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COMPLEX SYMMETRIC OPERATORS, SKEW SYMMETRIC OPERATORS AND REFLEXIVITY

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Abstract. Let $\mathscr{H}$ be a complex separable infinite-dimensional Hilbert space and C be a conjugation on $\mathscr{H}$. Let $\mathscr{C}$ and $\mathscr{S}$ denote respectively the set of C-symmetric operators and the set of C-skew-symmetric operators on $\mathscr{H}$. It is proved that $\mathscr{C}$ and $\mathscr{S}$ are Roberts orthogonal to each other, and some distance formulas from an operator to the sets $\mathscr{C}$, $\mathscr{S}$ are obtained. We exhibit the annihilating relation between $\mathscr{C}$ and $\mathscr{S}$ by describing their preannihilators. As applications, it is shown that $\mathscr{S}$ is hyperreflexive and not transitive.

1. Introduction

Throughout this paper, we let $\mathscr{H}$ denote a complex separable Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, and $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. Let C be a *conjugation* on $\mathscr{H}$, that is, C is conjugate-linear, invertible, $C^{-1}=C$ and $\langle Cx,Cy\rangle=\langle y,x\rangle$ for all $x,y\in\mathscr{H}$. An operator $T\in\mathscr{B}(\mathscr{H})$ is called C-*symmetric* if $CTC=T^*$, and T is called C-*skew-symmetric* if $CTC=-T^*$. If T is C-symmetric (C-skew-symmetric) for some conjugation C, then T is called complex symmetric (skew symmetric). Complex symmetric operators and skew symmetric operators are respectively natural generalizations of symmetric matrices and skew symmetric matrices in the Hilbert space setting.

The general study of complex symmetric operators was initiated by Garcia, Putinar and Wogen in [7, 8, 9, 10]. The class of complex symmetric operators includes normal operators, Hankel operators, binormal operators, truncated Toeplitz operators and many others. Recently, there has been a lot of work concerning complex symmetric operators. The class of skew symmetric operators is closely related to the study of complex symmetric operators. In view of [17, Lem. 1.4], a complex symmetric operator has many skew symmetric relatives, and vice versa. For example, the self-commutator of a complex symmetric operator is always skew symmetric; moreover, if T is skew symmetric, then T^{2k} is complex symmetric for any positive integer n. This observation provides a new approach to identifying new complex symmetric operators. One can see such an application to Toeplitz operators in [12]. Recently there has been

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growing interest in skew symmetric operators (see [19, 20, 21, 22, 26]); in particular, skew symmetric normal operators, partial isometries, compact operators and weighted shifts are classified (see [17, 16, 22]).

We let CSO and SSO denote respectively the set of complex symmetric operators on $\mathscr{H}$ and the set of skew symmetric operators on $\mathscr{H}$. Lately there has been some interest in the study of CSO and SSO as subsets of $\mathscr{B}(\mathscr{H})$ (see [9, 25, 4, 5, 11, 23, 21]). We remark that CSO and SSO are neither closed under addition nor closed under multiplication, although they are both closed under the adjoint operation. Thus neither CSO nor SSO possesses a linear structure.

Assume that C is a conjugation on $\mathscr{H}$. Denote

$$\mathscr{C}_C = \{X \in \mathscr{B}(\mathscr{H}) : CXC = X^*\}$$

and

$$\mathscr{S}_C = \{X \in \mathscr{B}(\mathscr{H}) : CXC = -X^*\}.$$

It is easy to see that $\mathscr{C}_C$ and $\mathscr{S}_C$ are two linear subspaces of $\mathscr{B}(\mathscr{H})$, and both closed in the weak operator topology. We remark that $\mathscr{C}_C$ is a typical linear subspace of $\mathscr{B}(\mathscr{H})$ included in CSO. In fact, if C_1 is another conjugation on $\mathscr{H}$, then there exists unitary $U \in \mathscr{B}(\mathscr{H})$ such that $C_1 = UCU^*$ (see [6, Lem. 2.11]). It is easy to check that $U\mathscr{C}_C U^* = \mathscr{C}_{C_1}$. Thus $\mathscr{C}_C$ and $\mathscr{C}_{C_1}$ has the same structure as linear subspaces of $\mathscr{B}(\mathscr{H})$. In addition, note that $\mathscr{C}$ is the union of all such linear spaces $\mathscr{C}_C$. Likewise, one can also see that $\mathscr{S}_C$ is a typical linear subspace of $\mathscr{B}(\mathscr{H})$ included in SSO. In this paper we are interested in the properties of $\mathscr{C}_C$ and $\mathscr{S}_C$ as linear subspaces of $\mathscr{B}(\mathscr{H})$.

In what follows, we let C be a fixed conjugation on $\mathscr{H}$ and, for convenience, we write $\mathscr{C}$ and $\mathscr{S}$ instead of $\mathscr{C}_C$ and $\mathscr{S}_C$ respectively. First we notice that $\mathscr{C}$ and $\mathscr{S}$ are complementary subspaces of $\mathscr{B}(\mathscr{H})$. In fact, it is trivial to see $\mathscr{C} \cap \mathscr{S} = \{0\}$. Given $T \in \mathscr{B}(\mathscr{H})$, we have $T = A + B$, where

$$A = \frac{T + CT^*C}{2} \quad \text{and} \quad B = \frac{T - CT^*C}{2}.$$

Easy to see $A \in \mathscr{C}$ and $B \in \mathscr{S}$. This shows that $\mathscr{B}(\mathscr{H}) = \mathscr{C} + \mathscr{S}$. The aim of this paper is to give some results which exhibit more connections between $\mathscr{C}$ and $\mathscr{S}$.

The rest of this paper is organized as follows.

In Section 2, we shall prove that $\mathscr{C}$ and $\mathscr{S}$ are Roberts orthogonal to each other. As applications, we obtain some distance formulas from an operator to the sets $\mathscr{C}$, $\mathscr{S}$, CSO and SSO.

In Section 3, we shall study the spaces $\mathscr{C}$ and $\mathscr{S}$ from the predual point of view. We shall characterize the preannihilators of $\mathscr{C}$ and $\mathscr{S}$. Our results exhibit the annihilating relations between $\mathscr{C}$ and $\mathscr{S}$.

In Section 4, we shall study the transitivity, reflexivity and hyperreflexivity of $\mathscr{S}$. This is partly inspired by a recent paper of Kliś-Garlicka and Ptak [15], where the reflexivity and transitivity of $\mathscr{C}$ are studied.

2. Orthogonality and distance formulas

In this section, we shall show that $\mathscr{C}$ and $\mathscr{S}$ are Roberts orthogonal to each other. As applications, we shall provide some distance formulas from an operator to the sets $\mathscr{C}$, $\mathscr{S}$, CSO and SSO.

Recall that two operators $A,B \in \mathscr{B}(\mathscr{H})$ are said to be *Roberts orthogonal*, if $\|A-\lambda B\| = \|A+\lambda B\|$ for all complex numbers λ (see [18]). It is known that Roberts orthogonality implies Birkhoff orthogonality (see [1]).

Given $T \in \mathscr{B}(\mathscr{H})$ and a subset $\mathscr{V}$ of $\mathscr{B}(\mathscr{H})$, we let $d(T,\mathscr{V})$ denote the standard distance from T to the set $\mathscr{V}$, that is, $d(T,\mathscr{V}) = \inf\{\|T-X\| : X \in \mathscr{V}\}$.

THEOREM 2.1. *Let $A \in \mathscr{C}$ and $B \in \mathscr{S}$. Then*

(i) A is Roberts orthogonal to B;

(ii) $\|A\| \leqslant \|A-B\|$ and $\|B\| \leqslant \|A-B\|$;

(iii) $d(A,\mathscr{S}) = \|A\|$ and $d(B,\mathscr{C}) = \|B\|$.

Proof. Note that C is isometric, $CAC = A^*$ and $CBC = -B^*$. Then for any complex number λ we have

$$\|A+\lambda B\| = \|C(A+\lambda B)C\| = \|A^* - \overline{\lambda} B^*\| = \|A-\lambda B\|. \tag{2.1}$$

So A is Roberts orthogonal to B.

By (2.1), we have

$$\|A\| = \frac{\|A+B+A-B\|}{2} \leqslant \frac{\|A+B\|+\|A-B\|}{2} = \|A-B\|$$

and

$$\|B\| = \frac{\|B+A+B-A\|}{2} \leqslant \frac{\|A+B\|+\|A-B\|}{2} = \|A-B\|.$$

Note that $A \in \mathscr{C}$, $B \in \mathscr{S}$ can be arbitrary and $0 \in \mathscr{C} \cap \mathscr{S}$. It follows immediately that

$$d(A,\mathscr{S}) = \|A\| \qquad \text{and} \qquad d(B,\mathscr{C}) = \|B\|. \tag{2.2}$$

This ends the proof. □

COROLLARY 2.2. *If $T \in \mathscr{B}(\mathscr{H})$, then*

$$d(T,\mathscr{C}) = \frac{\|T-CT^*C\|}{2} \qquad \text{and} \qquad d(T,\mathscr{S}) = \frac{\|T+CT^*C\|}{2}.$$

Proof. Denote

$$A = \frac{T+CT^*C}{2} \qquad \text{and} \qquad B = \frac{T-CT^*C}{2}.$$

Easy to see $A \in \mathscr{C}$, $B \in \mathscr{S}$ and $T = A + B$. Then, by (2.2), we have

$$d(T,\mathscr{C}) = d(B,\mathscr{C}) = \|B\| \qquad \text{and} \qquad d(T,\mathscr{S}) = d(A,\mathscr{C}) = \|A\|.$$

This ends the proof. $\square$

Note that $CSO = \cup_C \mathscr{C}_c$ and $SSO = \cup_C \mathscr{S}_c$. Then the following result is clear from Corollary 2.2.

COROLLARY 2.3. *If $T \in \mathscr{B}(\mathscr{H})$, then*

$$d(T,CSO) = \inf\left\{\frac{\|T - JT^*J\|}{2} : J \text{ is a conjugation on } \mathscr{H}\right\}$$

and

$$d(T,SSO) = \inf\left\{\frac{\|T + JT^*J\|}{2} : J \text{ is a conjugation on } \mathscr{H}\right\}.$$

REMARK 2.4. Let $T \in \mathscr{B}(\mathscr{H})$. Recall that an operator A is called a *transpose* of T if $A = JT^*J$ for some conjugation J on $\mathscr{H}$. Note that any two transposes of T are unitarily equivalent (see [11]). We let $\mathscr{Z}_T$ denote the set of all transposes of T. By Corollary 2.3, we have

$$d(T,CSO) = \frac{d(T,\mathscr{Z}_T)}{2} \qquad \text{and} \qquad d(T,SSO) = \frac{d(-T,\mathscr{Z}_T)}{2}.$$

COROLLARY 2.5. *If $T \in \mathscr{B}(\mathscr{H})$, then*

(i) $T \in \overline{CSO}$ if and only if there exist conjugations $\{C_n\}_{n=1}^\infty$ so that $C_nT^*C_n \to T$.

(ii) $T \in \overline{SSO}$ if and only if there exist conjugations $\{C_n\}_{n=1}^\infty$ so that $C_nT^*C_n \to -T$.

EXAMPLE 2.6. Let e be a unit vector in $\mathscr{H}$. Denote $T = e \otimes e$. Then T is positive and rank-one. It follows that $T \in CSO$. Now we shall use Corollary 2.3 to calculate $d(T,SSO)$.

If $\dim \mathscr{H} = 1$, then, by [13, page 217], $SSO = \{0\}$. Hence $d(T,SSO) = \|T\| = 1$. In what follows, we assume that $\dim \mathscr{H} \geqslant 2$.

Let J be a conjugation on $\mathscr{H}$. Note that T and JT^*J are both positive. Then

$$\begin{aligned}\|T + JT^*J\| &\geqslant \langle (T + JT^*J)e, e\rangle = \langle Te, e\rangle + \langle JT^*Je, e\rangle \\ &= 1 + \langle JT^*Je, e\rangle \geqslant 1.\end{aligned}$$

Since J can be arbitrary, in view of Corollary 2.3, it follows that $d(T,SSO) \geqslant \frac{1}{2}$.

Since $\dim \mathscr{H} \geqslant 2$, we can find another unit vector $f \in \mathscr{H}$ with $\langle e, f\rangle = 0$. By [24, Thm. 2.1], there exists a conjugation J_0 on $\mathscr{H}$ such that $J_0e = f$. Then

$$J_0T^*J_0 = J_0(e \otimes e)J_0 = (J_0e) \otimes (J_0e) = f \otimes f.$$

It follows that $\|T + J_0T^*J_0\| = \|e \otimes e + f \otimes f\| = 1$. In view of Corollary 2.3, it follows that $d(T,SSO) \leqslant \frac{1}{2}$. Therefore we obtain $d(T,SSO) = \frac{1}{2}$.

EXAMPLE 2.7. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$ and T be the unilateral shift on $\mathscr{H}$ defined as $Te_i = e_{i+1}$ for $i \geqslant 1$. We shall prove that $d(T, CSO) = 1$. Since $0 \in CSO$, it follows that $d(T, CSO) \leqslant \|T\| = 1$.

If $A \in \mathscr{B}(\mathscr{H})$ and $\|T - A\| < 1$, then $\|(T-A)T^*\| < 1$, $I - (T-A)T^*$ is invertible and

$$A = T - (T-A) = \Big(I - (T-A)T^*\Big)T.$$

Since T is Fredholm and $\operatorname{ind} T = -1$, it follows that A is also a Fredholm operator and $\operatorname{ind} A = \operatorname{ind} T = -1$. So A is not complex symmetric. Thus we obtain $d(T, CSO) \geqslant 1$. Therefore $d(T, CSO) = 1$.

3. Preannihilators

This section is devoted to the descriptions of the preannihilators of $\mathscr{C}$ and $\mathscr{S}$. To proceed, we first introduce some notation and terminology.

The set of all trace class operators on $\mathscr{H}$ will be denoted by $\mathscr{B}_1(\mathscr{H})$ with the norm $\|\cdot\|_1$. Then $\mathscr{B}(\mathscr{H})$ is the dual space of $\mathscr{B}_1(\mathscr{H})$ in the sense that each bounded linear functional l on $\mathscr{B}_1(\mathscr{H})$ corresponds uniquely to an operator $A \in \mathscr{B}(\mathscr{H})$ such that

$$l(X) = \operatorname{tr}(AX), \quad \forall X \in \mathscr{B}_1(\mathscr{H}),$$

where $\operatorname{tr}(\cdot)$ denotes the trace function. In this case, $\|l\| = \|A\|$. Let $\mathscr{V}$ be a linear subspace of $\mathscr{B}(\mathscr{H})$. We denote by $\mathscr{V}_\perp$ the preannihilator of $\mathscr{V}$, that is,

$$\mathscr{V}_\perp = \{X \in \mathscr{B}_1(\mathscr{H}) : \operatorname{tr}(AX) = 0, \forall A \in \mathscr{V}\}.$$

The main result of this section is the following theorem which describes the preannihilators of $\mathscr{C}$ and $\mathscr{S}$.

THEOREM 3.1.

(i) $\mathscr{C}_\perp = \mathscr{S} \cap \mathscr{B}_1(\mathscr{H})$.

(ii) $\mathscr{S}_\perp = \mathscr{C} \cap \mathscr{B}_1(\mathscr{H})$.

Note that $\mathscr{C} \cap \mathscr{B}_1(\mathscr{H})$ and $\mathscr{S} \cap \mathscr{B}_1(\mathscr{H})$ are complementary subspaces of $\mathscr{B}_1(\mathscr{H})$. Then the above theorem shows that $\mathscr{C} \cap \mathscr{B}_1(\mathscr{H})$ is the predual of $\mathscr{C}$, and $\mathscr{S} \cap \mathscr{B}_1(\mathscr{H})$ is the predual of $\mathscr{S}$.

We need an auxiliary result. The reader is referred to [6, Lem. 2.16] for a proof.

LEMMA 3.2. *Let* $T \in \mathscr{B}(\mathscr{H})$ *and* $\{e_n\}$ *be an orthonormal basis of* $\mathscr{H}$ *such that* $Ce_n = e_n$ *for all* n.

(i) If $T \in \mathscr{C}$ *then* $\langle Te_i, e_j\rangle = \langle Te_j, e_i\rangle$ *for all* i, j.

(ii) If $T \in \mathscr{S}$, *then* $\langle Te_i, e_j\rangle = -\langle Te_j, e_i\rangle$ *for all* i, j.

Now we are going to give the proof of Theorem 3.1.

Proof of Theorem 3.1. Since C is a conjugation on $\mathscr{H}$, by [6, Lem. 2.11], there exists an orthonormal basis $\{e_n\}$ such that $Ce_n = e_n$ for all n. For $i, j \geqslant 1$, denote

$$E_{i,j} = e_i \otimes e_j + e_j \otimes e_i, \qquad F_{i,j} = e_i \otimes e_j - e_j \otimes e_i.$$

Here $e_i \otimes e_j$ is defined as $(e_i \otimes e_j)(x) = \langle x, e_j \rangle e_i$ for $x \in \mathscr{H}$. Clearly $E_{i,j}, F_{i,j} \in \mathscr{B}_1(\mathscr{H})$.

Claim 1. $E_{i,j} \in \mathscr{C}$ for all i, j.

For $i, j \geqslant 1$ and $x \in \mathscr{H}$, note that

$$\begin{aligned} C(e_i \otimes e_j)x &= C(\langle x, e_j \rangle e_i) = \langle e_j, x \rangle e_i \\ &= \langle Cx, Ce_j \rangle e_i = \langle Cx, e_j \rangle e_i = (e_i \otimes e_j)Cx. \end{aligned}$$

Thus $C(e_i \otimes e_j) = (e_i \otimes e_j)C$ for all i, j. It follows that $CE_{i,j}C = E_{i,j} = E_{i,j}^*$. So $E_{i,j} \in \mathscr{C}$.

Claim 2. $F_{i,j} \in \mathscr{S}$ for all i, j.

From the proof of Claim 1, one can see that $CF_{i,j}C = F_{i,j} = -F_{i,j}^*$. So $F_{i,j} \in \mathscr{S}$.

(i) "$\subseteq$". Assume that $X \in \mathscr{C}_\perp$. Then it follows from Claim 1 that

$$0 = \operatorname{tr}(XE_{i,j}) = \operatorname{tr}(X(e_i \otimes e_j)) + \operatorname{tr}(X(e_j \otimes e_i)) = \langle Xe_i, e_j \rangle + \langle Xe_j, e_i \rangle,$$

that is, $\langle Xe_i, e_j \rangle = -\langle Xe_j, e_i \rangle$. Noting that

$$\langle Xe_i, e_j \rangle = \langle e_i, X^* e_j \rangle = \langle CX^* e_j, Ce_i \rangle = \langle CX^* Ce_j, e_i \rangle$$

and $\{e_i\}$ is an orthonormal basis of $\mathscr{H}$, it follows that $-X = CX^*C$. So $X \in \mathscr{S} \cap \mathscr{B}_1(\mathscr{H})$.

"$\supseteq$". Assume that $X \in \mathscr{S} \cap \mathscr{B}_1(\mathscr{H})$. For $n \geqslant 1$, denote by P_n the orthogonal projection of $\mathscr{H}$ onto $\vee\{e_i : 1 \leqslant i \leqslant n\}$, where $\vee$ denotes closed linear span. Then $\|P_n X P_n - X\|_1 \to 0$. It suffices to prove that $P_n X P_n \in \mathscr{C}_\perp$ for all n.

Now fix an $n \geqslant 1$. For any $Y \in \mathscr{C}$, one can verify that

$$\begin{aligned} \operatorname{tr}(YP_nXP_n) &= \sum_{i=1}^{\infty} \langle YP_nXP_n e_i, e_i \rangle \\ &= \sum_{i=1}^{n} \langle YP_nXe_i, e_i \rangle = \sum_{i=1}^{n} \langle P_nXe_i, P_nY^* e_i \rangle. \end{aligned}$$

Note that

$$P_nXe_i = \sum_{j=1}^{n} \langle P_nXe_i, e_j \rangle e_j \quad \text{and} \quad P_nY^* e_i = \sum_{j=1}^{n} \langle P_nY^* e_i, e_j \rangle e_j.$$

It follows that

$$\begin{aligned}\mathrm{tr}(YP_nXP_n) &= \sum_{i=1}^{n}\langle P_nXe_i, P_nY^*e_i\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\langle P_nXe_i, e_j\rangle\cdot\langle e_j, P_nY^*e_i\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\langle Xe_i, e_j\rangle\cdot\langle Ye_j, e_i\rangle \\ &= \Delta_1+\Delta_2+\Delta_3,\end{aligned}$$

where

$$\Delta_1 = \sum_{1\leqslant i<j\leqslant n}\langle Xe_i, e_j\rangle\cdot\langle Ye_j, e_i\rangle,$$

$$\Delta_2 = \sum_{1\leqslant j<i\leqslant n}\langle Xe_i, e_j\rangle\cdot\langle Ye_j, e_i\rangle$$

and

$$\Delta_3 = \sum_{i=1}^{n}\langle Xe_i, e_i\rangle\cdot\langle Ye_i, e_i\rangle.$$

Since $Y\in\mathscr{C}$ and $X\in\mathscr{S}$, it follows from Lemma 3.2 that

$$\langle Ye_i, e_j\rangle = \langle Ye_j, e_i\rangle, \quad \langle Xe_i, e_j\rangle = -\langle Xe_j, e_i\rangle.$$

Then $\langle Xe_i, e_i\rangle = 0$, $\Delta_3 = 0$ and

$$\begin{aligned}\Delta_2 &= \sum_{1\leqslant j<i\leqslant n}\langle Xe_i, e_j\rangle\cdot\langle Ye_j, e_i\rangle \\ &= -\sum_{1\leqslant j<i\leqslant n}\langle Xe_j, e_i\rangle\cdot\langle Ye_i, e_j\rangle \\ &= -\sum_{1\leqslant i<j\leqslant n}\langle Xe_i, e_j\rangle\cdot\langle Ye_j, e_i\rangle = -\Delta_1.\end{aligned}$$

This implies that $\mathrm{tr}(YP_nXP_n)=0$. Since $Y\in\mathscr{C}$ is arbitrary, we deduce that $P_nXP_n\in\mathscr{C}_\perp$. This proves the statement (i).

(ii) “$\subseteq$”. Assume that $A\in\mathscr{S}_\perp$. Then, by Claim 2, we have

$$0=\mathrm{tr}(AF_{i,j})=\mathrm{tr}(A(e_i\otimes e_j))-\mathrm{tr}(A(e_j\otimes e_i))=\langle Ae_i, e_j\rangle-\langle Ae_j, e_i\rangle,$$

that is, $\langle Ae_i, e_j\rangle=\langle Ae_j, e_i\rangle$. Noting that

$$\langle Ae_i, e_j\rangle=\langle e_i, A^*e_j\rangle=\langle CA^*e_j, Ce_i\rangle=\langle CA^*Ce_j, e_i\rangle$$

and $\{e_i\}$ is an orthonormal basis of $\mathscr{H}$, it follows that $A=CA^*C$. So $A\in\mathscr{C}\cap\mathscr{B}_1(\mathscr{H})$.

"$\supseteq$". Assume that $A \in \mathscr{C} \cap \mathscr{B}_1(\mathscr{H})$. For $n \geqslant 1$, denote by P_n the orthogonal projection of $\mathscr{H}$ onto $\vee\{e_i : 1 \leqslant i \leqslant n\}$. Then $\|P_nAP_n - A\|_1 \to 0$. It suffices to prove that $P_nAP_n \in \mathscr{S}_\perp$ for all n.

Now fix an $n \geqslant 1$. For any $B \in \mathscr{S}$, one can verify that

$$\begin{aligned}\operatorname{tr}(P_nAP_nB) &= \sum_{i=1}^{\infty} \langle P_nAP_nBe_i, e_i\rangle \\ &= \sum_{i=1}^{n} \langle AP_nBe_i, e_i\rangle = \sum_{i=1}^{n} \langle P_nBe_i, P_nA^*e_i\rangle.\end{aligned}$$

Note that

$$P_nBe_i = \sum_{j=1}^{n} \langle Be_i, e_j\rangle e_j \quad \text{and} \quad P_nA^*e_i = \sum_{j=1}^{n} \langle P_nA^*e_i, e_j\rangle e_j = \sum_{j=1}^{n} \langle e_i, Ae_j\rangle e_j.$$

It follows that

$$\operatorname{tr}(P_nAP_nB) = \sum_{i=1}^{n} \langle P_nBe_i, P_nA^*e_i\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle Be_i, e_j\rangle \cdot \langle Ae_j, e_i\rangle.$$

Since $A \in \mathscr{C}$ and $B \in \mathscr{S}$, by Lemma 3.2, we have

$$\langle Ae_j, e_i\rangle = \langle Ae_i, e_j\rangle, \quad \langle Be_i, e_j\rangle = -\langle Be_j, e_i\rangle.$$

Then, by the latter part of the proof of (i), one can see $\operatorname{tr}(P_nAP_nB) = 0$. Since $B \in \mathscr{S}$ is arbitrary, we deduce that $P_nAP_n \in \mathscr{S}_\perp$. This ends the proof. $\square$

REMARK 3.3. We let $\mathscr{B}_0(\mathscr{H})$ denote the set of all compact operators in $\mathscr{B}(\mathscr{H})$. It is well known that $\mathscr{B}_1(\mathscr{H})$ is the dual space of $\mathscr{B}_0(\mathscr{H})$ in the sense that each bounded linear functional l on $\mathscr{B}_0(\mathscr{H})$ corresponds uniquely to an operator $A \in \mathscr{B}_1(\mathscr{H})$ such that

$$l(X) = \operatorname{tr}(AX), \qquad \forall X \in \mathscr{B}_0(\mathscr{H}).$$

Using similar arguments in the proof of Theorem 3.1, one can prove that

$$(\mathscr{C} \cap \mathscr{B}_1(\mathscr{H}))_\perp = \mathscr{S} \cap \mathscr{B}_0(\mathscr{H}) \quad \text{and} \quad (\mathscr{S} \cap \mathscr{B}_1(\mathscr{H}))_\perp = \mathscr{C} \cap \mathscr{B}_0(\mathscr{H}).$$

4. Transitivity, reflexivity and hyperreflexivity

As applications of Theorem 3.1, we shall explore in this section the transitivity, reflexivity and hyperreflexivity of $\mathscr{S}$. We first introduce some notation.

Let $\mathscr{V}$ be a linear subspaces of $\mathscr{B}(\mathscr{H})$. The *reflexive closure* of $\mathscr{V}$ is given by

$$\operatorname{Ref} \mathscr{V} = \{T \in \mathscr{B}(\mathscr{H}) : Tx \in \overline{\mathscr{V}x}, \forall x \in \mathscr{H}\}.$$

A linear subspace $\mathscr{V}$ of $\mathscr{B}(\mathscr{H})$ is called *reflexive* if $\operatorname{Ref} \mathscr{V} = \mathscr{V}$, and $\mathscr{V}$ is called *transitive* if $\operatorname{Ref} \mathscr{V} = \mathscr{B}(\mathscr{H})$. For a linear subspace $\mathscr{V}$ of $\mathscr{B}(\mathscr{H})$, it is well known that the following are equivalent: (i) $\mathscr{V}$ is transitive, (ii) $\overline{\mathscr{V}x} = \mathscr{H}$ for all nonzero $x \in \mathscr{H}$, and (iii) $\mathscr{V}_\perp$ contains no rank-one operator. Recall that $\mathscr{V}$ is said to be *hyperreflexive* if there exists a constant $\delta > 0$ such that

$$d(A,\mathscr{V}) \leqslant \delta \cdot \sup\left\{\|QAP\| : P,Q \text{ are projections and } Q\mathscr{V}P = \{0\}\right\}, \quad \forall A \in \mathscr{B}(\mathscr{H}).$$

It is known that hyperreflexivity implies reflexivity.

In their paper [14], Kliś-Garlicka and Ptak studied several generalizations of hyperreflexivity. For $1 \leqslant k < \infty$, denote by $\mathscr{F}_k$ the set of operators on $\mathscr{H}$ of rank at most k. Given a linear subspace $\mathscr{V}$ of $\mathscr{B}(\mathscr{H})$ and $A \in \mathscr{B}(\mathscr{H})$, define

$$\alpha_k(A,\mathscr{V}) = \sup\{|\mathrm{tr}(AX)| : X \in \mathscr{V}_\perp \cap \mathscr{F}_k, \|X\|_1 = 1\}.$$

The subspace $\mathscr{V}$ is called *k-hyperreflexive* if there is a constant $\delta > 0$ such that

$$d(A,\mathscr{V}) \leqslant \delta \cdot \alpha_k(A,\mathscr{V}), \quad \forall A \in \mathscr{B}(\mathscr{H}).$$

In particular, 1-hyperreflexivity coincides with hyperreflexivity (see [2, Prop. 58.1]).

In [15], Kliś-Garlicka and Ptak proved that $\mathscr{C}$ is transitive and not reflexive; moreover, by describing the set $\mathscr{C}_\perp \cap \mathscr{F}_2$, they proved that $\mathscr{C}$ is 2-hyperreflexive.

The main results of this section is the following two theorems.

THEOREM 4.1. *$\mathscr{S}$ is not transitive.*

THEOREM 4.2. *$\mathscr{S}$ is hyperreflexive and hence reflexive.*

The proof of Theorem 4.1 is an immediate consequence of Theorem 3.1.

Proof of Theorem 4.1. Note that $\mathscr{C} \cap \mathscr{B}_1(\mathscr{H})$ contains many rank-one operators. In fact, for any nonzero $e \in \mathscr{H}$, one can check that $(Ce) \otimes e \in \mathscr{C} \cap \mathscr{B}_1(\mathscr{H})$. Thus, by Theorem 3.1 (ii), $\mathscr{S}_\perp$ contains rank-one operators, which implies that $\mathscr{S}$ is not transitive. □

REMARK 4.3. It is well known that a skew symmetric operator can not have an odd rank (see [13, page 217]). Thus, by Theorem 3.1 (i), $\mathscr{C}_\perp$ contains no rank-one operator, which implies that $\mathscr{C}$ is transitive. Hence Theorem 3.1 provides another view of Theorem 2.1 in [15].

To prove Theorem 4.2, we first make some preparation.

LEMMA 4.4. ([8], Thm. 3) *If $A \in \mathscr{B}(\mathscr{H})$ is C-symmetric and of rank one, then A is of the form $(Ce) \otimes e$, where $e \in \mathscr{H}$ and $e \neq 0$.*

LEMMA 4.5. *If $T \in \mathscr{B}(\mathscr{H})$ is C-symmetric, then*

$$\|T\| = \sup\{|\langle CTx, x\rangle| : x \in \mathscr{H}, \|x\| = 1\}.$$

Proof. Denote $m = \sup\{|\langle CTx,x\rangle| : x \in \mathscr{H}, \|x\| = 1\}$. It is obvious that $m \leqslant \|T\|$. So it suffices to prove $\|T\| \leqslant m$. Assume that $\lambda = \|T\|$. It is obvious that $\lambda \in \sigma(|T|)$. By Theorem 2 in [3], there exists a sequence of unit vectors $\{f_n\}$ such that $\lim_{n\to\infty}(T-\lambda C)f_n = 0$, that is, $\lim_{n\to\infty}(CT-\lambda)f_n = 0$. For each $n \geqslant 1$, note that

$$\|T\| = \lambda = |\langle \lambda f_n, f_n\rangle| \leqslant |\langle(\lambda - CT)f_n, f_n\rangle| + |\langle CTf_n, f_n\rangle|.$$

Then $\|T\| \leqslant \limsup_{n\to\infty} |\langle CTf_n, f_n\rangle| \leqslant m$. This ends the proof. $\square$

Proof of Theorem 4.2. Let $A \in \mathscr{B}(\mathscr{H})$. The proof is divided into two cases.

Case 1. $A \in \mathscr{C}$.

By Lemma 4.4, each operator in $\mathscr{C} \cap \mathscr{F}_1$ has the form $(Ce) \otimes e$ for some nonzero $e \in \mathscr{H}$. Then, in view of Theorem 3.1, we have $\mathscr{C} \cap \mathscr{F}_1 = \mathscr{S}_\perp \cap \mathscr{F}_1$ and

$$\begin{aligned}\alpha_1(A,\mathscr{S}) &= \sup\{|\mathrm{tr}(A(Ce\otimes e))| : e \in \mathscr{H}, \|e\| = 1\}\\ &= \sup\{|\langle ACe, e\rangle| : e \in \mathscr{H}, \|e\| = 1\}\\ &= \sup\{|\langle CA^*e, e\rangle| : e \in \mathscr{H}, \|e\| = 1\}\\ &= \|A^*\| = \|A\| && \text{(by Lem. 4.5)}\\ &= d(A,\mathscr{S}). && \text{(by Thm. 2.1)}\end{aligned}$$

Case 2. $A \notin \mathscr{C}$.

Set

$$A_1 = \frac{A + CA^*C}{2} \quad \text{and} \quad A_2 = \frac{A - CA^*C}{2}.$$

Then $A_1 \in \mathscr{C}$, $A_2 \in \mathscr{S}$ and $A = A_1 + A_2$. It follows that $d(A,\mathscr{S}) = d(A_1,\mathscr{S})$. On the other hand, noting that $\mathrm{tr}(A_2X) = 0$ for all $X \in \mathscr{S}_\perp$, we deduce $\alpha_1(A,\mathscr{S}) = \alpha_1(A_1,\mathscr{S})$. By the proof in Case 1, we deduce that

$$d(A,\mathscr{S}) = d(A_1,\mathscr{S}) = \alpha_1(A_1,\mathscr{S}) = \alpha_1(A,\mathscr{S}).$$

Thus, in either case, we have proved that $d(A,\mathscr{S}) = \alpha_1(A,\mathscr{S})$. This shows that T is hyperreflexive. $\square$

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