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D–NORM AND ITS ISOMETRIES ON $\mathfrak{c}_0$ SPACES

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Abstract. In this paper, based on the notion of *diameter,* we consider a natural preorder on $\mathfrak{c}_0(I)$ which is said "diametric majorization". Then by using this notion we define a norm on $\mathfrak{c}_0(I)$, where I is assumed to be an infinite set. This norm is equivalent to $\|\cdot\|_\infty$ and is said "d-norm". Finally, the structures of all bounded linear operators on $\mathfrak{c}_0(I)$ preserving diametric majorization and also isometries under the d-norm are both determined. We also give the relation between this isometries and isometries under the usual norm.

1. Introduction and preliminaries

Recently, many authors have discussed some various properties and structures of isometries on Banach spaces [5, 8]. For the collections of results in the topics of isometries we refer the reader to the monographs [3, 4].

In the following we point out to some important preliminaries.

DEFINITION 1.1. Let I be an infinite set (equipped with the discrete topology). The point $x_0 \in \mathbb{R}$ is called the limit of $f : I \to \mathbb{R}$ and is denoted by $\lim_{i\in I} f(i) = x_0$ (or more briefly $\lim f = x_0$) if for each neighborhood V of x_0 there exists a finite set $F \subseteq I$ such that $f(i) \in V$, for all $i \in I \setminus F$.

It is easily verified that if $\lim f = x_0$, then the set $\{i \in I; f(i) \neq x_0\}$ is at most a countable set. We will use the notation $\mathfrak{c}_0(I)$ for the set of all function $f : I \to \mathbb{R}$ with $\lim f = 0$. It is easily verified that every $f \in \mathfrak{c}_0(I)$ is bounded and $\mathfrak{c}_0(I)$ is a Banach space with the norm defined by $\|f\|_\infty = \sup_{i\in I} |f(i)|$. Each $f \in \mathfrak{c}_0(I)$ can be represented by $\sum_{i\in I} f(i)e_i$, where $e_i : I \to \mathbb{R}$ is defined as $e_i(j) = \delta_{ij}$, the Kronecker delta.

For a subset C of a metric space (X,d) the diameter of C is denoted by $\operatorname{diam}(C)$ and is defined as

$$\operatorname{diam}(C) := \sup\{d(x,y); x,y \in X\}.$$

For a function $f : I \to \mathbb{R}$, to simplify notations, we use $\operatorname{diam}(f)$, $\inf(f)$, and $\sup(f)$, instead of $\operatorname{diam}(\operatorname{Im}(f))$, $\inf_{i\in I}\{f(i)\}$ and $\sup_{i\in I}\{f(i)\}$, respectively. Also, the notation $\operatorname{co}(f)$ will be used for the convex combination of the set $\operatorname{Im}(f) := \{f(i); i \in I\}$.

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The idea of majorization is defined in various forms and on different spaces of finite and infinite dimension, such as $\mathbb{R}^n, M_{m\times n}(\mathbb{R}), \ell^p(I), \ell^1(I)^+, \ell_\infty, \mathfrak{c}_0, \mathfrak{c}$ and etc. See for examples [1, 2, 6, 7, 9]. In this paper, on the basis of the notion of $\mathrm{diam}(f)$, we define a relation on $\mathfrak{c}_0(I)$, that is said to be the diametric majorization. Moreover, we define a norm on $\mathfrak{c}_0(I)$, where I is assumed to be an infinite set, that is equivalent to $\|\cdot\|_\infty$ and discuss on the properties and characterization of all isometries under this norm. All bounded linear operators $T : \mathfrak{c}_0(I) \to \mathfrak{c}_0(I)$ which preserve diametric majorization, and their relations between this isometries are also determined.

2. Main results

Diametric majorization defines a relation on $\mathfrak{c}_0(I)$ that compares the distance occurred between the values of $\inf(f)$ and $\sup(f)$. More precisely,

DEFINITION 2.1. For $f,g \in \mathfrak{c}_0(I)$ we say that f is diametrically majorized by g and is denoted by $f \prec_d g$, whenever $\mathrm{diam}(f) \leqslant \mathrm{diam}(g)$.

The comparison under the relation diametric majorization for any arbitrary two elements is possible, i.e., for $f,g \in \mathfrak{c}_0(I)$ we have either $f \prec_d g$ or $g \prec_d f$.

DEFINITION 2.2. For $f \in \mathfrak{c}_0(I)$ define $\|f\|_d := \mathrm{diam}(f)$ and it is called the d-norm.

It is easily verified that the d-norm is a norm on $\mathfrak{c}_0(I)$, if and only if I is an infinite set. Moreover, since $\|\cdot\|_\infty \leqslant \|\cdot\|_d \leqslant 2\|\cdot\|_\infty$, this norm is a complete norm and is equivalent to the infinity norm. Also, notice that $\|f\|_d$ is equal to the length of (the interval) $\mathrm{co}(f)$, that is $\sup(f) - \inf(f)$.

A bounded linear operator $T : \mathfrak{c}_0(I) \to \mathfrak{c}_0(I)$ is called a diametric majorization preserver if $f \prec_d g$ implies $Tf \prec_d Tg$, for all $f \in \mathfrak{c}_0(I)$. The set of all such operators is denoted by $\mathscr{P}_d$. Also, T is said to be a diameter preserving isometry or (for short) d-isometry if T is an isometry, when $\mathfrak{c}_0(I)$ is equipped with the d-norm. The set of all d-isometries on $\mathfrak{c}_0(I)$ is denoted by $\mathscr{I}_d$.

The next theorem gives the relation between $\mathscr{I}_d$ and $\mathscr{P}_d$.

THEOREM 2.3. *The following statements are equivalent for a bounded linear operator $T : \mathfrak{c}_0(I) \to \mathfrak{c}_0(I)$.*

(i) *T preserves diametric majorization.*

(ii) *T is a scalar multiple of a d-isometry.*

Proof. (i)$\Rightarrow$(ii). Suppose that $T \in \mathscr{P}_d$. For each $f,g \in \mathfrak{c}_0(I)$ with $\|f\|_d = \|g\|_d \neq 0$, we have $\|Tf\|_d = \|Tg\|_d$. Hence $\|T\frac{f}{\|f\|_d}\|_d = \|T\frac{g}{\|g\|_d}\|_d$. Thus the value $c := \|T\frac{f}{\|f\|_d}\|_d$ is constant, independent of chosen $f \in \mathfrak{c}_0(I)$ (with $\|f\|_d \neq 0$). Now let

$f \in \mathfrak{c}_0(I)$. If $\|f\|_d = 0$, then $f \prec_d 0$, which implies $Tf \prec_d 0$. Therefore, $\|Tf\|_d = 0$. So, we have obviously

$$\|Tf\|_d = c\,\|f\|_d. \tag{1}$$

Now if $\|f\|_d \neq 0$, then

$$\|Tf\|_d = \|f\|_d \cdot \|T\frac{f}{\|f\|_d}\|_d = c\,\|f\|_d. \tag{2}$$

The conclusion follows by using (1), (2), and considering two cases: $c = 0$ and $c \neq 0$.

(ii)$\Rightarrow$(i). It is evident. □

Theorem 2.3 formulates that the structure of the elements in $\mathscr{P}_d$ on $\mathfrak{c}_0(I)$ is directly related to the structure of d-isometries. For this reason we shall focus on d-isometries on $\mathfrak{c}_0(I)$.

LEMMA 2.4. *Let $T \in \mathscr{I}_d$ and $i_0 \in I$. Then*

$$-1 \leqslant \sum_{j\in I^-} Te_j(i_0) \leqslant 0 \leqslant \sum_{j\in I^+} Te_j(i_0) \leqslant 1,$$

where $I^- := \{j \in I; Te_j(i_0) < 0\}$ and $I^+ := \{j \in I; Te_j(i_0) > 0\}$.

Proof. To prove $0 \leqslant \sum_{j\in I^+} Te_j(i_0) \leqslant 1$, it is sufficient to show that for each finite subset $F \subseteq I^+$, we have $0 \leqslant \sum_{j\in F} Te_j(i_0) \leqslant 1$. So, we assume that $F \subseteq I^+$. Take $f := \sum_{j\in F} e_j$. Since $\|f\|_d = 1$ and $T \in \mathscr{I}_d$ then $\|Tf\|_d = 1$. Also we have $Tf(i_0) \geqslant 0$, because $Te_j(i_0) \geqslant 0$ for all $j \in I^+$. Now if $Tf(i_0) > 1$, then by using the fact that 0 is a limit point of $\operatorname{Im}(f)$, we have $1 = \|Tf\|_d = \sup(Tf) - \inf(Tf) \geqslant Tf(i_0) - 0 = Tf(i_0) > 1$, which leads to a contradiction. So, $Tf(i_0) \in [0,1]$. A similar arguments shows $-1 \leqslant \sum_{j\in I^-} Te_j(i_0) \leqslant 0$. □

The next result concerns the limits of $\left(\inf Te_j\right)_{j\in I}$ and $\left(\sup Te_j\right)_{j\in I}$.

LEMMA 2.5. *Let $T \in \mathscr{I}_d$, $j_0 \in I$, and $\lambda := \sup Te_{j_0}$. Then*

$$\lambda = \lim_{j\in I} \lambda_j, \quad \text{and} \quad \lambda - 1 = \lim_{j\in I} \eta_j,$$

where $\lambda_j := \sup Te_j$, and $\eta_j := \inf Te_j$.

Proof. Because $\|Te_{j_0}\|_d = 1$, it follows that $\lambda \in [0,1]$. By replacing $-T$ by T (if necessary), we may assume that $0 < \lambda \leqslant 1$. Since $\lambda = \sup Te_{j_0} > 0$, we have $\lambda = \max Te_{j_0}$. So, there exists $i_1 \in I$ with $Te_{j_0}(i_1) = \lambda$. Now let $0 < \varepsilon < \lambda$ is arbitrary. Then $\inf Te_{j_0} = \lambda - 1$ and therefore there is $i_2 \in I$ such that

$$Te_{j_0}(i_2) \in [\lambda - 1, \lambda - 1 + \varepsilon).$$

Assume that $\varepsilon_0 := \min\{\varepsilon, 1 - \varepsilon\}$. Since $\lim_{i\in I} Te_{j_0}(i) = 0$, there exists a finite subset $F \subseteq I$ such that

$$\forall i \in I \setminus F, \quad |Te_{j_0}(i)| < \frac{\varepsilon_0}{2}.$$

It is easy to see that $i_1 \in F$. Moreover, without loss of generality, we may assume that $i_1 \in F$.

Since F is a finite set, there is a finite set $G \subseteq I$ such that

$$\forall j \in I \setminus G,\ \forall i \in F \quad |Te_j(i)| < \frac{\varepsilon_0}{2}.$$

Therefore, we have

$$\begin{aligned}\lambda - 1 - \frac{\varepsilon_0}{2} \leqslant Te_{j_0}(i_2) + Te_j(i_2) \leqslant (\lambda - 1 + \varepsilon) + \frac{\varepsilon_0}{2} \\ \leqslant \lambda - \frac{\varepsilon_0}{2} \leqslant Te_{j_0}(i_1) + Te_j(i_1) \leqslant \lambda + \frac{\varepsilon_0}{2} + \varepsilon.\end{aligned}$$

Let $j \in I \setminus G$ be a fixed element. According to the previous relations, if there exists $i \in I$ such that

$$Te_{j_0}(i) + Te_j(i) > \lambda + \frac{\varepsilon_0}{2} + \varepsilon,$$

or

$$Te_{j_0}(i) + Te_j(i) < \lambda - 1 - \frac{\varepsilon_0}{2},$$

then we have $1 = \|Te_{j_0} + Te_j\|_d > 1$. This contradiction implies that for all $i \in I$

$$\lambda - 1 - \frac{\varepsilon_0}{2} \leqslant Te_{j_0}(i) + Te_j(i) \leqslant \lambda + \frac{\varepsilon_0}{2} + \varepsilon,$$

or

$$\lambda - 1 - \frac{\varepsilon_0}{2} - Te_{j_0}(i) \leqslant Te_j(i) \leqslant \lambda + \frac{\varepsilon_0}{2} + \varepsilon - Te_{j_0}(i).$$

So, if $i \in I \setminus F$, then

$$\lambda - 1 - \varepsilon \leqslant \lambda - 1 - \varepsilon_0 \leqslant Te_j(i) \leqslant \lambda + \varepsilon_0 + \varepsilon \leqslant \lambda + 2\varepsilon, \tag{3}$$

and if $i \in F$, then

$$\frac{-\varepsilon}{2} \leqslant Te_j(i) \leqslant \frac{\varepsilon}{2}. \tag{4}$$

By using (3) and (4) we have

$$\lambda - 1 - \varepsilon \leqslant \min\left\{\lambda - 1 - \varepsilon, \frac{-\varepsilon}{2}\right\} \leqslant \inf Te_j \leqslant \sup Te_j \leqslant \max\left\{\lambda + 2\varepsilon, \frac{\varepsilon}{2}\right\} \leqslant \lambda + 2\varepsilon.$$

If $\sup Te_j < \lambda - \varepsilon$, then $\sup Te_j - \inf Te_j < 1$, which contradicts because $\sup Te_j - \inf Te_j = 1$. Therefore,

$$\lambda - \varepsilon \leqslant \lambda_j = \sup Te_j \leqslant \lambda + 2\varepsilon.$$

A similar argument shows

$$\lambda - 1 - \varepsilon \leqslant \eta_j = \inf Te_j \leqslant \lambda - 1 + 2\varepsilon.$$

Thus, we proved that

$$\forall\, 0 < \varepsilon < \lambda,\ \exists G \subseteq I (\text{finite set})\ \text{ s.t. } \forall j \in I \setminus G \quad |\lambda_j - \lambda| \leqslant 2\varepsilon\ ,\ |\eta_j - (\lambda - 1)| \leqslant 2\varepsilon,$$

i.e. $\lambda = \lim_{j \in I} \lambda_j$ and $\lambda - 1 = \lim_{j \in I} \eta_j$. $\square$

REMARK 2.6. Let $T \in \mathscr{I}_d$. Then the value of $\inf Te_j$ is independent of j because according to the previous lemma

$$\sup Te_{j_1} = \lim_{j\in I}\left(\sup Te_j\right) = \sup Te_{j_2},$$

holds for each $j_1, j_2 \in I$. A similar argument also holds for $\inf Te_j$.

THEOREM 2.7. *Suppose that $T \in \mathscr{I}_d$. Then one of the following conditions hold.*

(i) *For each $j \in I$, $\inf Te_j = -1$ and $\sup Te_j = 0$; or*

(ii) *For each $j \in I$, $\inf Te_j = 0$ and $\sup Te_j = 1$.*

Proof. By the previous remark the value of $\lambda := \sup Te_j$ is constant. It is clear that $\lambda \in [0,1]$. To show that $\lambda = 0$ or $\lambda = 1$, suppose on the contrary $\lambda \in (0,1)$. Since $\inf Te_j = \lambda - 1 < 0 < \lambda = \sup Te_j$, we conclude that $\lambda - 1 = \min Te_j = Te_{j_0}(i_1)$ and $\lambda = \max Te_j = Te_{j_0}(i_2)$, for some $i_1, i_2 \in I$. Now for each $0 < \varepsilon < \min\{\lambda, 1-\lambda\}$, there exists a finite set $F \subseteq I$ such that $|Te_{j_0}(i)| < \frac{\varepsilon}{2}$ for all $i \in I \setminus F$. It is clear that $i_1, i_2 \in F$. Moreover, there exists a finite set $G \subseteq I$ such that for all $i \in F$, and $j \in I \setminus G$, we have $|Te_j(i)| < \frac{\varepsilon}{2}$, because F is finite and $\lim_{j\in I} Te_j(i) = 0$ for all $i \in F$. Suppose that $j \in I \setminus G$. Then we have

$$a_1 := Te_{j_0}(i_1) - Te_j(i_1) = (\lambda - 1) - Te_j(i_1) \in \operatorname{Im}(Te_{j_0} - Te_j), \tag{5}$$

and

$$a_2 := Te_{j_0}(i_2) - Te_j(i_2) = \lambda - Te_j(i_2) \in \operatorname{Im}(Te_{j_0} - Te_j). \tag{6}$$

Now, there exist $i_3, i_4 \in I$ with $Te_j(i_3) = \lambda - 1$ and $Te_j(i_4) = \lambda$, since $\min Te_j = \lambda - 1$ and $\max Te_j = \lambda$. It is clear that $i_3, i_4 \in I \setminus F$. Also we have

$$\forall i \in F, \quad |Te_j(i)| < \frac{\varepsilon}{2} < \min\{\lambda, 1-\lambda\}.$$

Hence $Te_j(i) \notin \{\lambda, \lambda - 1\}$, and we also have

$$a_3 := Te_{j_0}(i_3) - Te_j(i_3) = Te_{j_0}(i_3) - \lambda \in \operatorname{Im}(Te_{j_0} - Te_j), \tag{7}$$

and

$$a_4 := Te_{j_0}(i_4) - Te_j(i_4) = Te_{j_0}(i_4) - \lambda \in \operatorname{Im}(Te_{j_0} - Te_j). \tag{8}$$

Relations (5)–(8) imply $a_1, a_2, a_3, a_4 \in \operatorname{Im}(Te_{j_0} - Te_j)$ and furthermore,

$$a_1 = \lambda - 1 - Te_j(i_1) \leqslant \lambda - 1 + \frac{\varepsilon}{2}\lambda - \frac{\varepsilon}{2} \leqslant \lambda - Te_j(i_2) = a_2.$$

So we have $a_1 \leqslant a_2$. A similar method implies $a_3 \leqslant a_4$. Thus

$$\operatorname{co}(Te_{j_0} - Te_j) \subseteq \left[\min\left(\lambda - 1 - \frac{\varepsilon}{2}, -\lambda - \frac{\varepsilon}{2}\right), \max\left(\lambda + \frac{\varepsilon}{2}, 1 - \lambda + \frac{\varepsilon}{2}\right)\right] \tag{9}$$

Now if $\lambda \leqslant 1-\lambda$, then the length of the interval

$$\left[\min\left(\lambda-1-\frac{\varepsilon}{2},-\lambda-\frac{\varepsilon}{2}\right),\max\left(\lambda+\frac{\varepsilon}{2},1-\lambda+\frac{\varepsilon}{2}\right)\right],$$

used in the previous relation, is equal to $2-2\lambda+\varepsilon$. So, according to (9) we have

$$2=\|Te_{j_0}-Te_j\|_d \leqslant 2-2\lambda+\varepsilon.$$

Thus $\lambda \leqslant \frac{\varepsilon}{2}$, which contradicts the choice ε, since ε was selected such that $0<\varepsilon<\min\{\lambda,1-\lambda\}$. In the case $\lambda \geqslant 1-\lambda$, using again (9), we obtain $2=\|Te_{j_0}-Te_j\|_d \leqslant 2\lambda+\varepsilon$. This case also contradicts the choice of ε. $\square$

In the following theorem, we obtain the structure of diameter preserving isometries on $\mathfrak{c}_0(I)$.

THEOREM 2.8. *Suppose that $T:\mathfrak{c}_0(I)\to\mathfrak{c}_0(I)$ is a bounded linear operator. Then $T\in\mathscr{I}_d$ if and only if one of the following conditions hold.*

(i) *For each $j\in I$, $\min Te_j=0$ and $\max Te_j=1$, and for each $i\in I$, $0\leqslant \sum_{k\in I} Te_k(i)\leqslant 1$;*
or

(ii) *For each $j\in I$, $\min Te_j=-1$ and $\max Te_j=0$, and for each $i\in I$, $-1\leqslant \sum_{k\in I} Te_k(i)\leqslant 0$.*

Proof. Using Theorem 2.7 for the constant value $\lambda:=\sup Te_j$, we have $\lambda=0$, or $\lambda=1$. By replacing $-T$ by T, we may assume that $\lambda=1$. So, $\inf Te_j=0$ and $\sup Te_j=1$, for all $j\in I$. Now suppose $j_0\in I$. Since $Te_{j_0}\in\mathfrak{c}_0(I)$ and $\sup Te_{j_0}>0$, the value of $\max Te_{j_0}$ exists and $\max Te_{j_0}=\sup Te_{j_0}=1$.

A similar argument for $j_1\neq j_0$, leads to $\max Te_{j_1}=1$. Thus there exists $i_1\in I$ with $Te_{j_1}(i_1)=1$. On the other hand $0\leqslant Te_{j_0}(i_1)\leqslant 1$. If $Te_{j_0}(i_1)>0$, then

$$1<Te_{j_0}(i_1)+Te_{j_1}(i_1)\leqslant \sum_{j\in I^+} Te_j(i_1),$$

which contradicts to Lemma 2.4. Thus $Te_{j_0}(i_1)=0$, and therefore $\min Te_{j_0}$ exists and is equal to 0. On the other hand,

$$\min Te_k=0\leqslant Te_k(i)\leqslant \max Te_k=1.$$

Thus using Lemma 2.4, again we have

$$0\leqslant \sum_{k\in I} Te_k(i)=\sum_{k\in I^+} Te_k(i)\leqslant 1.$$

Conversely, suppose that (i) satisfies and $f=(f_j)_{j\in I}\in\mathfrak{c}_0(I)$. For each $j_0\in I$, there is $i_0\in I$ such that $Te_{j_0}(i_0)=1$ because $\max Te_{j_0}=1$. So, we can imply that $Te_j(i_0)=0$ for all $j\neq j_0$ since $\sum_{j\in I} Te_j(i_0)=1$, $0\leqslant Te_j(i_0)\leqslant 1$, and $Te_{j_0}(i_0)=1$. Therefore

$$Tf(i_0)=\sum_{j\in I} Te_j(i_0)f_j=Te_{j_0}(i_0)f_{j_0}=f_{j_0},$$

which implies $\mathrm{Im}(f) \subseteq \mathrm{Im}(Tf)$. Thus

$$\|f\|_d \leqslant \|Tf\|_d. \tag{10}$$

On the other hand, for $f \in \mathfrak{c}_0(I)$, $\inf(f) \leqslant 0 \leqslant \sup(f)$ since I is an infinite set. So, we have

$$\inf(f) \leqslant \inf(f) \sum_{j\in I} Te_j(i) \leqslant Tf(i) = \sum_{j\in I} Te_j(i) f_j \leqslant \sup(f) \sum_{j\in I} Te_j(i) \leqslant \sup(f).$$

This implies

$$\|Tf\|_d \leqslant \|f\|_d. \tag{11}$$

From (10) and (11), T is a d-isometry. Now, if condition (ii) holds, then the operator $-T$ satisfies (i). Therefore, the previous part of this proof shows that $-T$ is a d-isometry. Then T is also a d-isometry. $\square$

Note that the previous theorem says that any d-isometry $T : \mathfrak{c}_0(I) \to \mathfrak{c}_0(I)$ is either positive (i.e. $Tf \geqslant 0$, for all $f \geqslant 0$) or negative operator (i.e. $Tf \leqslant 0$, for all $f \geqslant 0$).

The following remark compares the relation between isometries under the usual norm on $\mathfrak{c}_0(I)$ and d-isometries.

REMARK 2.9. It can be proved that, the operator $T : \mathfrak{c}_0(I) \to \mathfrak{c}_0(I)$ is an isometry(in the usual sense), if and only if T satisfies the following conditions.

(i) For each $j \in I$, $\|Te_j\|_\infty = 1$,

(ii) For each $i \in I, \sum_{j\in I} |Te_j(i)| \leqslant 1$.

Thus according to Theorem 2.8, every d-isometry $T : \mathfrak{c}_0(I) \to \mathfrak{c}_0(I)$ is an isometry, but the converse need not be true in general. For example, if $T : \mathfrak{c}_0 \to \mathfrak{c}_0$, is defined for each $f = (f_1, f_2, \dots) \in \mathfrak{c}_0$ by $T(f) = (\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} f_n, f_1, f_2, \dots)$, then T is an isometry however, $T \notin \mathscr{I}_d$, because $\|Te_1\|_d = \|(\frac{-1}{2}, 1, 0, 0, 0, \dots)\|_d = \frac{3}{2} \neq 1 = \|e_1\|_d$.

REMARK 2.10. Let I be a finite set with n elements. Lemma 2.4, Theorems 2.7 and 2.8 do not hold. Towards a counterexample for these results we define $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $T = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. Note that T satisfies:

$$\|Tf\|_d = |(x+3y) - (2x+2y)| = |x-y| = \|f\|_d,$$

for each $f = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, which shows that $T \in \mathscr{I}_d$.

In fact, an easy computation shows that if $n = 1$, then every linear map belongs to $\mathscr{P}_d$ and if $n = 2$, then $T \in \mathscr{I}_d$ if and only if T has the matrix form $T = \begin{bmatrix} a & b+c \\ a+c & b \end{bmatrix}$, for some $a, b, c \in \mathbb{R}$. But for $n \geqslant 3$ without being able to characterize $\mathscr{P}_d$, we give a large class of matrices in $\mathscr{P}_d$. In fact, if $\theta : \mathbb{R}^n \to \mathbb{R}$ is a linear operator and $P : \mathbb{R}^n \to \mathbb{R}^n$ is a permutation, then for $T : \mathbb{R}^n \to \mathbb{R}^n$ which is defined by

$$Tf = \theta(f)\mathbf{e} + \alpha P(f),$$

we have $T \in \mathscr{P}_d$, where $\mathbf{e} := \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. This claim can be proved easily.

Remark 2.10 shows a significant difference for the diametric majorization preservers when I is finite and infinite. More precisely, we give examples of diametric majorization preservers, but a complete characterization for these operators remains open.

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