

SOME PROPERTIES OF G-FRAMES FOR HILBERT SPACE OPERATORS

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Abstract. In this paper, we present some new characterizations of g -frames for a bounded operator. We discuss the g -frame for an operator K which has closed range and give a necessary and sufficient conditions for a family of bounded operators to be a K - g -frame. We also give a characterization of the dual for a K - g -frame. Moreover, We use quotient operators to characterize K - g -frames and find that the results on K - g -frames can be proved by theory of quotient operators.

1. Introduction

Frames were first introduced by Duffin and Schaeffer [7] in the study of nonharmonic Fourier series, and reintroduced in 1986 by Daubechies, et al. [5] and popularized from then on. Frames have established themselves by now as a standard notion in applied mathematics and engineering. Nice properties of frames have made them useful in filter bank theory [14], compressed sensing [3], coding theory [15, 16], probability statistics [8, 18], and signal and image processing [11].

Sun in [21] introduced the concept of g -frame in Hilbert space. G -frames are natural generalizations of frames which cover many other recent generalizations of frames, such as bounded quasi-projections [9], fusion frames [2] and pseudo-frames [17]. The author in [1] gave the notion of K - g -frame. It shows that K - g -frames possess higher generality than g -frames in the sense that the lower frame bound condition holds only for the elements in the range of K . Hence K - g -frames provide more flexibility and thus make the study of them interesting.

There are some important results on K -frames which have established by [10]. It shows that many properties for ordinary frames may not hold for K -frames, such as the corresponding synthesis operator for K -frames is not surjective, the frame operator for K -frames is not invertible and so on. We refer reader to [19, 20, 22] for more details about the results of K -frames. Since the structure of g -frame is more complicated than that of ordinary frame, it is necessary to generalize some of the known results in K -frames to K - g -frames. Moreover, we give some new characterizations of K - g -frames by the range of K .

Throughout the paper, \mathcal{H} and \mathcal{K} are two Hilbert spaces and $\{\mathcal{H}_i\}_{i \in I}$ is a sequence of closed subspaces of \mathcal{H} where I is a subset of \mathbb{Z} and $\mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ is the

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collection of all bounded linear operators from \mathcal{H} into \mathcal{H}_i . For $T \in \mathcal{L}(\mathcal{H})$, we denote $\mathcal{D}(T)$, $\mathcal{R}(T)$ and $\mathcal{N}(T)$ for domain, range and ker of T , respectively. And we denote by $I_{\mathcal{H}}$ the identity operator on \mathcal{H} .

We need recall some basic definitions and results of g -frames and K - g -frames in Hilbert space.

We call a sequence $\{\Lambda_i\}_{i \in I}$ a generalized frame, or simply a g -frame, for \mathcal{H} with respect to $\{\mathcal{H}_i\}_{i \in I}$ if there are two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

We call A and B the lower and upper frame bounds, respectively. If $\{\Lambda_i\}_{i \in I}$ possesses an upper frame bound, but not necessarily a lower bound, we call it a Bessel g -sequence with Bessel bound B .

We say $\{\Lambda_i\}_{i \in I}$ a g -frame sequence, if it is a g -frame for $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I}$.

Now define

$$\left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{\ell^2} := \left\{ \{f_i\}_{i \in I} \mid f_i \in \mathcal{H}_i, \|\{f_i\}_{i \in I}\|_2^2 = \sum_{i \in I} \|f_i\|^2 < \infty \right\},$$

with pointwise operators and inner product as

$$\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle.$$

In [21], Sun showed that every g -frame can be considered as a frame. More precisely, let $\{\Lambda_i\}_{i \in I}$ be a g -frame for \mathcal{H} and $\{e_{i,j}\}_{j \in J_i}$ be an orthonormal basis for \mathcal{H}_i , then there exists a frame $\{u_{i,j}\}_{i \in I, j \in J_i}$ of \mathcal{H} such that

$$u_{i,j} = \Lambda_i^* e_{i,j}, \tag{1}$$

and

$$\Lambda_i f = \sum_{j \in J_i} \langle f, u_{i,j} \rangle e_{i,j}, \quad \forall f \in \mathcal{H}.$$

We call $\{u_{i,j}\}_{i \in I, j \in J_i}$ the frame induced by $\{\Lambda_i\}_{i \in I}$ with respect to $\{e_{i,j}\}_{i \in I, j \in J_i}$. The next lemma is a characterization of g -frame by a frame.

LEMMA 1. [21] *Let $\{\Lambda_i\}_{i \in I}$ be a family of linear operators and $u_{i,j}$ be defined as in (1). Then $\{\Lambda_i\}_{i \in I}$ is a g -frame for \mathcal{H} if and only if $\{u_{i,j}\}_{i \in I, j \in J_i}$ is a frame for \mathcal{H} .*

DEFINITION 1. Let $K \in \mathcal{L}(\mathcal{H})$. We call a sequence $\{\Lambda_i\}_{i \in I}$ a K - g -frame for \mathcal{H} with respect to $\{\mathcal{H}_i\}_{i \in I}$ if there are two positive constants A and B such that

$$A\|K^* f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

We call A and B the lower and upper frame bounds, respectively.

We call $\{\Lambda_i\}_{i \in I}$ a tight K - g -frame if $A\|K^* f\|^2 = \sum_{i \in I} \|\Lambda_i f\|^2$, for every $f \in \mathcal{H}$. If we have only the second inequality, we call it a K -Bessel g -sequence. In [1], the synthesis operator of $\{\Lambda_i\}_{i \in I}$ is defined by

$$T_{\Lambda} : \left(\sum_{i \in I} \oplus \mathcal{H}_i \right)_{\ell^2} \longrightarrow \mathcal{H}, \quad T_{\Lambda}(\{f_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^* f_i,$$

and the K -g-frame operator S_Λ is defined as follows

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i \in I} \Lambda_i^* \Lambda_i f, \quad \forall f \in \mathcal{H}.$$

The following theorem is a characterization of K -g-frames in [1].

THEOREM 1. *Let $K \in \mathcal{L}(\mathcal{H})$. Then the following statements are equivalent:*

1. $\{\Lambda_i\}_{i \in I}$ is a K -g-frame;
2. $\{\Lambda_i\}_{i \in I}$ is a Bessel g-sequence for \mathcal{H} and there exists a Bessel g-sequence $\{\Gamma_i\}_{i \in I}$ for \mathcal{H} such that

$$Kf = \sum_{i \in I} \Lambda_i^* \Gamma_i f, \quad \forall f \in \mathcal{H}.$$

The following lemmas are fundamental results in the study of the K -g-frames.

LEMMA 2. [6] *Let $U, V \in \mathcal{L}(\mathcal{H})$. The following statements are equivalent:*

- (i) $\mathcal{R}(U) \subset \mathcal{R}(V)$.
- (ii) $UU^* \leq \lambda VV^*$ for some $\lambda \geq 0$.
- (iii) There exists $Q \in \mathcal{L}(\mathcal{H})$ such that $U = VQ$.

Moreover, if (i), (ii) and (iii) are valid, then there exists a unique operator Q such that

1. $\|Q\|^2 = \inf\{\mu : UU^* \leq \mu VV^*\}$,
2. $\mathcal{N}(U) = \mathcal{N}(Q)$, and
3. $\mathcal{R}(Q) \subset \overline{\mathcal{R}(V^*)}$.

LEMMA 3. [4] *Let \mathcal{H} be a Hilbert space, and suppose that $T \in \mathcal{L}(\mathcal{H})$ has a closed range. Then there exists an operator $T^\dagger \in \mathcal{L}(\mathcal{H})$ for which*

$$\mathcal{N}(T^\dagger) = \mathcal{R}(T)^\perp, \quad \mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp, \quad TT^\dagger f = f, \quad f \in \mathcal{R}(T).$$

We call the operator T^\dagger the pseudo-inverse of T .

2. Characterizations of K -g-frames

In this section, we give some equivalent characterizations of K -g-frames. First we need the following lemma which is a generalization of Theorem 3.5 in [22].

LEMMA 4. *Let $\{\Lambda_i\}_{i \in I}$ be a Bessel g-sequence for \mathcal{H} with frame operator S_Λ . Then $\{\Lambda_i\}_{i \in I}$ is a K -g-frame if and only if there exists $\lambda > 0$ such that $S_\Lambda \geq \lambda KK^*$.*

Proof. $\{\Lambda_i\}_{i \in I}$ is K -g-frame with frame bounds A, B and frame operator S_Λ if and only if

$$A\|K^*f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 = \langle S_\Lambda f, f \rangle \leq B\|f\|^2, \quad \forall f \in \mathcal{H},$$

that is,

$$\langle AKK^*f, f \rangle \leq \langle S_\Lambda f, f \rangle \leq \langle Bf, f \rangle, \quad \forall f \in \mathcal{H}.$$

So the conclusion holds. \square

The following theorem shows that every Bessel g-sequence can be a K -g-frame.

THEOREM 2. *Let $\{\Lambda_i\}_{i \in I}$ be a Bessel g-sequence for \mathcal{H} with frame operator S_Λ . Then $\{\Lambda_i\}_{i \in I}$ is a K -g-frame for \mathcal{H} if and only if $K = S_\Lambda^{1/2}T$, for some $T \in \mathcal{L}(\mathcal{H})$.*

Proof. By Lemma 4, $\{\Lambda_i\}_{i \in I}$ is a K -g-frame if and only if there exists $\lambda > 0$ such that

$$\lambda KK^* \leq S_\Lambda = S_\Lambda^{1/2}(S_\Lambda^{1/2})^*.$$

Therefore by Lemma 2 the conclusion hold. \square

COROLLARY 1. *Let $\{\Lambda_i\}_{i \in I}$ be a sequence of bounded operators for \mathcal{H} and u_{ij} be defined as in (1). Let $K \in \mathcal{L}(\mathcal{H})$, then $\{\Lambda_i\}_{i \in I}$ is a K -g-frame if and only if $\{u_{i,j}\}_{i \in I, j \in J_i}$ is a K -frame for \mathcal{H} .*

The following proposition gives a condition for a Bessel g-sequence to be a K -g-frame as well as other operator T .

PROPOSITION 1. *Let $K \in \mathcal{L}(\mathcal{H})$ and $\{\Lambda_i\}_{i \in I}$ be a K -g-frame for \mathcal{H} . Let $T \in \mathcal{L}(\mathcal{H})$ with $\mathcal{R}(T) \subset \mathcal{R}(K)$, then $\{\Lambda_i\}_{i \in I}$ is a T -g-frame for \mathcal{H} .*

Proof. Suppose that $\{\Lambda_i\}_{i \in I}$ is a K -g-frame for \mathcal{H} . Then there exist $0 < A \leq B < \infty$ such that

$$A\|K^*f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2}$$

Since $\mathcal{R}(T) \subset \mathcal{R}(K)$, by Lemma 4, there exists $\lambda > 0$ such that $TT^* \leq \lambda KK^*$. Then $\frac{1}{\lambda}\|T^*f\|^2 \leq \|K^*f\|^2$. From (2), we have

$$\frac{A}{\lambda}\|T^*f\|^2 \leq A\|K^*f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Hence $\{\Lambda_i\}_{i \in I}$ is a T -g-frame for \mathcal{H} . \square

Let $K = I$, we have the following corollary.

COROLLARY 2. *Let $\{\Lambda_i\}_{i \in I}$ be a g-frame for \mathcal{H} . Let $K \in \mathcal{L}(\mathcal{H})$, then $\{\Lambda_i\}_{i \in I}$ is a K -g-frame for \mathcal{H} .*

Proof. In fact, $\{\Lambda_i\}_{i \in I}$ can be viewed as an $I_{\mathcal{H}}$ -g-frame for \mathcal{H} . Since $\mathcal{R}(K) \subset \mathcal{R}(I_{\mathcal{H}})$, by proposition 1, the conclusion is hold. \square

The Corollary 2 is equivalent to Theorem 2.3 in [1].
 Now we characterize K -g-frame in terms of range of K .

THEOREM 3. *Let $K \in \mathcal{L}(\mathcal{H})$ with closed range and $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} \subset \mathcal{R}(K)$. Then $\{\Lambda_i\}_{i \in I}$ is a K -g-frame for \mathcal{H} if and only if $\{\Lambda_i\}_{i \in I}$ is a g-frame on $\mathcal{R}(K)$.*

Proof. Suppose that $\{\Lambda_i\}_{i \in I}$ is a K -g-frame for \mathcal{H} , then there exist $0 < A \leq B < \infty$ such that

$$A\|K^*f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Thus for all $f \in \mathcal{H}$, we have

$$\|K^*f\|^2 \leq \frac{1}{A} \sum_{i \in I} \|\Lambda_i f\|^2.$$

Since $\mathcal{R}(K)$ is closed, by Lemma 3, there exists K^\dagger of K such that $f = KK^\dagger f, \forall f \in \mathcal{R}(K)$. And then for all $f \in \mathcal{R}(K)$,

$$\|f\|^4 = |\langle KK^\dagger f, f \rangle|^2 = |\langle K^\dagger f, K^* f \rangle|^2 \leq \|K^\dagger f\|^2 \|K^* f\|^2 \leq \|K^\dagger\|^2 \|f\|^2 \frac{1}{A} \sum_{i \in I} \|\Lambda_i f\|^2.$$

Hence

$$\frac{A}{\|K^\dagger\|^2} \|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{R}(K).$$

Therefore, $\{\Lambda_i\}_{i \in I}$ is a g-frame on $\mathcal{R}(K)$.

Conversely, suppose that $\{\Lambda_i\}_{i \in I}$ is a g-frame on $\mathcal{R}(K)$, then there exist $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq D\|f\|^2, \quad \forall f \in \mathcal{R}(K).$$

Clearly, $\{\Lambda_i\}_{i \in I}$ is a Bessel g-sequence for \mathcal{H} . Now we prove that $\{\Lambda_i\}_{i \in I}$ is a K -g-frame for \mathcal{H} . We show that $\{\Lambda_i\}_{i \in I}$ has a lower bound for all $f \in \mathcal{H}$. Suppose that $\|K^*\| = \|K\| \neq 0$, for $f \in \mathcal{H}$, we have

$$\|K^*f\| \leq \|K^*\| \|f\| = \|K\| \|f\|,$$

and then $\|f\| \geq \frac{1}{\|K\|} \|K^*f\|$, for all $f \in \mathcal{H}$. Since $\mathcal{R}(K)$ is closed, we have $\mathcal{H} = \mathcal{R}(K) \oplus \mathcal{R}(K)^\perp$. For any $f \in \mathcal{H}$, let $f = f_1 + f_2$, where $f_1 \in \mathcal{R}(K)$ and $f_2 \in \mathcal{R}(K)^\perp$. So

$$\sum_{i \in I} \|\Lambda_i f_1\|^2 \geq C\|f_1\|^2 \geq \frac{C}{\|K\|^2} \|K^*f_1\|^2.$$

Since $f_2 \in \mathcal{R}(K)^\perp, K^*f_2 = 0$ and

$$\frac{C}{\|K\|^2} \|K^*f_2\|^2 = 0 \leq C\|f_2\|^2 \leq \sum_{i \in I} \|\Lambda_i f_2\|^2.$$

And since $\overline{\text{span}}\{\Lambda_i^*(\mathcal{H}_i)\}_{i \in I} \subset \mathcal{R}(K)$, for any $f \in \mathcal{H}$, we have

$$\frac{C}{\|K\|^2} \|K^*f\|^2 \leq C\|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2. \quad \square$$

REMARK 1. If $\mathcal{R}(K) = \mathcal{H}$, then $\{\Lambda_i\}_{i \in I}$ is a K -g-frame for \mathcal{H} as well as a g-frame for \mathcal{H} .

Let $\{\Lambda_i\}_{i \in I}$ be a K -g-frame, we know that the frame operator of $\{\Lambda_i\}_{i \in I}$ may be not invertible, so there is no classical canonical dual for $\{\Lambda_i\}_{i \in I}$. Next, we will give a characterization of duals for a K -g-frame.

DEFINITION 2. Suppose $K \in \mathcal{L}(\mathcal{H})$ and $\{\Lambda_i\}_{i \in I}$ is a K -frame for \mathcal{H} . A g-Bessel sequence $\{\Gamma_i\}_{i \in I}$ for \mathcal{H} is called a K -dual g-frame of $\{\Lambda_i\}_{i \in I}$ if

$$Kf = \sum_{i \in I} \Lambda_i^* \Gamma_i f, \quad \forall f \in \mathcal{H}.$$

The following theorem provides a necessary and sufficient conditions for a Bessel g-sequence to be a K -dual g-frame. Note that $\{\delta_i\}_{i \in I}$ denotes the canonical basis of $\ell^2(I)$. Let $\{e_{ij}\}_{i \in I, j \in J_i}$ be an orthonormal basis for \mathcal{H}_i , then roughly speaking $\{e_{ij}\delta_i\}_{i \in I, j \in J_i}$ is an orthonormal basis of $(\sum_{i \in I} \oplus \mathcal{H}_i)_{\ell^2}$, because for any $\{f_i\}_{i \in I} \in (\sum_{i \in I} \oplus \mathcal{H}_i)_{\ell^2}$,

$$f_i = \sum_{i \in I} f_i \delta_i = \sum_{i \in I} \sum_{j \in J_i} \langle f_i, e_{ij} \rangle e_{ij} \delta_i.$$

THEOREM 4. Suppose that $K \in \mathcal{L}(\mathcal{H})$ and $\{\Lambda_i\}_{i \in I}$ is a K -g-frame for \mathcal{H} with the synthesis operator T_Λ . Then the dual $\{\Gamma_i\}_{i \in I}$ is a K -dual g-frame of $\{\Lambda_i\}_{i \in I}$ if and only if there exists a bounded operator $\Phi : (\sum_{i \in I} \oplus \mathcal{H}_i) \rightarrow \mathcal{H}$ such that $K^* = \Phi T_\Lambda^*$ and $\Gamma_i^* e_{i,j} = \Phi(e_{ij} \delta_i)$, $j \in J_i$, $i \in I$. Moreover, K -dual g-frame $\{\Gamma_i\}_{i \in I}$ is a K^* -g-frame.

Proof. Suppose that $\{\Gamma_i\}_{i \in I}$ is a K -dual g-frame of $\{\Lambda_i\}_{i \in I}$. Then the synthesis operator for $\{\Gamma_i\}_{i \in I}$ satisfies the conditions. In fact, $\{\Gamma_i\}_{i \in I}$ is a Bessel g-sequence and for all $f \in \mathcal{H}$, we have

$$K^* f = \sum_{i \in I} \Gamma_i^* \Lambda_i f.$$

Let Φ be the synthesis operator of $\{\Gamma_i\}_{i \in I}$, then

$$\Phi(e_{ij} \delta_i) = \sum_{i \in I} \Gamma_i^* e_{ij} \delta_i = \sum_{i \in I} u_{ij} \delta_i = \sum_{j \in J_i} u_{ij} = \sum_{i \in J_i} \langle e_{ij}, e_{ij} \rangle u_{ij} = \Gamma_i^* e_{ij}.$$

So a calculation as above shows that

$$K^* f = \sum_{i \in I} \Gamma_i^* \Lambda_i f = \sum_{i \in I} \Gamma_i^* \left(\sum_{j \in J_i} \langle f, u_{ij} \rangle e_{ij} \right) = \Phi \left(\sum_{i,j} \langle \Lambda_i f, e_{ij} \rangle e_{ij} \delta_i \right) = \Phi T_\Lambda^* f.$$

So $K^* = \Phi T_\Lambda^*$.

Conversely, if $\{f_i\}_{i \in I} \in (\sum_{i \in I} \oplus \mathcal{H}_i)_{\ell^2}$, then we have

$$\{f_i\}_{i \in I} = \sum_{i \in I} f_i \delta_i = \sum_{i \in I} \sum_{j \in J_i} \langle f_i, e_{ij} \rangle e_{ij} \delta_i.$$

Roughly speaking $\{e_{i,j} \delta_i\}_{i \in I, j \in J_i}$ is an orthonormal basis of $(\sum_{i \in I} \oplus \mathcal{H}_i)_{\ell^2}$. Let $u_{i,j}$ be defined as in (1). If $K^* = \Phi T_\Lambda^*$ and $\Gamma_i^* e_{i,j} = \Phi(e_{ij} \delta_i)$, $j \in J_i$, $i \in I$ then for all $f \in \mathcal{H}$

we have

$$\begin{aligned} K^*f &= \Phi T_\Lambda^* f = \Phi \left(\sum_{i,j} \langle \Lambda_i f, e_{ij} \rangle e_{ij} \delta_i \right) = \sum_{i \in I} \sum_{j \in J} \langle f, \Lambda_i^* e_{ij} \rangle \Phi(e_{ij} \delta_i) \\ &= \sum_{i \in I} \sum_{j \in J} \langle f, u_{ij} \rangle \Gamma_i^* e_{ij} = \sum_{i \in I} \Gamma_i^* \left(\sum_{j \in J_i} \langle f, u_{ij} \rangle e_{ij} \right) = \sum_{i \in I} \Gamma_i^* \Lambda_i f. \end{aligned}$$

Consequently, $Kf = \sum_{i \in I} \Lambda_i^* \Gamma_i f$, meaning that $\{\Gamma_i\}_{i \in I}$ is a K -dual g-frame of $\{\Lambda_i\}_{i \in I}$.

Moreover, let B and D be the Bessel bounds for $\{\Lambda_i\}_{i \in I}$ and $\{\Gamma_i\}_{i \in I}$, respectively. For any $f \in \mathcal{H}$, we have

$$\|(K^*)^* f\|^2 = \|Kf\|^2 = \|T_\Lambda \Phi^* f\|^2 \leq \|T_\Lambda\|^2 \|\Phi^* f\|^2 \leq B \sum_{i \in I} \|\Gamma_i f\|^2 \leq D \|f\|^2,$$

thus,

$$\frac{1}{B} \|(K^*)^* f\|^2 \leq \sum_{i \in I} \|\Gamma_i f\|^2 \leq \frac{D}{B} \|f\|^2, \quad \forall f \in \mathcal{H}.$$

Hence, K -dual g-frame $\{\Gamma_i\}_{i \in I}$ is a K^* -g-frame. \square

REMARK 2. When $K = I_{\mathcal{H}}$, the K -g-frame is exactly g-frame, in this case, the K -dual is exactly the canonical dual g-frame.

We end this section by giving the following results concerning the constructions of new K -frames.

THEOREM 5. Let $K \in \mathcal{L}(\mathcal{H})$ and let $\{\Lambda_i\}_{i \in I}$ be a K -g-frame. For $T \in \mathcal{L}(\mathcal{H})$ with $TK^* = K^*T$, then $\Lambda_i T$ is a K -g-frame for \mathcal{H} .

Proof. Suppose that $\{\Lambda_i\}_{i \in I}$ is a K -g-frame with bounds A and B . Now for any $f \in \mathcal{H}$, we have

$$\sum_{i \in I} \|\Lambda_i T f\|^2 \geq A \|K^* T f\|^2 = A \|TK^* f\|^2 \geq A \|T\| \|K^* f\|^2,$$

and

$$\sum_{i \in I} \|\Lambda_i T f\|^2 \leq B \|T f\|^2 \leq B \|T\|^2 \|f\|^2.$$

Hence, $\Lambda_i T$ is a K -g-frame for \mathcal{H} . \square

COROLLARY 3. Let $K \in \mathcal{L}(\mathcal{H})$ and let $\{\Lambda_i\}_{i \in I}$ be a K -g-frame. For $T \in \mathcal{L}(\mathcal{H})$, then $\Lambda_i T^*$ is a TK -g-frame for \mathcal{H} .

COROLLARY 4. Let $\{\Lambda_i\}_{i \in I}$ be a g-frame. For $K \in \mathcal{L}(\mathcal{H})$, then $\Lambda_i K^*$ is a K -g-frame for \mathcal{H} .

3. K -g-frames with operator quotient

In this section, we characterize K -g-frame by operator quotient.

DEFINITION 3. [12] Let U and V be bounded (linear) operators on a Hilbert space \mathcal{H} with the kernel condition

$$\mathcal{N}(V) \subset \mathcal{N}(U).$$

Then the quotient $[U/V]$ is a map from $\mathcal{R}(V)$ to $\mathcal{R}(U)$ defined by $Vf \mapsto Uf$ for all $f \in \mathcal{H}$.

We note that $W = [U/V]$ is a linear operator on \mathcal{H} if and only if $\mathcal{N}(V) \subset \mathcal{N}(U)$. In this case $\mathcal{D}(W) = \mathcal{R}(V)$, $\mathcal{R}(W) \subset \mathcal{R}(U)$ and $WV = U$. The quotient $[U/V]$ is called a semiclosed operator and its collection is closed under sum and product [13]. The authors of [20] fined that there is a relationship between K -frames and operator quotient operator. So we present few results on K -g-frame techniques on quotients of bounded operators. These results are inspired by the results in [20]. But there are some different properties in our results because g -frames are more complicated than ordinary frames.

THEOREM 6. Let $K \in \mathcal{H}$ and $\{\Lambda_i\}_{i \in I}$ be a Bessel g -sequence in \mathcal{H} with the frame operator S_Λ . Then $\{\Lambda_i\}_{i \in I}$ is a K -g-frame if and only if the quotient operator $[K^*/S_\Lambda^{1/2}]$ is bounded.

Proof. \implies : Since $\{\Lambda_i\}_{i \in I}$ is a K -g-frame for \mathcal{H} , there exists a constant $A > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 = \langle S_\Lambda f, f \rangle, \quad \forall f \in \mathcal{H}.$$

That is, $A\|K^*f\|^2 \leq \|S_\Lambda^{1/2}f\|^2$ for all $f \in \mathcal{H}$. Define $W : \mathcal{R}(S_\Lambda^{1/2}) \rightarrow \mathcal{R}(K^*)$ by

$$W(S_\Lambda^{1/2}f) = K^*f, \quad \forall f \in \mathcal{H}.$$

Then W is well-defined because $\mathcal{N}(S_\Lambda^{1/2}) \subset \mathcal{N}(K^*)$. For all $f \in \mathcal{H}$, we have

$$\|WS_\Lambda^{1/2}f\| = \|K^*f\| \leq \frac{1}{\sqrt{A}}\|S_\Lambda^{1/2}f\|.$$

So W is bounded. From the notion of quotient of bounded operators, W can be expressed as $[K^*/S_\Lambda^{1/2}]$.

\impliedby : Suppose that the quotient operator $[K^*/S_\Lambda^{1/2}]$ is bounded. Then there exists $\lambda > 0$ such that

$$\|K^*f\|^2 \leq \lambda \|S_\Lambda^{1/2}f\|^2, \quad \forall f \in \mathcal{H}.$$

Thus

$$\frac{1}{\lambda} \|K^*f\|^2 \leq \|S_\Lambda^{1/2}f\|^2 = \langle S_\Lambda f, f \rangle = \sum_{i \in I} \|\Lambda_i f\|^2,$$

for all $f \in \mathcal{H}$. Hence $\{\Lambda_i\}_{i \in I}$ is a K -g-frame for \mathcal{H} . \square

Let $K = I_{\mathcal{H}}$, we get the following corollary.

COROLLARY 5. Let $\{\Lambda_i\}_{i \in I}$ be a Bessel g -sequence for \mathcal{H} with the frame operator S_Λ . Then $\{\Lambda_i\}_{i \in I}$ is a g -frame if and only if the frame operator S_Λ is bounded.

THEOREM 7. Let $\{\Lambda_i\}_{i \in I}$ be a K - g -frame with the frame operator S_Λ and $T \in \mathcal{L}(\mathcal{H})$. Then the following are equivalent:

- (1) $\{\Lambda_i T^*\}_{i \in I}$ is a TK - g -frame;
- (2) $[(TK)^*/S_\Lambda^{1/2}T^*]$ is bounded;
- (3) $[(TK)^*/(TS_\Lambda T^*)^{1/2}]$ is bounded.

Proof. (1) \Rightarrow (2); Suppose that $\{\Lambda_i T\}_{i \in I}$ is a TK - g -frame. Then there exist $\lambda > 0$ such that

$$\lambda \|(TK)^* f\|^2 \leq \sum_{i \in I} \|\Lambda_i T^* f\|^2 = \langle S_\Lambda T^* f, T^* f \rangle = \|S_\Lambda^{1/2} T^* f\|^2, \quad \forall f \in \mathcal{H}.$$

Hence $[(TK)^*/S_\Lambda^{1/2}T^*]$ is bounded.

(2) \Rightarrow (3); Suppose $[(TK)^*/S_\Lambda^{1/2}T^*]$ is bounded. Then there exists $\mu > 0$ such that

$$\|(TK)^* f\|^2 \leq \mu \|S_\Lambda^{1/2} T^* f\|^2, \quad \forall f \in \mathcal{H}.$$

Since

$$\begin{aligned} \|(TS_\Lambda T^*)^{1/2} f\|^2 &= \left\langle (TS_\Lambda T^*)^{1/2} f, (TS_\Lambda T^*)^{1/2} f \right\rangle = \langle (TS_\Lambda T^*) f, f \rangle \\ &= \langle S_\Lambda T^* f, T^* f \rangle = \|S_\Lambda^{1/2} T^* f\|^2, \end{aligned}$$

for all $f \in \mathcal{H}$, we have

$$\frac{1}{\mu} \|(TK)^* f\|^2 \leq \|(TS_\Lambda T^*)^{1/2} f\|^2.$$

Therefore $[(TK)^*/(TS_\Lambda T^*)^{1/2}]$ is bounded.

(3) \Rightarrow (1); Suppose $[(TK)^*/(TS_\Lambda T^*)^{1/2}]$ is bounded. Then there exists $\mu > 0$ such that

$$\|(TK)^* f\|^2 \leq \mu \|(TS_\Lambda T^*)^{1/2} f\|^2, \quad \forall f \in \mathcal{H}.$$

Consider

$$\sum_{i \in I} \|\Lambda_i T f\|^2 = \langle S_\Lambda T^* f, T f \rangle = \langle TS_\Lambda T^* f, f \rangle,$$

So $TS_\Lambda T^*$ is positive and self-adjoint, its square root exists, and it is denoted by $(TS_\Lambda T^*)^{1/2}$. Hence

$$\sum_{i \in I} \|\Lambda_i T f\|^2 = \|(TS_\Lambda T^*)^{1/2} f\|^2 \geq \frac{1}{\mu} \|(TK)^* f\|^2, \quad \forall f \in \mathcal{H}.$$

Hence $\{\Lambda_i T\}_{i \in I}$ is a TK - g -frame. \square

COROLLARY 6. Let $K \in \mathcal{L}(\mathcal{H})$ and $\{\Lambda_i\}_{i \in I}$ be a g -frame for \mathcal{H} . Then the following are equivalent:

1. $\{\Lambda_i K^*\}_{i \in I}$ is a K - g -frame for \mathcal{H} ;
2. $[K^*/S^{1/2}]$ is bounded.

COROLLARY 7. Let $K \in \mathcal{L}(\mathcal{H})$ and $\{\Theta_i\}_{i \in I}$ be a g -orthonormal basis for \mathcal{H} . Then the following are equivalent:

1. $\{\Theta_i K^*\}$ is a K - g -frame for \mathcal{H} ;
2. $[K^*/I_{\mathcal{H}}]$ is bounded.

REMARK 3. The Theorem 7 proofs the conclusions in Corollary 3 and Corollary 4.

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