$$

LEMMA 3.2. *Let r be an arbitrary positive integer, $\{U_n^{(a,b)}\}_{n\in N}$ be the Horadam sequence with $U_{k+1}^{(a,b)} \neq 0$. If $\alpha = \frac{-BU_k^{(a,b)}}{U_{k+1}^{(a,b)}}$ or $\beta = \frac{-BU_k^{(a,b)}}{U_{k+1}^{(a,b)}}$, then we have*

$$U_{k+1}^{(a,b)}U_{r+k+1}^{(a,b)} - U_{k+2}^{(a,b)}U_{r+k}^{(a,b)} = 0. \tag{8}$$

Proof. Obviously (8) is valid for $B = 0$. In the rest we consider $B \neq 0$. If $\alpha = \frac{-BU_k^{(a,b)}}{U_{k+1}^{(a,b)}}$, then we have

$$\frac{A+\sqrt{A^2+4B}}{2} = \frac{-BU_k^{(a,b)}}{U_{k+1}^{(a,b)}},$$

After applying simple algebraic transformations, we get

$$1 = \frac{B(U_k^{(a,b)})^2}{(U_{k+1}^{(a,b)})^2} + \frac{AU_k^{(a,b)}}{U_{k+1}^{(a,b)}} = \frac{U_k^{(a,b)}(BU_k^{(a,b)} + AU_{k+1}^{(a,b)})}{(U_{k+1}^{(a,b)})^2} = \frac{U_k^{(a,b)}U_{k+2}^{(a,b)}}{(U_{k+1}^{(a,b)})^2}.$$

Hence $U_k^{(a,b)}U_{k+2}^{(a,b)}=(U_{k+1}^{(a,b)})^2$.

On the other hand, since $\alpha+\beta=A$ and $\alpha-\beta=\sqrt{A^2+4B}$, we obtain

$$c_1=\frac{a}{2}+\frac{2b-aA}{2\sqrt{A^2+4B}}=\frac{a}{2}+\frac{2b-a(\alpha+\beta)}{2(\alpha-\beta)}=\frac{b-a\beta}{\alpha-\beta},$$
$$c_2=\frac{a}{2}-\frac{2b-aA}{2\sqrt{A^2+4B}}=\frac{a}{2}-\frac{2b-a(\alpha+\beta)}{2(\alpha-\beta)}=\frac{a\alpha-b}{\alpha-\beta}.$$

By applying Binet's formula (2), one has

$$U_k^{(a,b)}U_{k+2}^{(a,b)}-\left(U_{k+1}^{(a,b)}\right)^2=c_1c_2(\alpha\beta)^k(\alpha-\beta)^2=(-B)^k(a^2B+abA-b^2).$$

Therefore it follows that

$$\begin{aligned}U_{k+1}^{(a,b)}U_{r+k+1}^{(a,b)}-U_{k+2}^{(a,b)}U_{r+k}^{(a,b)}&=c_1c_2(\alpha\beta)^{k+1}(\alpha-\beta)(\alpha^{r-1}-\beta^{r-1})\\&=\frac{(-B)^{k+1}(a^2B+abA-b^2)}{\sqrt{A^2+4B}}(\alpha^{r-1}-\beta^{r-1})=0.\end{aligned}$$

Similarly, we can verify the validity of (8) for $\beta=\frac{-BU_k^{(a,b)}}{U_{k+1}^{(a,b)}}$. Thus the proof is completed. $\square$

THEOREM 3.3. *Let $\{U_n^{(a,b)}\}_{n\in N}$ be the Horadam sequence satisfying $U_{k+1}^{(a,b)}\neq 0$. Then the inverse of the matrix $\mathscr{U}_n^{(k)}$ is the matrix $\mathscr{R}_n=[r_{i,j}]_{n\times n}$ defined by*

$$r_{i,j}=\begin{cases}\frac{(-1)^kB^{k+1}(a^2B+abA-b^2)}{(U_{k+1}^{(a,b)})^3}\left(\frac{-BU_k^{(a,b)}}{U_{k+1}^{(a,b)}}\right)^{i-j-2}, & \text{if } i>j+1,\\ -\frac{U_{k+2}^{(a,b)}}{(U_{k+1}^{(a,b)})^2}, & \text{if } i=j+1,\\ \frac{1}{U_{k+1}^{(a,b)}}, & \text{if } i=j,\\ 0, & \text{otherwise,}\end{cases}$$

where k is an arbitrary integer satisfying $0\leqslant k<n$.

Proof. Let us denote $\mathscr{U}_n^{(k)}=[u_{i,j}^{(k)}]$, and $\mathscr{H}_n=[h_{i,j}]$ as the matrix product $\mathscr{U}_n^{(k)}\mathscr{R}_n$. Clearly $h_{i,j}=0$ for $i<j$.

For the case $i=j$, we get

$$h_{i,j}=u_{i,i}^{(k)}r_{i,i}=U_{k+1}^{(a,b)}\cdot\frac{1}{U_{k+1}^{(a,b)}}=1.$$

For $i=j+1$, we have

$$\begin{aligned}h_{i,j}&=u_{j+1,j}^{(k)}r_{j,j}+u_{j+1,j+1}^{(k)}r_{j+1,j}\\&=U_{k+2}^{(a,b)}\cdot\frac{1}{U_{k+1}^{(a,b)}}+U_{k+1}^{(a,b)}\cdot\left[-\frac{U_{k+2}^{(a,b)}}{(U_{k+1}^{(a,b)})^2}\right]\\&=0.\end{aligned}$$

For $i > j+1$, we obtain

$$\begin{aligned}
h_{i,j} = &\ u_{i,j}^{(k)} r_{j,j} + u_{i,j+1}^{(k)} r_{j+1,j} + \sum_{l=2}^{i-j} u_{i,i-l+2}^{(k)} r_{i-l+2,j} \\
= &\ \frac{u_{i,j}^{(k)}}{U_{k+1}^{(a,b)}} - \frac{U_{k+2}^{(a,b)}}{(U_{k+1}^{(a,b)})^2} u_{i,j+1}^{(k)} \\
&+ \frac{(-1)^k B^{k+1}(a^2B + abA - b^2)}{(U_{k+1}^{(a,b)})^3} \sum_{l=2}^{i-j} u_{i,i-l+2}^{(k)} \left(\frac{-BU_k^{(a,b)}}{U_{k+1}^{(a,b)}} \right)^{i-j-l}.
\end{aligned}$$

Let $r = i-j$. Then we have $u_{i,j}^{(k)} = U_{k+r+1}^{(a,b)}$ and $u_{i,j+1}^{(k)} = U_{k+r}^{(a,b)}$, hence

$$\begin{aligned}
h_{i,j} = &\ \frac{U_{k+r+1}^{(a,b)}}{U_{k+1}^{(a,b)}} - \frac{U_{k+2}^{(a,b)}}{(U_{k+1}^{(a,b)})^2} U_{k+r}^{(a,b)} \\
&+ \frac{(-1)^k B^{k+1}(a^2B + abA - b^2)}{(U_{k+1}^{(a,b)})^3} \sum_{l=2}^{r} U_{l+k-1}^{(a,b)} \left(\frac{-BU_k^{(a,b)}}{U_{k+1}^{(a,b)}} \right)^{r-l} \\
= &\ \frac{U_{k+1}^{(a,b)} U_{k+r+1}^{(a,b)} - U_{k+2}^{(a,b)} U_{k+r}^{(a,b)}}{(U_{k+1}^{(a,b)})^2} + \frac{(-1)^k B^{k+1}(a^2B + abA - b^2)}{(U_{k+1}^{(a,b)})^3} Con(r,k).
\end{aligned}$$

Obviously $h_{i,j} = 0$ for $B = 0$. If $\alpha = \frac{-BU_k^{(a,b)}}{U_{k+1}^{(a,b)}}$ or $\beta = \frac{-BU_k^{(a,b)}}{U_{k+1}^{(a,b)}}$, by applying (8), one can verify $h_{i,j} = 0$.

If $B \neq 0$ and $\alpha, \beta \neq \frac{-BU_k^{(a,b)}}{U_{k+1}^{(a,b)}}$, using the result of Theorem 2.1, we obtain

$$\begin{aligned}
h_{i,j} = &\ \frac{U_{k+1}^{(a,b)} U_{k+r+1}^{(a,b)} - U_{k+2}^{(a,b)} U_{k+r}^{(a,b)}}{(U_{k+1}^{(a,b)})^2} \\
&+ \frac{(-1)^k B^{k+1}(a^2B + abA - b^2)}{(U_{k+1}^{(a,b)})^3} \cdot \left(\frac{U_{k+1}^{(a,b)} [U_k^{(a,b)} U_{k+r}^{(a,b)} - U_{k+1}^{(a,b)} U_{k+r-1}^{(a,b)}]}{U_k^{(a,b)} U_{k+2}^{(a,b)} - [U_{k+1}^{(a,b)}]^2} \right) \\
= &\ \frac{U_{k+1}^{(a,b)} U_{k+r+1}^{(a,b)} - U_{k+2}^{(a,b)} U_{k+r}^{(a,b)}}{(U_{k+1}^{(a,b)})^2} + \frac{B(U_k^{(a,b)} U_{k+r}^{(a,b)} - U_{k+1}^{(a,b)} U_{k+r-1}^{(a,b)})}{(U_{k+1}^{(a,b)})^2} \\
= &\ \frac{U_{k+1}^{(a,b)} (U_{k+r+1}^{(a,b)} - BU_{k+r-1}^{(a,b)}) - U_{k+r}^{(a,b)} (U_{k+2}^{(a,b)} - BU_k^{(a,b)})}{(U_{k+1}^{(a,b)})^2} \\
= &\ 0.
\end{aligned}$$

So we prove that $\mathscr{H}_n$ is the $n \times n$ identity matrix. Similarly, one can verify $\mathscr{R}_n \mathscr{U}_n^{(k)}$ is also the identity matrix. Thus, the proof is completed. $\square$

If we take $k=0$ in Theorem 3.3, we obtain the inverse of the nonsingular matrix $\mathscr{U}_n^{(a,b,0)}$.

COROLLARY 3.4. [13] *Let $\{U_n^{(a,b)}\}_{n\in N}$ be the Horadam sequence satisfying $b\neq 0$. Then the inverse of the matrix $\mathscr{U}_n^{(a,b,0)}$ is the matrix $\mathscr{X}_n=[x_{i,j}]_{n\times n}$ defined by*

$$x_{i,j}=\begin{cases}(-1)^{i-j}\cdot\frac{a^2B+abA-b^2}{b^{i-j+1}}a^{i-j-2}B^{i-j-1}, & \text{if } i>j+1,\\ -\frac{aB+bA}{b^2}, & \text{if } i=j+1,\\ \frac{1}{b}, & \text{if } i=j,\\ 0, & \text{otherwise.}\end{cases}$$

If we take $A=B=1$ in Theorem 3.3, we get the inverse of the matrix $\mathscr{G}_n^{(k)}$ involving the generalized Fibonacci numbers.

COROLLARY 3.5. [14] *Let $\{G_n\}_{n\in N}$ be the generalized Fibonacci sequence satisfying $G_{k+1}\neq 0$. Then the inverse of the matrix $\mathscr{G}_n^{(k)}$ is the matrix $\mathscr{Y}_n=[y_{i,j}]_{n\times n}$ defined by*

$$y_{i,j}=\begin{cases}\frac{(-1)^k(a^2+ab-b^2)}{(G_{k+1})^3}\left(\frac{-G_k}{G_{k+1}}\right)^{i-j-2}, & \text{if } i>j+1,\\ -\frac{G_{k+2}}{(G_{k+1})^2}, & \text{if } i=j+1,\\ \frac{1}{G_{k+1}}, & \text{if } i=j,\\ 0, & \text{otherwise,}\end{cases}$$

where k is an arbitrary integer satisfying $0\leqslant k<n$.

At last, we derive the Moore-Penrose inverse for the singular matrix $\mathscr{U}_n^{(s,k)}$ $(s<0)$ given by Definition 3.1.

THEOREM 3.6. *Let $s<0$, $k\geqslant 0$ be arbitrary integers, and $\{U_n^{(a,b)}\}_{n\in N}$ be the Horadam sequence satisfying $U_{k+1}^{(a,b)}\neq 0$. Then the Moore-Penrose inverse of the matrix $\mathscr{U}_n^{(s,k)}$ is the $n\times n$ block matrix $\mathscr{Q}_n$ given by*

$$\mathscr{Q}_n=\begin{pmatrix}\mathscr{O}_{(n+s)\times(-s)} & \mathscr{R}_{n+s}\\ \mathscr{O}_{(-s)\times(-s)} & \mathscr{O}_{(-s)\times(n+s)}\end{pmatrix},$$

where $\mathscr{R}_{n+s}=[r_{i,j}]$ is an $(n+s)\times(n+s)$ matrix given by

$$r_{i,j}=\begin{cases}\frac{(-1)^kB^{k+1}(a^2B+abA-b^2)}{(U_{k+1}^{(a,b)})^3}\left(\frac{-BU_k^{(a,b)}}{U_{k+1}^{(a,b)}}\right)^{i-j-2}, & \text{if } i>j+1,\\ -\frac{U_{k+2}^{(a,b)}}{(U_{k+1}^{(a,b)})^2}, & \text{if } i=j+1,\\ \frac{1}{U_{k+1}^{(a,b)}}, & \text{if } i=j,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Since the matrix $\mathscr{U}_n^{(s,k)}$ can be expressed as

$$\mathscr{U}_n^{(s,k)} = \begin{pmatrix} \mathscr{O}_{(-s)\times(n+s)} & \mathscr{O}_{(-s)\times(-s)} \\ \mathscr{U}_{n+s}^{(k)} & \mathscr{O}_{(n+s)\times(-s)} \end{pmatrix},$$

where $\mathscr{U}_{n+s}^{(k)}$ is given by (7). Hence the Moore-Penrose inverse of the matrix $\mathscr{U}_n^{(s,k)}$ has the following representation

$$[\mathscr{U}_n^{(s,k)}]^{\dagger} = \begin{pmatrix} \mathscr{O}_{(n+s)\times(-s)} & [\mathscr{U}_{n+s}^{(k)}]^{-1} \\ \mathscr{O}_{(-s)\times(-s)} & \mathscr{O}_{(-s)\times(n+s)} \end{pmatrix}.$$

Thus, we get the desired result by using the result of Theorem 3.3. □

If we take $s=-1$, $k=1$ in Theorem 3.6, we obtain the Moore-Penrose inverse of the matrix $\mathscr{U}_n^{(a,b,-1)}$.

COROLLARY 3.7. [12] *Let $\{U_n^{(a,b)}\}_{n\in N}$ be the Horadam sequence satisfying $aB+bA\neq 0$. Then the Moore-Penrose inverse of the matrix $\mathscr{U}_n^{(a,b,-1)}$ is the matrix $\mathscr{V}_n=[v_{i,j}]_{n\times n}$ defined by*

$$v_{i,j} = \begin{cases} -(-bB)^{i-j-1}\cdot\frac{B^2(a^2B+abA-b^2)}{(aB+bA)^{i-j+2}}, & \textit{if} \quad i>j,\ i\neq n,\ j\neq 1, \\ -\frac{aAB+(A^2+B)b}{(aB+bA)^2}, & \textit{if} \quad i=j,\ i\notin\{1,n\}, \\ \frac{1}{aB+bA}, & \textit{if} \quad i+1=j, \\ 0, & \textit{otherwise.} \end{cases}$$

If we take $A=B=1$ in Theorem 3.6, we get the Moore-Penrose inverse of the singular matrix $\mathscr{G}_n^{(s,k)}(s<0)$.

COROLLARY 3.8. [14] *Let $s<0$, $k\geqslant 0$ be arbitrary integers, and $\{G_n\}_{n\in N}$ be the generalized Fibonacci sequence satisfying $G_{k+1}\neq 0$. Then the Moore-Penrose of the matrix $\mathscr{G}_n^{(s,k)}$ is the $n\times n$ block matrix $\mathscr{W}_n$ given by*

$$\mathscr{W}_n = \begin{pmatrix} \mathscr{O}_{(n+s)\times(-s)} & \mathscr{Y}_{n+s} \\ \mathscr{O}_{(-s)\times(-s)} & \mathscr{O}_{(-s)\times(n+s)} \end{pmatrix},$$

where $\mathscr{Y}_{n+s}=[y_{i,j}]$ is an $(n+s)\times(n+s)$ matrix defined by

$$y_{i,j} = \begin{cases} \frac{(-1)^k(a^2+ab-b^2)}{(G_{k+1})^3}\left(\frac{-G_k}{G_{k+1}}\right)^{i-j-2}, & \textit{if} \quad i>j+1, \\ -\frac{G_{k+2}}{(G_{k+1})^2}, & \textit{if} \quad i=j+1, \\ \frac{1}{G_{k+1}}, & \textit{if} \quad i=j, \\ 0, & \textit{otherwise.} \end{cases}$$

REFERENCES

[1] M. AKBULAK, D. BOZKURT, *On the norms of Toeplitz matrices involving Fibonacci and Lucas numbers*, Hacet. J. Math. Stat. **37** (2008) 89–95.

[2] R. CHAN, M. NG, *Conjugate gradient methods for Toeplitz systems*, SIAM Rev. **38** (1996) 427–482.

[3] P. COURRIEU, *Fast computation of Moore-Penrose inverse matrices*, Neural Information Processing-Letters and Reviews **8** (2005) 25–29.

[4] S. FALCÓN, Á. PLAZA, *The k-Fibonacci sequence and the Pascal 2-triangle*, Chaos, Solitons Fract. **33** (2007) 38–49.

[5] S. FALCÓN, Á. PLAZA, *On k-Fibonacci numbers of arithmetic indexes*, Appl. Math. Comput. **208** (2009) 180–185.

[6] A. B. ISRAEL, T. N. E. GREVILLE, *Generalized Inverses: Theory and Applications*, Second Ed., Springer, New York, 2003.

[7] T. KAILATH, A. SAYED, *Displacement structure: Theory and Applications*, SIAM Rev. **37** (1995) 297–386.

[8] G.-Y. LEE, J.-S. KIM, S.-G. LEE, *Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices*, Fibonacci Quart. **40** (2002) 203–211.

[9] G.-Y. LEE, J.-S. KIM, S.-H. CHO, *Some combinatorial identities via Fibonacci numbers*, Discrete Appl. Math. **130** (2003) 527–534.

[10] M. MILADINOVIĆ, P. STANIMIROVIĆ, *Singular case of generalized Fibonacci and Lucas matrices*, J. Korean Math. Soc. **48** (2011) 33–48.

[11] S.-Q. SHEN, *On the norms of Toeplitz matrices involving k-Fibonacci and k-Lucas numbers*, Int. J. Contemp. Math. Science **7** (2012) 363–368.

[12] S.-Q. SHEN, J.-J. HE, *Moore-Penrose inverse of generalized Fibonacci matrix and its applications*, Int. J. Comput. Math. **93** (2016) 1756–1770.

[13] P. STANIMIROVIĆ, J. NIKOLOV, I. STANIMIROVIĆ, *A generalization of Fibonacci and Lucas matrices*, Discrete Appl. Math. **156** (2008) 2606–2619.

[14] P. STANIMIROVIĆ, M. MILADINOVIĆ, *Inversion of the generalized Fibonacci matrix by convolution*, Int. J. Comput. Math. **88** (2011) 1519–1532.

[15] K. USLU, N. TASKARA, H. KOSE, *The generalized k-Fibonacci and k-Lucas numbers*, Ars Combinatoria **99** (2011) 25–32.

[16] Z. ZHANG, Y. ZHANG, *The Lucas matrix and some combinatorial identities*, Indian J. Pure Appl. Math. **38** (2007) 457–465.

(Received October 16, 2016)

Shouqiang Shen
School of Mathematics and Statistics
Central South University
New Campus, Changsha, Hunan, 410083, P. R. China
e-mail: shenshouqiang@126.com

Weijun Liu
School of Mathematics and Statistics
Central South University
New Campus, Changsha, Hunan, 410083, P. R. China
e-mail: wjliu6210@126.com

Lihua Feng
School of Mathematics and Statistics
Central South University
New Campus, Changsha, Hunan, 410083, P. R. China
e-mail: fenglh@163.com