

ERRATUM TO: ON THE DJL CONJECTURE FOR ORDER 6

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Abstract. In this note an erratum is provided to the article “On the DJL conjecture for order 6” by Naomi Shaked-Monderer, published in *Operators and Matrices* 11(1), 2017, 71–88. We will demonstrate and correct two errors in this article. The first error is in the statement of a proposition, which omits a certain category of extreme matrices. The second error is in the proof of a lemma. Fortunately the lemma itself is correct, and in this note we will in fact show that a stronger result holds.

This note looks at two errors in the paper [1]. The same terms and notations are used in this note as in the original paper, and so it has been decided not to reintroduce them here.

In Section 1 of this note we look at the first error which is in [1, Proposition 2.11], where a certain category of extremal matrix is omitted. This error is briefly explained and a corrected lemma is given.

In Section 2 we look at the second error in the proof of [1, Lemma 3.3]. Fortunately it is only the proof that is incorrect and not the result. We provide a counter example to an important aspect of the proof, and then go on to prove a stronger result from which [1, Lemma 3.3] directly follows.

1. First error

The first error in [1] is Proposition 2.11. In this it states that if $A \in \mathcal{COP}_n$ is an extremal copositive matrix with $a_{ii} = 0$ for some i , then $a_{ij} = 0$ for all j . However this is incorrect as it omits the case of extremal copositive matrices which are nonnegative but not positive semidefinite [2]. A summary of known results on the extremal copositive matrices is provided by [3, Theorem 8.20], and using this summary we get the following corrected form of Proposition 2.11:

LEMMA 1. *Let $A \in \mathcal{COP}_n$ be an extremal copositive matrix with at least one negative entry.*

1. *If $a_{ii} = 0$, then $a_{ij} = 0$ for every $i \neq j$, and $A(i) \in \mathcal{COP}_{n-1}$ is extremal.*
2. *If $\text{diag}(A) = \mathbf{1}$, then $a_{ij} \in [-1, 1]$ for every $i \neq j$.*

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2. Second error

The second error in [1] is in the proof of Lemma 3.3, which comes from a misinterpretation of the results from [4]. This error is slightly more difficult to demonstrate and correct, and we will break down doing this into two parts. First, in Subsection 2.1, we will give a counter example demonstrating that an aspect of the proof is incorrect. Then, in Subsection 2.2, we will prove a new (stronger) result from which [1, Lemma 3.3] directly follows.

2.1. Counter example

The incorrect claim in the proof of [1, Lemma 3.3] can be summed up as follows:

CLAIM 2. *Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}_+^n$ be minimal zeros of $M \in \mathcal{COP}_n$ with supports $\sigma_{\mathbf{v}}, \sigma_{\mathbf{w}}$ respectively, and let \mathbf{x} be a zero of M with support $\sigma_{\mathbf{v}} \cup \sigma_{\mathbf{w}}$. Then $\mathbf{x} = a\mathbf{v} + b\mathbf{w}$ for some $\mathbf{v}, \mathbf{w} \in \mathbb{R}_+$.*

This incorrect claim comes from a misinterpretation of the following (correct) result:

LEMMA 3. ([4, Corollary 3.4], [1, Proposition 2.18]) *For $A \in \mathcal{COP}_n$ and a zero of this given by $\mathbf{u} \in \mathbb{R}_+^n$, there exists minimal zeros $\mathbf{w}_1, \dots, \mathbf{w}_m \in \mathbb{R}_+^n$ such that $\mathbf{u} = \sum_{i=1}^m a_i \mathbf{w}_i$ for some $\mathbf{a} \in \mathbb{R}^m$ entrywise strictly positive. Letting σ be the support of \mathbf{u} and σ_i the support of \mathbf{w}_i for all i we then have $\sigma = \bigcup_{i=1}^m \sigma_i$.*

In spite of this (correct) lemma, and in contradiction to the claim, if a zero support is equal to the union of some minimal zero supports, then the corresponding zero is not necessarily a nonnegative combination of the corresponding minimal zeros. Instead it can be a nonnegative combination of a different set of minimal zeros. We now give an example of this occurring.

EXAMPLE 4. Consider the matrix

$$M = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}^T \in \mathcal{COP}_n.$$

Up to multiplication by a positive scalar, the minimal zeros of this are given uniquely by the vectors:

$$\mathbf{t} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

whose supports are respectively

$$\sigma_{\mathbf{t}} = \{1, 3\}, \quad \sigma_{\mathbf{u}} = \{2, 4\}, \quad \sigma_{\mathbf{v}} = \{2, 3\}, \quad \sigma_{\mathbf{w}} = \{1, 4\}.$$

Now considering the vector $\mathbf{x} = \mathbf{t} + 2\mathbf{u} = (1 \ 2 \ 1 \ 2)^\top$, we have that \mathbf{x} is a zero of M with support equal to $\sigma_{\mathbf{v}} \cup \sigma_{\mathbf{w}} = \{1, 2, 3, 4\}$, however clearly $\mathbf{x} \neq a\mathbf{v} + b\mathbf{w}$ for all $a, b \in \mathbb{R}$.

2.2. New result

We will now prove the following new result, from which [1, Lemma 3.3] directly follows:

THEOREM 5. *Let $M \in \mathcal{COP}_n$ and for $m \geq 3$ let $\sigma_1, \dots, \sigma_m$ be zero supports of M such that $\sigma_{i,j} := \sigma_i \cup \sigma_j$ is also a zero support of M for all $1 \leq i < j \leq m$. Then $\bigcup_{i=1}^m \sigma_i$ is a zero support of M .*

Proof. For $1 \leq i \leq m$ let $\mathbf{w}_i \in \mathbb{R}_+^n$ be a zero of M with support σ_i .

For all $1 \leq i, j \leq m$ we have that the support of σ_i is contained in the $\sigma_{i,j}$, and thus we trivially have

$$\mathbf{w}_i[\sigma_{i,j}]^\top M[\sigma_{i,j}] \mathbf{w}_i[\sigma_{i,j}] = \mathbf{w}_i^\top M \mathbf{w}_i = 0. \quad (1)$$

Therefore, using [3, Theorems 6.3 and 6.4] and the fact that $\sigma_{i,j}$ is a zero support of M , we have that $M[\sigma_{i,j}]$ is a positive semidefinite matrix and $M[\sigma_{i,j}] \mathbf{w}_i[\sigma_{i,j}] = \mathbf{0}$.

Therefore, for all $1 \leq i, j \leq m$ we have

$$\mathbf{w}_j^\top M \mathbf{w}_i = \mathbf{w}_j[\sigma_{i,j}]^\top M[\sigma_{i,j}] \mathbf{w}_i[\sigma_{i,j}] = \mathbf{w}_j[\sigma_{i,j}]^\top \mathbf{0} = 0.$$

Now letting $\mathbf{z} = \sum_{i=1}^m \mathbf{w}_i \in \mathbb{R}_+^n$, we have that the support of \mathbf{z} is equal to $\bigcup_{i=1}^m \sigma_i$ and $\mathbf{z}^\top M \mathbf{z} = (\sum_{j=1}^m \mathbf{w}_j)^\top M (\sum_{i=1}^m \mathbf{w}_i) = \sum_{i,j=1}^m \mathbf{w}_j^\top M \mathbf{w}_i = 0$. \square

An immediate corollary of this result is [1, Lemma 3.3], which for the sake of completeness is included below:

COROLLARY 6. ([1, Lemma 3.3]) *Let $M \in \mathcal{COP}_n$, and let $\sigma_1, \sigma_2, \sigma_3$ be three minimal supports of M , such that $\sigma_i \cup \sigma_j$ is a zero support for every $1 \leq i \neq j \leq 3$. Then $\sigma_1 \cup \sigma_2 \cup \sigma_3$ is a zero support of M .*

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