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# ON THE APPROXIMATION OF $C_0$–SEMIGROUPS ON THE DUAL OF A BANACH SPACE

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*Abstract.* The main purpose of this paper is to present satisfactory versions of the Chernoff product formula and of the Lie-Trotter product formula for $C_0$-semigroups on the dual of a Banach space. Also, an application of the Lie-Trotter product formula is given for the diffusion operator on $L^\infty$.

## 1. Introduction

The abstract theory of strongly continuous semigroups of linear operators in Banach spaces or in locally convex spaces can be found in many works devoted to the subject, such as Yosida [15], Davies [3], Pazy [13], Goldstein [5] or Engel and Nagel [4]. It is well known that approximation is an important method used to study a complicated operator $\mathscr{L}$ and the $C_0$-semigroup it generates. Several interesting points of view concerning the famous Yosida type approximation are given in [12].

In general, the semigroup generated by $\mathscr{L}$ can be obtained as the limit of the known semigroups generated by some operators $\mathscr{A}_n$, which approximate the operator $\mathscr{L}$. Unfortunately, there is a major defect because we have to assume that the limit operator $\mathscr{L}$ is a-priori known to be a generator of a $C_0$-semigroup. But, in applications, in general we don't know if $\mathscr{L}$ coincides with the generator $\mathscr{A}$ of a $C_0$-semigroup. We only know that $\mathscr{L}$ is contained in the generator $\mathscr{A}$, in the sense that $\mathscr{D}(\mathscr{L}) \subset \mathscr{D}(\mathscr{A})$ and $\mathscr{L} = \mathscr{A}/_{\mathscr{D}(\mathscr{L})}$. So, we need to approximate the operator $\mathscr{L}$ by operators $\mathscr{A}_n$ which are generators of some $C_0$-semigroups and then conclude that the closure of $\mathscr{L}$ coincides with the generator $\mathscr{A}$.

One of the most important results in this direction is given by Trotter-Kato theorem. The Trotter-Kato theorem has many important applications and some approximation formulas such as Chernoff product formula and Lie-Trotter product formula can be derived. In fact, all these results give the theoretical starting point for many approximation schemes in abstract operator theory. Moreover, the discrete product formulas are very useful for numerical applications.

The main purpose of this paper is to present several satisfactory versions of the Chernoff product formula and of the Lie-Trotter product formula for $C_0$-semigroups on the dual of a Banach space.

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## 2. Framework

In the sequel, we follow closely [8] and [10]. It is well known that, in general, for a $C_0$-semigroup $\{\mathbf{T}_t\}_{t\geqslant 0}$ on a Banach space $(\mathscr{E},\|\cdot\|)$, its adjoint semigroup $\{\mathbf{T}_t^*\}_{t\geqslant 0}$ is no longer strongly continuous on the dual space $(\mathscr{E}^*,\|\cdot\|^*)$ with respect to the strong topology of $\mathscr{E}^*$. Indeed, let $\mathscr{D}(\mathbf{A}^*)$ be the domain of the adjoint $\mathbf{A}^*$ of the generator $\mathbf{A}$ of the $C_0$-semigroup $\{\mathbf{T}_t\}_{t\geqslant 0}$. Because in our case $\mathscr{E}$ and $\mathscr{E}^*$ are sequentially complete, by the famous theorem of Phillips (see [15, Chapter IX, p. 273]) it follows that $\{\mathbf{T}_t^*\}_{t\geqslant 0}$ is a $C_0$-semigroup on the closure $\overline{\mathscr{D}(\mathbf{A}^*)}$ of $\mathscr{D}(\mathbf{A}^*)$ with respect to the strong dual topology. But $\overline{\mathscr{D}(\mathbf{A}^*)}\neq\mathscr{E}^*$ because the domain $\mathscr{D}(\mathbf{A}^*)$ is not dense in general in $\mathscr{E}^*$. Anyway, one can consider the $C_0$-semigroup $\{\mathbf{T}_t^*\}_{t\geqslant 0}$ only on a subspace of $\mathscr{E}^*$ with respect to the strong dual topology.

In order to solve this problem concerning the adjoint of a $C_0$-semigroup, starting from a very practical point of view, Wu and Zhang [14] introduced on $\mathscr{E}^*$ a topology for which the usual semigroups in literature become $C_0$-semigroups. That is *the topology of uniform convergence on compact subsets of* $(\mathscr{E},\|\cdot\|)$, denoted by $\mathscr{C}(\mathscr{E}^*,\mathscr{E})$. For $f_0\in\mathscr{E}^*$, a basis of neighborhoods with respect to the topology of $\mathscr{C}(\mathscr{E}^*,\mathscr{E})$ is given by

$$\mathscr{V}(f_0;K,\varepsilon):=\left\{f\in\mathscr{E}^*\ \middle|\ \sup_{x\in K}|\langle x,f\rangle-\langle x,f_0\rangle|<\varepsilon\right\}$$

where $K$ runs over all compact subsets of $(\mathscr{E},\|\cdot\|)$ and $\varepsilon>0$.

In this context, by [14, Theorem 1.4, p. 564] one can see that if $\{\mathbf{T}_t\}_{t\geqslant 0}$ is a $C_0$-semigroup on $(\mathscr{E},\|\cdot\|)$ with generator $\mathbf{A}$, then $\{\mathbf{T}_t^*\}_{t\geqslant 0}$ is a $C_0$-semigroup on $(\mathscr{E}^*,\mathscr{C}(\mathscr{E}^*,\mathscr{E}))$ with generator $\mathbf{A}^*$. This is a satisfactory variant of Phillips's theorem concerning the adjoint of a $C_0$-semigroup on a Banach space.

Remark that $(\mathscr{E}^*,\mathscr{C}(\mathscr{E}^*,\mathscr{E})):=\mathscr{E}^*_{\mathscr{C}}$ is a complete locally convex vector space and the dual space $(\mathscr{E}^*_{\mathscr{C}})^*$ of $\mathscr{E}^*_{\mathscr{C}}$ is $(\mathscr{E},\|\cdot\|)$ (see [14, Lemma 1.10, p. 567]). Unfortunately, $\mathscr{E}^*_{\mathscr{C}}$ is neither barrelled, nor bornological in general (see [14, Remark 1.11, p. 568]).

Recall that a family $\mathscr{T}$ of linear continuous operators on $\mathscr{E}^*$ is called *equicontinuous* with respect to the topology $\mathscr{C}(\mathscr{E}^*,\mathscr{E})$ if for any neighborhood $\mathscr{V}(0;K,\varepsilon)$, there exists a neighborhood $\mathscr{V}(0;H,\delta)$ such that $f\in\mathscr{V}(0;H,\delta)$ and $T\in\mathscr{T}$ implies $Tf\in\mathscr{V}(0;K,\varepsilon)$.

Therefore we have all ingredients to consider $C_0$-semigroups on the dual space $\mathscr{E}^*_{\mathscr{C}}$ and we introduce

DEFINITION 1. We say that a family $\{T(t)\}_{t\geqslant 0}$ of linear continuous operators is a $C_0$-semigroup on $\mathscr{E}^*_{\mathscr{C}}$ if the following properties hold:

(i) $T(0)=I$;

(ii) $T(t+s)=T(t)T(s)$, for all $t,s\geqslant 0$;

(iii) $\lim_{t\searrow 0}T(t)f=f$, for all $f\in\mathscr{E}^*$;

(iv) there exists $\omega\in\mathbb{R}$ such that the family $\{e^{-\omega t}T(t)\,:\,t\in[0,\infty)\}$ is equicontinuous with respect to the topology $\mathscr{C}(\mathscr{E}^*,\mathscr{E})$.

The *infinitesimal generator* of the $C_0$-semigroup $\{T(t)\}_{t\geqslant 0}$ is a linear operator $\mathscr{A}$ defined on the domain

$$\mathscr{D}(\mathscr{A}) = \left\{ f \in \mathscr{E}^* \,\middle|\, \lim_{t\searrow 0} \frac{T(t)f - f}{t} \text{ exists in } \mathscr{E}^*_{\mathscr{C}} \right\}$$

by

$$\mathscr{A} f = \lim_{t\searrow 0} \frac{T(t)f - f}{t}, \quad \forall f \in \mathscr{D}(\mathscr{A}).$$

We can see that if $\mathscr{A}$ is the infinitesimal generator of a $C_0$-semigroup $\{T(t)\}_{t\geqslant 0}$, then $\mathscr{A}$ is a densely defined and closed operator on $\mathscr{E}^*_{\mathscr{C}}$ and the resolvent $R(\lambda;\mathscr{A}) = (\lambda I - \mathscr{A})^{-1}$, for any $\lambda \in \rho(\mathscr{A})$ (the resolvent set of $\mathscr{A}$) satisfies the equality

$$R(\lambda;\mathscr{A})f = \int_0^\infty e^{-\lambda t} T(t) f \, dt, \quad \forall \lambda > \omega \text{ and } \forall f \in \mathscr{E}^*.$$

Also, several results concerning the existence and uniqueness of $C_0$-semigroups and a Desch-Schappacher perturbation theorem have been obtained by the author in [8] and, respectively, in [7].

The classical version of Trotter-Kato theorem on locally convex spaces can be reformulated for the space $\mathscr{E}^*_{\mathscr{C}}$ (see [14, Theorem 2.7, p. 576] for details).

THEOREM 1. (Trotter-Kato) *Let $\{T_n(t)\}_{t\geqslant 0}$, $n \in \mathbb{N}$, be a sequence of $C_0$-semigroups on $\mathscr{E}^*_{\mathscr{C}}$ with generators $(\mathscr{A}_n, \mathscr{D}(\mathscr{A}_n))$ and such that for some $\omega \in \mathbb{R}$, the family*

$$\left\{ e^{-\omega t} T_n(t) \,:\, n \in \mathbb{N}, t \in [0,\infty) \right\}$$

*is equicontinuous in $\mathscr{E}^*_{\mathscr{C}}$.*

*Assume that there exists an operator $(\mathscr{L}, \mathscr{D})$ satisfying next properties:*

*(i) $\mathscr{D}$ is dense in $\mathscr{E}^*_{\mathscr{C}}$;*

*(ii) for some $\lambda > \omega$, $(\lambda I - \mathscr{L})(\mathscr{D})$ is dense in $\mathscr{E}^*_{\mathscr{C}}$;*

*(iii) $\mathscr{D} \subset \mathscr{D}(\mathscr{A}_n)$, for all $n \in \mathbb{N}$, and $\mathscr{A}_n \varphi \longrightarrow \mathscr{L}\varphi$, for all $\varphi \in \mathscr{D}$.*

*Then the closure $\overline{\mathscr{L}}$ of $\mathscr{L}$ is the generator of some $C_0$-semigroup $\{T(t)\}_{t\geqslant 0}$. Moreover,*

$$T(t)f = \lim_{n\to\infty} T_n(t)f, \quad \forall f \in \mathscr{E}^* \tag{1}$$

*uniformly for $t$ in compact intervals of $[0,\infty)$.*

REMARK 1. By Hahn-Banach theorem, property (ii) in Theorem 1 is equivalent with $\ker(\lambda I - \mathscr{L}^*) = 0$, for some $\lambda > \omega$, i.e., if $x \in \mathscr{D}(\mathscr{L}^*)$ satisfies $(\lambda I - \mathscr{L}^*)x = 0$, then $x = 0$ (which is a Liouville type property).

## 3. Main results

In the following, we will continue to use notations from the previous section. Our main results, strongly inspired by [4], [1], [14] and [9], are a Chernoff product formula and a Lie-Trotter product formula for $C_0$-semigroups on $\mathscr{E}^*_{\mathscr{C}}$, the dual of the Banach space $(\mathscr{E}, \|\cdot\|)$.

By using similar arguments as in the case of the Banach spaces, we can formulate next technical lemma.

LEMMA 1. *Let $T$ be a linear continuous operator on $\mathscr{E}^*_{\mathscr{C}}$ and $N \geqslant 1$ such that the family $\left\{N^{-k}T^k : k \in \mathbb{N}\right\}$ is $\mathscr{E}^*_{\mathscr{C}}$-equicontinuous.*

*Then, for any compact $K \subset \mathscr{E}$, there exists a compact $H \subset \mathscr{E}$ such that*

$$\sup_{x\in K}\left|\left\langle x, e^{n(T-I)}f - T^n f\right\rangle\right| \leqslant N^{n-1}e^{(N-1)n}\sqrt{n^2(N-1)^2 + nN}\sup_{x\in H}|\langle x, Tf - f\rangle| \qquad (2)$$

*for any $n \in \mathbb{N}$ and for all $f \in \mathscr{E}^*$.*

*Proof.* Let $K \subset \mathscr{E}$ be a compact set. By the equicontinuity of the family $\{N^{-k}T^k : k \in \mathbb{N}\}$, there exists a compact $H \subset \mathscr{E}$ such that for all $k, n \in \mathbb{N}$ with $k \geqslant n$ it follows that

$$\begin{aligned}
&\sup_{x\in K}\left|\left\langle x, T^k f - T^n f\right\rangle\right| \\
&= \sup_{x\in K}\left|\left\langle x, \sum_{i=n}^{k-1}\left(T^{i+1}f - T^i f\right)\right\rangle\right| \leqslant \sum_{i=n}^{k-1} N^i \sup_{x\in H}|\langle x, Tf - f\rangle| \\
&\leqslant \sup_{x\in H}|\langle x, Tf - f\rangle| \sum_{i=n}^{k-1} N^{k-1} = (k-n)N^{k-1}\sup_{x\in H}|\langle x, Tf - f\rangle| \\
&\leqslant |k-n|N^{n+k-1}\sup_{x\in H}|\langle x, Tf - f\rangle|, \quad \forall f \in \mathscr{E}^*,
\end{aligned}$$

and similarly for $k > n$, by taking into account the symmetry.

Therefore, for all $k, n \in \mathbb{N}$, we have

$$\sup_{x\in K}\left|\left\langle x, T^k f - T^n f\right\rangle\right| \leqslant |k-n|N^{n+k-1}\sup_{x\in H}|\langle x, Tf - f\rangle|, \quad \forall f \in \mathscr{E}^*.$$

This allows us to estimate

$$\begin{aligned}
\sup_{x\in K}\left|\left\langle x, e^{n(T-I)}f - T^n f\right\rangle\right| &= \sup_{x\in K}\left|\left\langle x, e^{-n}\sum_{k=0}^{\infty}\frac{n^k}{k!}\left(T^k f - T^n f\right)\right\rangle\right| \\
&\leqslant N^{n-1}e^{-n}\sup_{x\in H}|\langle x, Tf - f\rangle|\sum_{k=0}^{\infty}\frac{(nN)^k}{k!}|k-n|.
\end{aligned}$$

By using the Cauchy-Schwarz inequality, we have

$$\sum_{k=0}^{\infty}\frac{(nN)^k}{k!}|k-n| = \sum_{k=0}^{\infty}\left(\sqrt{\frac{(nN)^k}{k!}}\right)\left(\sqrt{\frac{(nN)^k}{k!}}|k-n|\right)$$
$$\leqslant \sqrt{\sum_{k=0}^{\infty}\frac{(nN)^k}{k!}}\sqrt{\sum_{k=0}^{\infty}\frac{(nN)^k}{k!}(k-n)^2} = e^{nN}\sqrt{n^2(1-N)^2+nN},$$

and, by consequence,

$$\sup_{x\in K}\left|\left\langle x, e^{n(T-I)}f - T^n f\right\rangle\right| \leqslant N^{n-1}e^{(N-1)n}\sqrt{n^2(N-1)^2+nN}\sup_{x\in H}|\langle x, Tf-f\rangle|$$

all $n\in\mathbb{N}$ and $f\in\mathscr{E}^*$. □

Now, we are able to state the main result of this paper, that is the next version of the Chernoff product formula for $C_0$-semigroups on $\mathscr{E}^*_{\mathscr{C}}$.

THEOREM 2. (Chernoff product formula) *Let $\{F(\tau)\}_{\tau\geqslant 0}$ be a family of linear continuous operators on $\mathscr{E}^*_{\mathscr{C}}$ satisfying $F(0)=I$ and assume that there exist $\omega\in\mathbb{R}$ and $b>0$ such that the family*

$$\left\{e^{-k\omega\tau}F^k(\tau)\,:\,k\in\mathbb{N},\,\tau\in[0,b]\right\}$$

*is equicontinuous with respect to the topology $\mathscr{C}(\mathscr{E}^*,\mathscr{E})$ (stability condition).*

*Let $(\mathscr{L},\mathscr{D})$ be a linear operator satisfying next properties:*

*(i) $\mathscr{D}$ is dense in $\mathscr{E}^*_{\mathscr{C}}$;*

*(ii) for some $\lambda>\omega$, $(\lambda I-\mathscr{L})(\mathscr{D})$ is dense in $\mathscr{E}^*_{\mathscr{C}}$;*

*(iii) for all $\varphi\in\mathscr{D}$ we have*

$$\mathscr{L}\varphi = \lim_{\tau\searrow 0}\frac{F(\tau)\varphi-\varphi}{\tau}.$$

*Then the closure $\overline{\mathscr{L}}$ of $\mathscr{L}$ is the generator of some $C_0$-semigroup $\{T(t)\}_{t\geqslant 0}$ on $\mathscr{E}^*_{\mathscr{C}}$ and*

$$T(t)f = \lim_{n\to\infty}\left[F\left(\frac{t}{n}\right)\right]^n f, \quad \forall f\in\mathscr{E}^*, \tag{3}$$

*uniformly for $t$ in compact intervals of $[0,\infty)$.*

*Proof.* For $\tau>0$, consider the linear continuous operators $\mathscr{A}_\tau$ on $\mathscr{E}^*_{\mathscr{C}}$ given by

$$\mathscr{A}_\tau = \frac{F(\tau)-I}{\tau}$$

and observe that for any $\varphi\in\mathscr{D}$ we have

$$\lim_{\tau\searrow 0}\mathscr{A}_\tau\varphi = \lim_{\tau\searrow 0}\frac{F(\tau)\varphi-\varphi}{\tau} = \mathscr{L}\varphi.$$

Let $\{T_\tau(t)\}_{t\geqslant 0}$, $\tau\in(0,\infty)$, be $C_0$-semigroups on $\mathscr{E}^*_{\mathscr{C}}$ with generators $\mathscr{A}_\tau$, given by

$$T_\tau(t)=e^{t\mathscr{A}_\tau}.$$

Taking into account that

$$\begin{aligned}T_\tau(t)&=e^{t\mathscr{A}_\tau}=e^{t\frac{F(\tau)-I}{\tau}}=e^{\frac{t}{\tau}F(\tau)-\frac{t}{\tau}}\\&=e^{-\frac{t}{\tau}}e^{\frac{t}{\tau}F(\tau)}=e^{-\frac{t}{\tau}}\sum_{k=0}^{\infty}\frac{\left(\frac{t}{\tau}\right)^k}{k!}F^k(\tau),\end{aligned}$$

by the $\mathscr{E}^*_{\mathscr{C}}$-equicontinuity of the family $\left\{e^{-k\omega\tau}F^k(\tau)\,:\,k\in\mathbb{N},\,\tau\in[0,b]\right\}$ it follows the $\mathscr{E}^*_{\mathscr{C}}$-equicontinuity of the family $\left\{e^{-|\omega|e^{|\omega|b}t}T_\tau(t)\,:\,\tau\in(0,b],\,t\in[0,\infty)\right\}$.

By Theorem 1 it follows that the closure $\overline{\mathscr{L}}$ of $\mathscr{L}$ is the generator of some $C_0$-semigroup $\{T(t)\}_{t\geqslant 0}$ and

$$T(t)f=\lim_{\tau\searrow 0}e^{t\mathscr{A}_\tau}f,\quad \forall f\in\mathscr{E}^*,$$

uniformly for $t$ in compact intervals of $[0,\infty)$, and therefore

$$T(t)f=\lim_{n\to\infty}e^{t\mathscr{A}_{\frac{t}{n}}}f,\quad \forall f\in\mathscr{E}^*,$$

uniformly for $t$ in compact intervals of $[0,\infty)$.

On the other hand, by Lemma 1, for any compact $K\subset\mathscr{E}$, there exists a compact $H\subset\mathscr{E}$ such that for all $\varphi\in\mathscr{D}$ one estimate

$$\begin{aligned}&\sup_{x\in K}\left|\left\langle x,e^{t\mathscr{A}_{\frac{t}{n}}}\varphi-\left[F\left(\frac{t}{n}\right)\right]^n\varphi\right\rangle\right|=\sup_{x\in K}\left|\left\langle x,e^{n\left[F\left(\frac{t}{n}\right)-I\right]}\varphi-\left[F\left(\frac{t}{n}\right)\right]^n\varphi\right\rangle\right|\\&\leqslant e^{|\omega|\frac{t}{n}(n-1)}e^{\left(e^{|\omega|\frac{t}{n}}-1\right)n}\sqrt{n^2\left(e^{|\omega|\frac{t}{n}}-1\right)^2+ne^{|\omega|\frac{t}{n}}}\sup_{x\in H}\left|\left\langle x,F\left(\frac{t}{n}\right)\varphi-\varphi\right\rangle\right|\\&=e^{|\omega|\frac{t}{n}(n-1)+\left(e^{|\omega|\frac{t}{n}}-1\right)n}\sqrt{n^2\left(e^{|\omega|\frac{t}{n}}-1\right)^2+ne^{|\omega|\frac{t}{n}}}\,\frac{t}{n}\sup_{x\in H}\left|\left\langle x,\frac{F\left(\frac{t}{n}\right)\varphi-\varphi}{\frac{t}{n}}\right\rangle\right|\\&=e^{|\omega|\frac{t}{n}(n-1)+\frac{e^{|\omega|\frac{t}{n}}-1}{\frac{t}{n}}t}\sqrt{t^2\left(e^{|\omega|\frac{t}{n}}-1\right)^2+\frac{t^2}{n}e^{|\omega|\frac{t}{n}}}\sup_{x\in H}\left|\left\langle x,\frac{F\left(\frac{t}{n}\right)\varphi-\varphi}{\frac{t}{n}}\right\rangle\right|\longrightarrow 0\end{aligned}$$

as $n\to\infty$, uniformly for $t$ in compact intervals of $[0,\infty)$.

Then for any compact $K\subset\mathscr{E}$ we have

$$\begin{aligned}&\sup_{x\in K}\left|\left\langle x,T(t)\varphi-\left[F\left(\frac{t}{n}\right)\right]^n\varphi\right\rangle\right|\\&\leqslant\sup_{x\in K}\left|\left\langle x,T(t)\varphi-e^{t\mathscr{A}_{\frac{t}{n}}}\varphi\right\rangle\right|+\sup_{x\in K}\left|\left\langle x,e^{t\mathscr{A}_{\frac{t}{n}}}\varphi-\left[F\left(\frac{t}{n}\right)\right]^n\varphi\right\rangle\right|\longrightarrow 0,\quad\forall\varphi\in\mathscr{D}\end{aligned}$$

as $n \to \infty$, uniformly for $t$ in compact intervals of $[0,\infty)$.

Therefore

$$T(t)\varphi = \lim_{n\to\infty}\left[F\left(\frac{t}{n}\right)\right]^n \varphi, \quad \forall \varphi \in \mathscr{D},$$

uniformly for $t$ in compact intervals of $[0,\infty)$.

Since $\mathscr{D}$ is dense in $\mathscr{E}^*_{\mathscr{C}}$ and the family $\left\{e^{-k\omega\tau}F^k(\tau) : k \in \mathbb{N},\ \tau \in [0,b]\right\}$ is equicontinuous, it follows that

$$T(t)f = \lim_{n\to\infty}\left[F\left(\frac{t}{n}\right)\right]^n f, \quad \forall f \in \mathscr{E}^*,$$

uniformly for $t$ in compact intervals of $[0,\infty)$. □

Another important result concerning approximation of the perturbed $C_0$-semigroups on $\mathscr{E}^*_{\mathscr{C}}$ is the next version of Lie-Trotter product formula.

COROLLARY 1. (Lie-Trotter product formula) *Let $\{T_1(t)\}_{t\geqslant 0}$ and $\{T_2(t)\}_{t\geqslant 0}$ be some $C_0$-semigroups on $\mathscr{E}^*_{\mathscr{C}}$ with generators $(\mathscr{A}_1,\mathscr{D}(\mathscr{A}_1))$ and respectively $(\mathscr{A}_2,\mathscr{D}(\mathscr{A}_2))$ and let $\omega \in \mathbb{R}$ and $b > 0$ be such that the family*

$$\left\{e^{-k\omega t}[T_1(t)T_2(t)]^k : k \in \mathbb{N}, t \in [0,b]\right\}$$

*is equicontinuous with respect to the topology $\mathscr{C}(\mathscr{E}^*,\mathscr{E})$ (stability condition).*

*Let $(\mathscr{L},\mathscr{D})$ be a linear operator satisfying next properties:*

*(i) $\mathscr{D}$ is dense in $\mathscr{E}^*_{\mathscr{C}}$;*

*(ii) for some $\lambda > \omega$, $(\lambda I - \mathscr{L})(\mathscr{D})$ is dense in $\mathscr{E}^*_{\mathscr{C}}$;*

*(iii) $\mathscr{L}\varphi = \mathscr{A}_1\varphi + \mathscr{A}_2\varphi$, for all $\varphi \in \mathscr{D} \subset \mathscr{D}(\mathscr{A}_1)\cap\mathscr{D}(\mathscr{A}_2)$.*

*Then the closure $\overline{\mathscr{L}}$ of $\mathscr{L}$ is the generator of some $C_0$-semigroup $\{T(t)\}_{t\geqslant 0}$ on $\mathscr{E}^*_{\mathscr{C}}$ and*

$$T(t)f = \lim_{n\to\infty}\left[T_1\left(\frac{t}{n}\right)T_2\left(\frac{t}{n}\right)\right]^n f, \quad \forall f \in \mathscr{E}^*, \tag{4}$$

*and the limit is uniform for $t$ in compact intervals of $[0,\infty)$.*

*Proof.* In order to apply the Chernoff product formula, it suffices to consider the family $\{F(\tau)\}_{\tau\geqslant 0}$ given by

$$F(\tau) = T_1(\tau)T_2(\tau), \quad \forall \tau \geqslant 0.$$

Then $F(0) = I$ and for all $\varphi \in \mathscr{D}$, we have:

$$\begin{aligned}
&\lim_{\tau\searrow 0}\frac{F(\tau)\varphi-\varphi}{\tau} = \lim_{\tau\searrow 0}\frac{T_1(\tau)T_2(\tau)\varphi-\varphi}{\tau}\\
&= \lim_{\tau\searrow 0}\frac{T_1(\tau)T_2(\tau)\varphi - T_1(\tau)\varphi}{\tau} + \lim_{\tau\searrow 0}\frac{T_1(\tau)\varphi-\varphi}{\tau}\\
&= \lim_{\tau\searrow 0}T_1(\tau)\frac{T_2(\tau)\varphi-\varphi}{\tau} + \mathscr{A}_1\varphi = \mathscr{A}_2\varphi + \mathscr{A}_1\varphi = \mathscr{L}\varphi.
\end{aligned}$$

By Theorem 2 it follows that the closure $\overline{\mathscr{L}}$ of $\mathscr{L}$ is the generator of some $C_0$-semigroup $\{T(t)\}_{t\geqslant 0}$ on $\mathscr{E}^*_{\mathscr{C}}$ and

$$T(t)f = \lim_{n\to\infty}\left[T_1\left(\frac{t}{n}\right)T_2\left(\frac{t}{n}\right)\right]^n f, \quad \forall f\in\mathscr{E}^*,$$

and the limit is uniform for $t$ in compact intervals of $[0,\infty)$. □

All the above results can be slightly proved in the case of the dual of a locally convex vector space.

Remark that from the point of view of approximation, the stability condition in all above results is very important. Kühnemund and Wacker [6] have proven that the Lie-Trotter product formula does not hold for arbitrary sums of generators. Also, a very interesting analysis concerning the stability of Chernoff product formula is given by McAllister, Neubrander, Reiser and Zhuang [11].

EXAMPLE 1. Let us to consider the diffusion operator

$$\mathscr{L}\varphi := \frac{1}{2}\Delta\varphi + b\cdot\nabla\varphi =: \mathscr{A}\varphi + \mathscr{B}\varphi, \quad \forall\varphi\in C_0^\infty(\mathbb{R}^N), \tag{5}$$

where $\cdot$ denotes the usual inner product in $\mathbb{R}^N$, $N\geqslant 2$, $C_0^\infty(\mathbb{R}^N)$ is the space of all infinitely differentiable functions whith compact support and $b:\mathbb{R}^N\to\mathbb{R}^N$ is the velocity vector field which is supposed to be locally Lipschitzian.

Let $(B_t)_{t\geqslant 0}$ be the Brownian Motion in $\mathbb{R}^N$ defined on a filtered probability space $\left(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant 0},(\mathbb{P}_x)_{x\in\mathbb{R}^N}\right)$, with $\mathbb{P}_x(B_0=x)=1$ for any initial point $x\in\mathbb{R}^N$. Let $\partial$ be the point at infinity of $\mathbb{R}^N$ and consider the diffusion $(X_t)_{0\leqslant t<\tau_e}$ generated by $\mathscr{L}$, where $\tau_e=\inf\{t\geqslant 0 \mid X_t=\partial\}$ is the explosion time, described by the Itô stochastic differential equation

$$dX_t = dB_t + b(X_t)dt. \tag{6}$$

Then the corresponding transition semigroup $\{P_t\}_{t\geqslant 0}$ is given by

$$P_t f(x) := \mathbb{E}^x 1_{\{t<\tau_e\}} f(X_t),$$

where $\mathbb{E}^x$ is the expectation with respect to $\mathbb{P}_x$. Remark that, in general, $\{P_t\}_{t\geqslant 0}$ fails to be of the class $C_0$ on $(L^\infty(\mathbb{R}^N,dx),\|\,.\,\|_\infty)$. Therefore, the Lie-Trotter's product formula in its classical formulation does not apply.

Consider on $L^\infty(\mathbb{R}^N,dx)$ the topology of uniform convergence on compact subsets of $(L^1(\mathbb{R}^N,dx),\|\,.\,\|_1)$, denoted by $\mathscr{C}(L^\infty,L^1)$. It is well known (see [14]) that $\{P_t\}_{t\geqslant 0}$ is a $C_0$-semigroup on $(L^\infty(\mathbb{R}^N,dx),\mathscr{C}\left(L^\infty,L^1\right)) =: L^\infty_{\mathscr{C}}$.

We shall prove that the Lie-Trotter product formula holds for $\{P_t\}_{t\geqslant 0}$ on $L^\infty_{\mathscr{C}}$. Remark that $\mathscr{D}:=C_0^\infty(\mathbb{R}^N)$ is dense in $L^\infty_{\mathscr{C}}$.

Let's consider the equation

$$dX_t = dB_t$$

which corresponds to the diffusion term in (6). Then the Brownian Motion semigroup $\{T_t\}_{t\geqslant 0}$ given by $T_t f(x) := \mathbb{E}^x f(B_t)$ is a $C_0$-semigroup on $L^\infty_{\mathscr{C}}$ and its generator contains the operator $(\mathscr{A},\mathscr{D})$ from (5) (see [14]).

Consider now the equation

$$dX_t = b(X_t)dt$$

corresponding to the drift term in (6). Then the corresponding transition semigroup $\{S_t\}_{t\geqslant 0}$ given by $S_t f(x) := f(X_t)1_{\{t<\tau_e\}}$ is a $C_0$-semigroup on $L^\infty_{\mathscr{C}}$ and its generator contains the operator $(\mathscr{B},\mathscr{D})$ from (5) (see [10]).

Since $\{T_t\}_{t\geqslant 0}$ and $\{S_t\}_{t\geqslant 0}$ are $C_0$-semigroups, it is not difficult to prove the $L^\infty_{\mathscr{C}}$-equicontinuity of the family

$$\left\{e^{-k\omega t}[T_t S_t]^k \,:\, k\in\mathbb{N},\, t\in[0,b]\right\},$$

for some $\omega\in\mathbb{R}$ and $b>0$.

For the density of $(\lambda I-\mathscr{L})(\mathscr{D})$, $\lambda>\omega$, in $L^\infty_{\mathscr{C}}$ it is enough to show that if $u\in L^1(\mathbb{R}^N,dx)$ satisfies

$$\int_{\mathbb{R}^N} u(\lambda I-\mathscr{L})\varphi\,dx = 0, \quad \forall\varphi\in C_0^\infty(\mathbb{R}^N),$$

then $u=0$. For this purpose, on can follow step by step the proof of Theorem 6.1 in [14] under assumption of the existence of some measurable locally bounded function $\beta:\mathbb{R}^+\to\mathbb{R}$ such that

$$\frac{b(x)x}{|x|}\geqslant\beta(|x|), \quad x\in\mathbb{R}^N-\{0\}$$

and

$$\int_0^\infty m'(x)\,dx\int_0^\infty s'(r)\,dr = +\infty,$$

where

$$s'(r)=e^{-\int_1^r\left[2\beta(r)+\frac{N-1}{r}\right]dt} \quad\text{and}\quad m'(x)=\frac{1}{s'(x)}.$$

Now we are able to approximate $\{P_t\}_{t\geqslant 0}$ by the Lie-Trotter product of $\{T_t\}_{t\geqslant 0}$ and $\{S_t\}_{t\geqslant 0}$:

$$P_t f = \lim_{n\to\infty}\left[T_{\frac{t}{n}}S_{\frac{t}{n}}\right]^n f, \quad \forall f\in L^\infty(\mathbb{R}^N,dx),$$

and the limit is uniform for $t$ in compact intervals of $[0,\infty)$.

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