

PRODUCTS OF RADIAL DERIVATIVE AND WEIGHTED COMPOSITION OPERATORS FROM WEIGHTED BERGMAN–ORLICZ SPACES TO WEIGHTED–TYPE SPACES

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Abstract. Let $H(\mathbb{B}^n)$ be the space of all holomorphic functions on the unit ball \mathbb{B}^n of \mathbb{C}^n , φ a holomorphic self-map of \mathbb{B}^n , $u \in H(\mathbb{B}^n)$, and \mathfrak{R} the radial derivative operator on $H(\mathbb{B}^n)$. Two operators on $H(\mathbb{B}^n)$ are defined by $\mathfrak{R}W_{u,\varphi}f(z) = \mathfrak{R}(u(z)f(\varphi(z)))$ and $W_{u,\varphi}\mathfrak{R}f(z) = u(z)\mathfrak{R}f(\varphi(z))$, which are called the products of radial derivative operators and weighted composition operators. In this paper, the boundedness and compactness of the operators $\mathfrak{R}W_{u,\varphi}$ and $W_{u,\varphi}\mathfrak{R}$ from weighted Bergman-Orlicz spaces to a class of weighted-type spaces are characterized.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane \mathbb{C} and $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ the unit ball of the complex vector space \mathbb{C}^n . Let $H(\mathbb{D})$ be the space of all analytic functions on \mathbb{D} , $H(\mathbb{B}^n)$ the space of all holomorphic functions on \mathbb{B}^n and $S(\mathbb{B}^n)$ the class of all holomorphic self-maps of \mathbb{B}^n .

Let $\varphi \in S(\mathbb{B}^n)$ and $u \in H(\mathbb{B}^n)$. The weighted composition operator $W_{u,\varphi}$ is defined on $H(\mathbb{B}^n)$ by

$$W_{u,\varphi}f(z) = u(z)f(\varphi(z)).$$

When $u(z) \equiv 1$ on \mathbb{B}^n , the operator $W_{u,\varphi}$ is reduced to the composition operator, usually denoted by C_φ , while if $\varphi(z) = z$, the operator $W_{u,\varphi}$ is reduced to the multiplication operator, usually denoted by M_u . It is clear that the weighted composition operator is the product of composition operator and multiplication operator. Weighted composition operators between various spaces of holomorphic functions on different domains have been studied by numerous authors (see, e.g., [4, 6, 8, 9, 11, 12, 16, 21, 23, 24, 25, 30, 31, 38, 39, 43] and the references therein).

Let D be the differentiation operator on $H(\mathbb{D})$ defined by

$$Df(z) = f'(z).$$

The products of differentiation and composition operators DC_φ and $C_\varphi D$ have been studied, for example, in [2, 7, 10, 13, 14, 17, 27, 32, 34], while some generalizations of

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these product-type operators have been studied, for example, in [15, 29, 35, 36, 37, 40]. Recently, in [22] Sharma has studied the following six operators:

$$M_u C_\varphi D, M_u D C_\varphi, C_\varphi M_u D, C_\varphi D M_u, D M_u C_\varphi, D C_\varphi M_u. \tag{1}$$

The operators in (1) have been also studied by Stević, Sharma and Bhat in a unified manner in [41] and [42]. Quite recently, the products of differentiation and weighted composition operators $DW_{u,\varphi}$ and $W_{u,\varphi}D$ have also been studied in [5]. A natural problem is to consider the products of radial derivative and weighted composition operators on some subspaces of $H(\mathbb{B}^n)$. Let \mathfrak{R} be the radial derivative operator on $H(\mathbb{B}^n)$, that is

$$\mathfrak{R}f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

The products of radial derivative and weighted composition operators on some subspaces of $H(\mathbb{B}^n)$ are defined as follows

$$\mathfrak{R}W_{u,\varphi}f(z) = \mathfrak{R}(u(z)f(\varphi(z)))$$

and

$$W_{u,\varphi}\mathfrak{R}f(z) = u(z)\mathfrak{R}f(\varphi(z)).$$

This paper is devoted to studying the operators $\mathfrak{R}W_{u,\varphi}$ and $W_{u,\varphi}\mathfrak{R}$ from weighted Bergman-Orlicz spaces to a class of weighted-type spaces. Weighted composition operators and the integral-type operators defined and studied in [26, 28, 33], from weighted Bergman-Orlicz spaces to a class of weighted-type spaces have been studied in [18]. Here it must be mentioned that Stević in [35] introduced a more general operator $\mathfrak{R}_{u,\varphi}^m$, called the weighted iterated radial composition operator. Clearly, for $m = 1$ the operator $\mathfrak{R}_{u,\varphi}^m$ becomes the operator $W_{u,\varphi}\mathfrak{R}$. This paper can be regarded as a continuation of the investigation of concrete operators between these spaces.

Let dV be the Lebesgue measure on the unit ball \mathbb{B}^n , $d\sigma$ the normalized surface measure on $\mathbb{S}^n = \partial\mathbb{B}^n$ (the boundary of \mathbb{B}^n). Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in \mathbb{C}^n , $\langle z, w \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n$ and $|z|^2 = \langle z, z \rangle$. For $\alpha > -1$, by dV_α we denote the normalized Lebesgue measure $c_\alpha(1 - |z|^2)^\alpha dV(z)$ (constant c_α is chosen such that $\nu_\alpha(\mathbb{B}^n) = 1$).

We now present some facts from [18]. The function $\Phi \not\equiv 0$ is called a growth function, if it is a continuous and nondecreasing function from the interval $[0, \infty)$ onto itself. Clearly, these conditions, among others, imply that $\Phi(0) = 0$.

The function Φ is of positive upper type $q \geq 1$, if there exists $C > 0$ such that $\Phi(st) \leq Ct^q\Phi(s)$ for every $s > 0$ and $t \geq 1$. We denote by \mathfrak{U}^q the set of growth functions Φ of positive upper type q (for some $q \geq 1$), such that the function $t \mapsto \Phi(t)/t$ is non-decreasing on $(0, \infty)$. The function Φ is of positive lower type $p > 0$, if there exists $C > 0$ such that $\Phi(st) \leq Ct^p\Phi(s)$ for every $s > 0$ and $0 < t \leq 1$. By \mathfrak{L}_p we denote the set of growth functions Φ of positive lower type p (for some $0 < p \leq 1$), such that the function $t \mapsto \Phi(t)/t$ is non-increasing on $(0, \infty)$.

Let Φ be a growth function. The weighted Bergman-Orlicz space $A_\alpha^\Phi(\mathbb{B}^n)$ consists of all $f \in H(\mathbb{B}^n)$ such that

$$\|f\|_{A_\alpha^\Phi(\mathbb{B}^n)} = \int_{\mathbb{B}^n} \Phi(|f(z)|) d\nu_\alpha(z) < \infty.$$

On $A_\alpha^\Phi(\mathbb{B}^n)$ is defined the following quasi-norm

$$\|f\|_{A_\alpha^\Phi(\mathbb{B}^n)}^{lux} = \inf \left\{ \lambda > 0 : \int_{\mathbb{B}^n} \Phi\left(\frac{|f(z)|}{\lambda}\right) d\nu_\alpha(z) \leq 1 \right\}.$$

If $\Phi \in \mathcal{U}^q$ or $\Phi \in \mathcal{L}_p$, then the quasi-norm on $A_\alpha^\Phi(\mathbb{B}^n)$ is finite and call the Luxembourg norm.

The classical weighted Bergman space $A_\alpha^p(\mathbb{B}^n)$, $p > 0$, $\alpha > -1$, corresponds to $\Phi(t) = t^p$ and consists of all $f \in H(\mathbb{B}^n)$ such that

$$\|f\|_{A_\alpha^p(\mathbb{B}^n)}^p = \int_{\mathbb{B}^n} |f(z)|^p d\nu_\alpha(z) < \infty.$$

We say that a function $\omega : (0, 1] \rightarrow (0, \infty)$ belongs to class Ω_1 , if ω is non-increasing, $1/\omega$ is of some positive lower type and the function $t\omega$ is increasing. For example, the function $\omega(t) = 1/t^\alpha$, $0 < \alpha < 1$, belongs to class Ω_1 . We say that a function $\omega : (0, 1] \rightarrow (0, \infty)$ belongs to class Ω_2 , if $\omega \in \mathcal{L}_p$, and satisfies the condition:

$$\int_t^1 \frac{\omega(s)}{s^2} ds \lesssim \frac{\omega(t)}{t} \quad (0 < t < 1).$$

Let ω be a positive function defined on $(0, 1]$. An $f \in H(\mathbb{B}^n)$ is said to be in $H_\omega^\infty(\mathbb{B}^n)$, if

$$\|f\|_{H_\omega^\infty(\mathbb{B}^n)} = \sup_{z \in \mathbb{B}^n} \frac{|f(z)|}{\omega(1 - |z|)} < \infty.$$

It is easy to see that $H_\omega^\infty(\mathbb{B}^n)$ is a Banach space with the norm $\|\cdot\|_{H_\omega^\infty(\mathbb{B}^n)}$. The space $H_\omega^\infty(\mathbb{B}^n)$ with $\omega \in \Omega_1$ is not quite often used in the literature. It seems to first appear in [3] as far as we know.

An $f \in H(\mathbb{B}^n)$ belongs to $\Lambda_\omega(\mathbb{B}^n)$, if $\Re f \in H_{\omega/t}^\infty(\mathbb{B}^n)$, that is

$$b_{\Lambda_\omega(\mathbb{B}^n)} = \sup_{z \in \mathbb{B}^n} \frac{(1 - |z|)|\Re f(z)|}{\omega(1 - |z|)} < \infty.$$

$\Lambda_\omega(\mathbb{B}^n)$ is a Banach space under the norm

$$\|f\|_{\Lambda_\omega(\mathbb{B}^n)} = |f(0)| + b_{\Lambda_\omega(\mathbb{B}^n)}.$$

For the relations between $H_\omega^\infty(\mathbb{B}^n)$ and $\Lambda_\omega(\mathbb{B}^n)$, we have that, if $\omega \in \Omega_1$, then $H_\omega^\infty(\mathbb{B}^n) = \Lambda_\omega(\mathbb{B}^n)$; if $\omega \in \Omega_2$, then $H_\omega^\infty(\mathbb{B}^n)$ embeds continuously into $\Lambda_\omega(\mathbb{B}^n)$.

Let X and Y be topological vector spaces whose topologies are given by translation invariant metrics d_X and d_Y , respectively. Let $L : X \rightarrow Y$ be a linear operator. The operator $L : X \rightarrow Y$ is bounded if there exists a positive constant K such that

$$d_Y(Lf, 0) \leq K d_X(f, 0)$$

for all $f \in X$. The operator $L : X \rightarrow Y$ is compact if it maps bounded sets into relatively compact sets.

Throughout this paper, positive constant C may differ from one occurrence to the other. The notation $a \lesssim b$ means that $a \leq Cb$ for some positive constant C .

2. Preliminary results

We first have the following compactness criteria. Since the proof is similar to that of Proposition 3.11 in [1], it is omitted.

LEMMA 2.1. *Let $\alpha > -1$, ω a positive function defined on $(0, 1]$, and $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$. Let $\varphi \in S(\mathbb{B}^n)$, $u \in H(\mathbb{B}^n)$ and $T \in \{\mathfrak{R}W_{u,\varphi}, W_{u,\varphi}\mathfrak{R}\}$. Then the bounded operator $T : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is compact if and only if for every bounded sequence $\{f_j\}$ in $A_\alpha^\Phi(\mathbb{B}^n)$ such that $f_j \rightarrow 0$ uniformly on any compact subset of \mathbb{B}^n as $j \rightarrow \infty$, it follows that*

$$\lim_{j \rightarrow \infty} \|T f_j\|_{H_\omega^\infty(\mathbb{B}^n)} = 0.$$

We need the following estimate. See Lemma 2.16 in [19] for a complete proof.

LEMMA 2.2. *Let $\alpha > -1$ and $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$. Then there exists a positive constant C independent of $f \in A_\alpha^\Phi(\mathbb{B}^n)$ and $z \in \mathbb{B}^n$ such that*

$$|f(z)| \leq C\Phi^{-1}\left(\frac{1}{(1-|z|^2)^{n+1+\alpha}}\right) \|f\|_{A_\alpha^\Phi(\mathbb{B}^n)}^{lux}.$$

We also need the following estimate for derivative of functions in weighted Bergman-Orlicz spaces. See Lemma 4.8 in [20] for a complete proof.

LEMMA 2.3. *Let $\alpha > -1$ and $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$. Then there exist two positive constants $C_n = C(\alpha, n)$ and $D_n = D(\alpha, n)$ independent of $f \in A_\alpha^\Phi(\mathbb{B}^n)$ and $z \in \mathbb{B}^n$ such that*

$$|\nabla f(z)| \leq \frac{C_n}{1-|z|^2} \Phi^{-1}\left(\frac{D_n}{(1-|z|^2)^{n+1+\alpha}}\right) \|f\|_{A_\alpha^\Phi(\mathbb{B}^n)}^{lux}.$$

REMARK 2.1. By the proofs of Lemma 2.16 in [19] and Lemma 4.8 in [20], we can choose the constant $C := \max\{1, D_n\}$ instead of 1 and D_n in Lemmas 2.2 and 2.3.

LEMMA 2.4. *Let $\alpha > -1$ and $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$. Then for every $t \geq 0$ and $w \in \mathbb{B}^n$, the following function is in $A_\alpha^\Phi(\mathbb{B}^n)$*

$$f_{w,t}(z) = \Phi^{-1}\left(\frac{C}{(1-|w|^2)^{n+1+\alpha}}\right) \left(\frac{1-|w|^2}{1-\langle z, w \rangle}\right)^{2(n+1+\alpha)+t},$$

where C is an arbitrary positive constant. Moreover,

$$\sup_{w \in \mathbb{B}^n} \|f_{w,t}\|_{A_\alpha^\Phi(\mathbb{B}^n)}^{lux} \lesssim 1.$$

Proof. Let

$$g_w(z) = \left(\frac{1 - |w|^2}{1 - \langle z, w \rangle} \right)^{2(n+1+\alpha)+t}.$$

We first have

$$\int_{\mathbb{B}^n} \Phi(|f_{w,t}(z)|) d\nu_\alpha(z) = \int_{\mathbb{B}^n} \Phi\left(\Phi^{-1}\left(\frac{C}{(1 - |w|^2)^{n+1+\alpha}}\right) |g_w(z)|\right) d\nu_\alpha(z) = I + J,$$

where

$$I = \int_{\{z \in \mathbb{B}^n : |g_w(z)| \leq 1\}} \Phi\left(\Phi^{-1}\left(\frac{C}{(1 - |w|^2)^{n+1+\alpha}}\right) |g_w(z)|\right) d\nu_\alpha(z)$$

and

$$J = \int_{\{z \in \mathbb{B}^n : |g_w(z)| > 1\}} \Phi\left(\Phi^{-1}\left(\frac{C}{(1 - |w|^2)^{n+1+\alpha}}\right) |g_w(z)|\right) d\nu_\alpha(z).$$

Let us start by considering the case where $\Phi \in \mathfrak{U}^q$. Since $\Phi(r)/r$ is non-decreasing on $(0, \infty)$, we have

$$\frac{\Phi(r|g_w(z)|)}{r|g_w(z)|} \leq \frac{\Phi(r)}{r}$$

for any $z \in \{z \in \mathbb{B}^n : |g_w(z)| \leq 1\}$, which shows

$$\begin{aligned} I &= \int_{\{z \in \mathbb{B}^n : |g_w(z)| \leq 1\}} \Phi\left(\Phi^{-1}\left(\frac{C}{(1 - |w|^2)^{n+1+\alpha}}\right) |g_w(z)|\right) d\nu_\alpha(z) \\ &\leq C \int_{\mathbb{B}^n} \frac{(1 - |w|^2)^{n+1+\alpha+t}}{|1 - \langle z, w \rangle|^{2(n+1+\alpha)+t}} d\nu_\alpha(z) \\ &\lesssim 1, \end{aligned}$$

where we use Theorem 1.12 in [45]. Using that Φ is of positive upper type q and $q \geq 1$, we obtain

$$\begin{aligned} J &= \int_{\{z \in \mathbb{B}^n : |g_w(z)| > 1\}} \Phi\left(\Phi^{-1}\left(\frac{C}{(1 - |w|^2)^{n+1+\alpha}}\right) |g_w(z)|\right) d\nu_\alpha(z) \\ &\lesssim \int_{\{z \in \mathbb{B}^n : |g_w(z)| > 1\}} \Phi\left(\Phi^{-1}\left(\frac{C}{(1 - |w|^2)^{n+1+\alpha}}\right)\right) |g_w(z)|^q d\nu_\alpha(z) \\ &\leq \frac{C}{(1 - |w|^2)^{n+1+\alpha}} \int_{\mathbb{B}^n} \frac{(1 - |w|^2)^{2q(n+1+\alpha+\frac{1}{2})}}{|1 - \langle z, w \rangle|^{2q(n+1+\alpha+\frac{1}{2})}} d\nu_\alpha(z) \\ &\lesssim 1. \end{aligned}$$

We now consider the case where $\Phi \in \mathfrak{L}_p$. Using that Φ is of lower type p and Theorem 1.12 in [45], we have

$$\begin{aligned} I &= \int_{\{z \in \mathbb{B}^n : |g_w(z)| \leq 1\}} \Phi\left(\Phi^{-1}\left(\frac{C}{(1 - |w|^2)^{n+1+\alpha}}\right) |g_w(z)|\right) d\nu_\alpha(z) \\ &\leq \frac{C}{(1 - |w|^2)^{n+1+\alpha}} \int_{\mathbb{B}^n} |g_w(z)|^p d\nu_\alpha(z) \end{aligned}$$

$$\begin{aligned}
 &= \frac{C}{(1 - |w|^2)^{n+1+\alpha}} \int_{\mathbb{B}^n} \left(\frac{1 - |w|^2}{|1 - \langle z, w \rangle|} \right)^{[2(n+1+\alpha)+t]p} d\nu_\alpha(z) \\
 &\lesssim 1.
 \end{aligned}$$

We consider the second integral. Using the fact that $\Phi^{-1}(r)/r$ is non-decreasing on $(0, \infty)$, we obtain

$$\frac{\Phi^{-1}(r)}{r} \leq \frac{\Phi^{-1}(r|g_w(z)|)}{r|g_w(z)|}$$

for any $z \in \{z \in \mathbb{B}^n : |g_w(z)| > 1\}$, which shows

$$|g_w(z)|\Phi^{-1}(r) \leq \Phi^{-1}(r|g_w(z)|)$$

for any $z \in \{z \in \mathbb{B}^n : |g_w(z)| > 1\}$. Hence, we have

$$\begin{aligned}
 J &= \int_{\{z \in \mathbb{B}^n : |g_w(z)| > 1\}} \Phi\left(\Phi^{-1}\left(\frac{C}{(1 - |w|^2)^{n+1+\alpha}}\right)|g_w(z)|\right) d\nu_\alpha(z) \\
 &\leq \frac{1}{(1 - |w|^2)^{n+1+\alpha}} \int_{\mathbb{B}^n} \left(\frac{1 - |w|^2}{|1 - \langle z, w \rangle|} \right)^{2(n+1+\alpha)+t} d\nu_\alpha(z) \\
 &\lesssim 1.
 \end{aligned}$$

This finishes the proof of the lemma. \square

3. Boundedness and compactness of $\mathfrak{R}W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$

Let $\varphi = (\varphi_1, \dots, \varphi_n) \in S(\mathbb{B}^n)$. By $(\Re\varphi_1, \dots, \Re\varphi_n)$ we denote $\Re\varphi$, and by ∇f the gradient of function f . We assume that holomorphic self-map φ satisfies the following condition:

There is a $\delta \in (0, 1)$ such that

$$|\Re\varphi(z)| \leq \frac{1}{\delta} |\langle \Re\varphi(z), \varphi(z) \rangle| \tag{2}$$

on $K = \{z \in \mathbb{B}^n : |\varphi(z)| \geq \delta\}$.

It is easily seen that if $n = 1$, all the holomorphic self-maps of \mathbb{D} satisfy this condition. While if $n > 1$, we can also find some holomorphic self-maps φ to satisfy this condition. For example, if $\Re\varphi(z) = \varphi(z)$, then φ satisfies the condition. For such holomorphic self-maps, one can see, for example, $\varphi(z) = (z_1, z_2/2, \dots, z_n/n)$.

Now, we characterize the boundedness of $\mathfrak{R}W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$.

THEOREM 3.1. *Let $\alpha > -1$, $\varphi \in S(\mathbb{B}^n)$ satisfying condition (2), $u \in H(\mathbb{B}^n)$, $\Phi \in \mathfrak{L}^q \cup \mathfrak{L}_p$, and ω be a positive function defined on $(0, 1]$. Then the following statements are equivalent:*

- (i) *The operator $\mathfrak{R}W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is bounded.*
- (ii) *The functions u and φ satisfy the following conditions*

$$M_1 := \sup_{z \in \mathbb{B}^n} \frac{|\Re u(z)|}{\omega(1 - |z|)} \Phi^{-1}\left(\frac{C}{(1 - |\varphi(z)|^2)^{n+1+\alpha}}\right) < \infty \tag{3}$$

and

$$M_2 := \sup_{z \in \mathbb{B}^n} \frac{|u(z)| |\Re \varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right) < \infty, \tag{4}$$

where C is the positive constant in Remark 2.1.

Proof. (i) \Rightarrow (ii). Suppose that (i) holds. First for each $h \in H(\mathbb{B}^n)$, we have

$$\frac{\partial(h(\varphi(z)))}{\partial z_j} = \sum_{i=1}^n \frac{\partial h}{\partial z_i}(\varphi(z)) \frac{\partial \varphi_i}{\partial z_j}(z),$$

from which it follows that

$$\Re(h(\varphi(z))) = \langle \nabla h(\varphi(z)), \overline{\Re \varphi(z)} \rangle. \tag{5}$$

In Lemma 2.4, let $t = 0$, C the constant in Remark 2.1, and replace w with $\varphi(w)$. Then we obtain the function $f_w(z) := f_{\varphi(w),0}(z)$. From some calculations, it follows that

$$f_w(\varphi(w)) = \Phi^{-1} \left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}} \right) \tag{6}$$

and

$$\frac{\partial f_w}{\partial z_j}(z) = c_{\alpha,n} \Phi^{-1} \left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}} \right) \left(\frac{1-|\varphi(w)|^2}{1-\langle z, \varphi(w) \rangle} \right)^{2(n+1+\alpha)} \frac{\overline{\varphi_j(w)}}{1-\langle z, \varphi(w) \rangle}, \tag{7}$$

where $c_{\alpha,n} = 2(n + \alpha + 1)$. From (7), we have

$$\frac{\partial f_w}{\partial z_j}(\varphi(w)) = c_{\alpha,n} \Phi^{-1} \left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}} \right) \frac{\overline{\varphi_j(w)}}{1-|\varphi(w)|^2}. \tag{8}$$

From (8), we obtain

$$\nabla f_w(\varphi(w)) = c_{\alpha,n} \Phi^{-1} \left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}} \right) \frac{\overline{\varphi(w)}}{1-|\varphi(w)|^2}. \tag{9}$$

Hence, from (5) and (9) we deduce that

$$|\Re(f_w(\varphi(w)))| = c_{\alpha,n} \Phi^{-1} \left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}} \right) \frac{|\langle \Re \varphi(w), \varphi(w) \rangle|}{1-|\varphi(w)|^2}. \tag{10}$$

From (6), (10) and the boundedness of $\Re W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$, we have

$$\begin{aligned} & \Phi^{-1} \left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}} \right) \frac{|\Re u(w)|}{\omega(1-|w|)} \\ & - c_{\alpha,n} \Phi^{-1} \left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}} \right) \frac{|u(w)| |\langle \Re \varphi(w), \varphi(w) \rangle|}{\omega(1-|w|)(1-|\varphi(w)|^2)} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\omega(1-|w|)} |\Re u(w) f_w(\varphi(w))| - \frac{1}{\omega(1-|w|)} |u(w) \Re(f_w(\varphi(w)))| \\
 &\leq \frac{1}{\omega(1-|w|)} |\Re u(w) f_w(\varphi(w)) + u(w) \Re(f_w(\varphi(w)))| \\
 &= \frac{1}{\omega(1-|w|)} |\Re W_{u,\varphi} f_w(w)| \leq \|\Re W_{u,\varphi} f_w\|_{H_\omega^\infty(\mathbb{B}^n)} \leq C \|\Re W_{u,\varphi}\|. \tag{11}
 \end{aligned}$$

From (11), we obtain the following inequality

$$\begin{aligned}
 &\Phi^{-1}\left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\right) \frac{|\Re u(w)|}{\omega(1-|w|)} \\
 &\leq C \|\Re W_{u,\varphi}\| + c_{\alpha,n} \Phi^{-1}\left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\right) \frac{|u(w)| |\langle \Re \varphi(w), \varphi(w) \rangle|}{\omega(1-|w|)(1-|\varphi(w)|^2)}. \tag{12}
 \end{aligned}$$

On the other hand, from Lemma 2.4 it follows that the function

$$g_w(z) := f_{\varphi(w),1}(z) - f_{\varphi(w),0}(z)$$

is in $A_\alpha^\Phi(\mathbb{B}^n)$ and $\|g_w\|_{A_\alpha^\Phi(\mathbb{B}^n)}^{lux} \leq C$. From some calculations, we similarly obtain

$$|\Re(g_w(\varphi(w)))| = \Phi^{-1}\left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\right) \frac{|\langle \Re \varphi(w), \varphi(w) \rangle|}{1-|\varphi(w)|^2} \tag{13}$$

and $g_w(\varphi(w)) = 0$. From this, (13) and the boundedness of $\Re W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$, it follows that

$$\begin{aligned}
 &\Phi^{-1}\left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}}\right) \frac{|u(w)| |\langle \Re \varphi(w), \varphi(w) \rangle|}{\omega(1-|w|)(1-|\varphi(w)|^2)} \\
 &= \frac{1}{\omega(1-|w|)} |u(w) \Re(g_w(\varphi(w)))| \\
 &= \frac{1}{\omega(1-|w|)} |\Re u(w) g_w(\varphi(w)) + u(w) \Re(g_w(\varphi(w)))| \\
 &= \frac{1}{\omega(1-|w|)} |\Re W_{u,\varphi} g_w(w)| \\
 &\leq \|\Re W_{u,\varphi} g_w\|_{H_\omega^\infty(\mathbb{B}^n)}, \tag{14}
 \end{aligned}$$

which shows

$$\sup_{z \in \mathbb{B}^n} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) \frac{|u(z)| |\langle \Re \varphi(z), \varphi(z) \rangle|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \leq \|\Re W_{u,\varphi} g_w\|_{H_\omega^\infty(\mathbb{B}^n)}. \tag{15}$$

Hence, from (12) and (15), it follows that

$$M_1 = \sup_{z \in \mathbb{B}^n} \frac{|\Re u(z)|}{\omega(1-|z|)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) < \infty.$$

For δ and K in (2), from (15) we have

$$\begin{aligned} & \sup_{z \in K} \frac{|u(z)| |\Re \varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right) \\ & \leq \frac{1}{\delta} \sup_{z \in K} \frac{|u(z)| |\langle \Re \varphi(z), \varphi(z) \rangle|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right) \\ & \leq \frac{1}{\delta} \| \Re W_{u, \varphi} \delta_w \|_{H_\omega^\infty(\mathbb{B}^n)} \\ & \leq C \| \Re W_{u, \varphi} \| \\ & < \infty. \end{aligned} \tag{16}$$

Take functions $f(z) = 1$ and $f(z) = z_j$, respectively. From the boundedness of $\Re W_{u, \varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$, we get

$$\sup_{z \in \mathbb{B}^n} \frac{|\Re u(z)|}{\omega(1-|z|)} < \infty \tag{17}$$

and

$$\sup_{z \in \mathbb{B}^n} \frac{1}{\omega(1-|z|)} |\varphi_j(z) \Re u(z) + u(z) \Re \varphi_j(z)| < \infty. \tag{18}$$

From (17), (18) and the boundedness of $\varphi_j(z)$, we have

$$\sup_{z \in \mathbb{B}^n} \frac{|u(z)| |\Re \varphi_j(z)|}{\omega(1-|z|)} < \infty. \tag{19}$$

Since

$$|\Re \varphi(z)| \leq |\Re \varphi_1(z)| + |\Re \varphi_2(z)| + \dots + |\Re \varphi_n(z)|,$$

by (19) we have

$$\begin{aligned} & \sup_{z \in \mathbb{B}^n \setminus K} \frac{|u(z)| |\Re \varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right) \\ & \leq c_{\alpha, \delta} \sup_{z \in \mathbb{B}^n} \frac{|u(z)| |\Re \varphi(z)|}{\omega(1-|z|)} \\ & \leq c_{\alpha, \delta} \sum_{j=1}^n \sup_{z \in \mathbb{B}^n} \frac{|u(z)| |\Re \varphi_j(z)|}{\omega(1-|z|)} \\ & < \infty, \end{aligned} \tag{20}$$

where

$$c_{\alpha, \delta} = \frac{1}{1-\delta^2} \Phi^{-1} \left(\frac{C}{(1-\delta^2)^{n+1+\alpha}} \right).$$

Consequently, from (16) and (20) we get

$$M_2 = \sup_{z \in \mathbb{B}^n} \frac{|u(z)| |\Re \varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right) < \infty.$$

(ii) \Rightarrow (i). Suppose that (3) and (4) hold. Then for every $f \in A_\alpha^\Phi(\mathbb{B}^n)$, from Lemma 2.2, Lemma 2.3 and Remark 2.1 we have

$$\begin{aligned} \|\mathfrak{R}W_{u,\varphi}f\|_{H_\omega^\infty(\mathbb{B}^n)} &= \sup_{z \in \mathbb{B}^n} \frac{1}{\omega(1-|z|)} |\mathfrak{R}u(z)f(\varphi(z)) + u(z)\mathfrak{R}(f(\varphi(z)))| \\ &\leq \sup_{z \in \mathbb{B}^n} \frac{|\mathfrak{R}u(z)||f(\varphi(z))|}{\omega(1-|z|)} + \sup_{z \in \mathbb{B}^n} \frac{|u(z)||\mathfrak{R}(f(\varphi(z)))|}{\omega(1-|z|)} \\ &= \sup_{z \in \mathbb{B}^n} \frac{|\mathfrak{R}u(z)||f(\varphi(z))|}{\omega(1-|z|)} + \sup_{z \in \mathbb{B}^n} \frac{|u(z)||\langle \nabla f(\varphi(z)), \overline{\mathfrak{R}\varphi(z)} \rangle|}{\omega(1-|z|)} \\ &\leq C \sup_{z \in \mathbb{B}^n} \frac{|\mathfrak{R}u(z)|}{\omega(1-|z|)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) \|f\|_{A_\alpha^\Phi(\mathbb{B}^n)}^{lux} \\ &\quad + \sup_{z \in \mathbb{B}^n} \frac{|u(z)||\mathfrak{R}\varphi(z)||\nabla f(\varphi(z))|}{\omega(1-|z|)} \\ &\leq C \sup_{z \in \mathbb{B}^n} \frac{|\mathfrak{R}u(z)|}{\omega(1-|z|)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) \|f\|_{A_\alpha^\Phi(\mathbb{B}^n)}^{lux} \\ &\quad + C_n \sup_{z \in \mathbb{B}^n} \frac{|u(z)||\mathfrak{R}\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) \|f\|_{A_\alpha^\Phi(\mathbb{B}^n)}^{lux} \\ &= (CM_1 + C_nM_2) \|f\|_{A_\alpha^\Phi(\mathbb{B}^n)}^{lux}, \end{aligned}$$

which shows that the operator $\mathfrak{R}W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is bounded. \square

From the fact $H_\omega^\infty(\mathbb{B}^n) \hookrightarrow \Lambda_\omega(\mathbb{B}^n)$ when $\omega \in \Omega_2$, and Theorem 3.1, we can obtain the following result.

PROPOSITION 3.2. *Let $\alpha > -1$, $\varphi \in S(\mathbb{B}^n)$ satisfying condition (2), $u \in H(\mathbb{B}^n)$, $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, and $\omega \in \Omega_2$. If u and φ satisfy the following conditions:*

$$\sup_{z \in \mathbb{B}^n} \frac{|\mathfrak{R}u(z)|}{\omega(1-|z|)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) < \infty$$

and

$$\sup_{z \in \mathbb{B}^n} \frac{|u(z)||\mathfrak{R}\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) < \infty,$$

where C is the positive constant in Remark 2.1, then the operator $\mathfrak{R}W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is bounded.

Next we prove the following compactness criteria.

THEOREM 3.3. *Let $\alpha > -1$, $\varphi \in S(\mathbb{B}^n)$ satisfying condition (2), $u \in H(\mathbb{B}^n)$, $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, and ω a positive function defined on $(0, 1]$. Then the following statements are equivalent:*

(i) *The operator $\mathfrak{R}W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is compact.*

(ii) The functions u and φ are such that $\Re u \in H_\omega^\infty(\mathbb{B}^n)$, for each $j \in \{1, 2, \dots, n\}$

$$L_j := \sup_{z \in \mathbb{B}^n} \frac{|u(z)| |\Re \varphi_j(z)|}{\omega(1 - |z|)} < \infty, \tag{21}$$

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{|\Re u(z)|}{\omega(1 - |z|)} \Phi^{-1} \left(\frac{C}{(1 - |\varphi(z)|^2)^{n+1+\alpha}} \right) = 0, \tag{22}$$

and

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{|u(z)| |\Re \varphi(z)|}{\omega(1 - |z|)(1 - |\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1 - |\varphi(z)|^2)^{n+1+\alpha}} \right) = 0, \tag{23}$$

where C is the positive constant in Remark 2.1.

Proof. (i) \Rightarrow (ii). Suppose that (i) holds. Then $\Re W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is bounded. For any $f \in A_\alpha^\Phi(\mathbb{B}^n)$, it follows that

$$\|\Re W_{u,\varphi} f\|_{H_\omega^\infty(\mathbb{B}^n)} = \sup_{z \in \mathbb{B}^n} \frac{1}{\omega(1 - |z|)} |\Re u(z) f(\varphi(z)) + u(z) \Re(f(\varphi(z)))| < \infty. \tag{24}$$

Taking $f(z) = 1$ and $f(z) = z_j$ in (24), we obtain that $\Re u \in H_\omega^\infty(\mathbb{B}^n)$ and (21).

Next consider a sequence $\{\varphi(z_j)\}$ in \mathbb{B}^n such that $|\varphi(z_j)| \rightarrow 1$ as $j \rightarrow \infty$. If such sequence does not exist, then (22) and (23) obviously hold. Using this sequence, we define the functions $f_j(z) = f_{\varphi(z_j),0}(z)$. Then the sequence $\{f_j\}$ is uniformly bounded in $A_\alpha^\Phi(\mathbb{B}^n)$ and uniformly converges to zero on any compact subset of \mathbb{B}^n as $j \rightarrow \infty$. Since f_j is defined by replacing $\varphi(w)$ by $\varphi(z_j)$ in f_w in the proof of Theorem 3.1, from (12) we have

$$\begin{aligned} & \frac{|\Re u(z_j)|}{\omega(1 - |z_j|)} \Phi^{-1} \left(\frac{C}{(1 - |\varphi(z_j)|^2)^{n+1+\alpha}} \right) \\ & - c_{\alpha,n} \frac{|u(z_j)| |\langle \Re \varphi(z_j), \varphi(z_j) \rangle|}{\omega(1 - |z_j|)(1 - |\varphi(z_j)|^2)} \Phi^{-1} \left(\frac{C}{(1 - |\varphi(z_j)|^2)^{n+1+\alpha}} \right) \\ & \leq \|\Re W_{u,\varphi} f_j\|_{H_\omega^\infty(\mathbb{B}^n)}, \end{aligned} \tag{25}$$

We also define the function

$$g_j(z) = f_{\varphi(z_j),1}(z) - f_{\varphi(z_j),0}(z)$$

for each $j \in \mathbb{N}$. The sequence $\{g_j\}$ is uniformly bounded in $A_\alpha^\Phi(\mathbb{B}^n)$ and uniformly converges to zero on any compact subset of \mathbb{B}^n as $j \rightarrow \infty$. From (15), we have

$$\frac{|u(z_j)| |\langle \Re \varphi(z_j), \varphi(z_j) \rangle|}{\omega(1 - |z_j|)(1 - |\varphi(z_j)|^2)} \Phi^{-1} \left(\frac{C}{(1 - |\varphi(z_j)|^2)^{n+1+\alpha}} \right) \leq \|\Re W_{u,\varphi} g_j\|_{H_\omega^\infty(\mathbb{B}^n)}. \tag{26}$$

From (25) and (26), it follows that

$$\frac{|\Re u(z_j)|}{\omega(1-|z_j|)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z_j)|^2)^{n+1+\alpha}} \right) \leq \| \Re W_{u,\varphi} f_j \|_{H_\omega^\infty(\mathbb{B}^n)} + c_{\alpha,n} \| \Re W_{u,\varphi} g_j \|_{H_\omega^\infty(\mathbb{B}^n)}.$$

From the compactness of $\Re W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$, it follows that (22) holds.

Since $|\varphi(z_j)| \rightarrow 1$ as $j \rightarrow \infty$, there exists a $J \in \mathbb{N}$ such that $|\varphi(z_j)| > \delta$ for $j \geq J$, that is, $\varphi(z_j) \in K$ for $j \geq J$. Similar to (16), we can obtain

$$\frac{|u(z_j)| |\Re \varphi(z_j)|}{\omega(1-|z_j|)(1-|\varphi(z_j)|^2)} \leq \frac{1}{\delta} \| \Re W_{u,\varphi} g_j \|_{H_\omega^\infty(\mathbb{B}^n)}.$$

This implies that (23) holds.

(ii) \Rightarrow (i). We first check that $\Re W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is bounded. For this, we observe that (22) and (23) imply that for every $\varepsilon > 0$, there is an $\eta \in (0, 1)$ such that

$$\frac{|\Re u(z)|}{\omega(1-|z|)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right) < \varepsilon \tag{27}$$

and

$$\frac{|u(z)| |\Re \varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right) < \varepsilon, \tag{28}$$

for any $z \in K_\eta = \{z \in \mathbb{B}^n : |\varphi(z)| > \eta\}$.

Write

$$I(z) := \frac{|\Re u(z)|}{\omega(1-|z|)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right)$$

and

$$J(z) := \frac{|u(z)| |\Re \varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right).$$

Then from (21), (27) and (28) we have

$$\begin{aligned} M_1 &= \sup_{z \in \mathbb{B}^n} I(z) = \sup_{z \in \mathbb{B}^n \setminus K_\eta} I(z) + \sup_{z \in K_\eta} I(z) \\ &\leq \| \Re u \|_{H_\omega^\infty(\mathbb{B}^n)} \Phi^{-1} \left(\frac{C}{(1-\eta^2)^{n+1+\alpha}} \right) + \varepsilon \end{aligned}$$

and

$$\begin{aligned} M_2 &= \sup_{z \in \mathbb{B}^n} J(z) = \sup_{z \in \mathbb{B}^n \setminus K_\eta} J(z) + \sup_{z \in K_\eta} J(z) \\ &\leq \sum_{j=1}^n L_j \Phi^{-1} \left(\frac{C}{(1-\eta^2)^{n+1+\alpha}} \right) + \varepsilon. \end{aligned}$$

From Theorem 3.1, it follows that $\mathfrak{R}W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is bounded.

To prove that $\mathfrak{R}W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is compact, by Lemma 2.1 we just need to prove that, if $\{f_j\}$ is a sequence in $A_\alpha^\Phi(\mathbb{B}^n)$ such that $\|f_j\|_{A_\alpha^\Phi(\mathbb{B}^n)} \leq M$ and $\{f_j\}$ uniformly converges to zero on any compact subset of \mathbb{B}^n as $j \rightarrow \infty$, then

$$\lim_{j \rightarrow \infty} \|\mathfrak{R}W_{u,\varphi} f_j\|_{H_\omega^\infty(\mathbb{B}^n)} = 0.$$

For any $\varepsilon > 0$ and the associated η in (27) and (28), by using again $\mathfrak{R}u \in H_\omega^\infty(\mathbb{B}^n)$, (27), (28), Lemmas 2.2, 2.3 and Remark 2.1, we have

$$\begin{aligned} & \|\mathfrak{R}W_{u,\varphi} f_j\|_{H_\omega^\infty(\mathbb{B}^n)} \\ &= \sup_{z \in \mathbb{B}^n} \frac{1}{\omega(1-|z|)} \left| \mathfrak{R}(u(z)f_j(\varphi(z))) \right| \\ &= \sup_{z \in \mathbb{B}^n} \frac{1}{\omega(1-|z|)} \left| \mathfrak{R}u(z)f_j(\varphi(z)) + u(z)\mathfrak{R}(f_j(\varphi(z))) \right| \\ &= \sup_{z \in \mathbb{B}^n} \frac{1}{\omega(1-|z|)} \left| \mathfrak{R}u(z)f_j(\varphi(z)) + u(z)\langle \nabla f_j(\varphi(z)), \overline{\mathfrak{R}\varphi(z)} \rangle \right| \\ &\leq \sup_{z \in \mathbb{B}^n} \frac{|\mathfrak{R}u(z)|}{\omega(1-|z|)} |f_j(\varphi(z))| + \sup_{z \in \mathbb{B}^n} \frac{|u(z)|}{\omega(1-|z|)} \left| \langle \nabla f_j(\varphi(z)), \overline{\mathfrak{R}\varphi(z)} \rangle \right| \\ &\leq \sup_{z \in \mathbb{B}^n} \frac{|\mathfrak{R}u(z)|}{\omega(1-|z|)} |f_j(\varphi(z))| + \sup_{z \in \mathbb{B}^n} \frac{|u(z)| |\mathfrak{R}\varphi(z)|}{\omega(1-|z|)} |\nabla f_j(\varphi(z))| \\ &\leq \sup_{z \in \mathbb{B}^n \setminus K_\eta} \frac{|\mathfrak{R}u(z)|}{\omega(1-|z|)} |f_j(\varphi(z))| + \sup_{z \in K_\eta} \frac{|\mathfrak{R}u(z)|}{\omega(1-|z|)} |f_j(\varphi(z))| \\ &\quad + \sup_{z \in \mathbb{B}^n \setminus K_\eta} \frac{|u(z)| |\mathfrak{R}\varphi(z)|}{\omega(1-|z|)} |\nabla f_j(\varphi(z))| + \sup_{z \in K_\eta} \frac{|u(z)| |\mathfrak{R}\varphi(z)|}{\omega(1-|z|)} |\nabla f_j(\varphi(z))| \\ &\leq \|\mathfrak{R}u\|_{H_\omega^\infty(\mathbb{B}^n)} \sup_{\{z:|z|\leq\eta\}} |f_j(z)| + CM \sup_{z \in K_\eta} \frac{|\mathfrak{R}u(z)|}{\omega(1-|z|)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right) \\ &\quad + \sum_{j=1}^n L_j \sup_{\{z:|z|\leq\eta\}} |\nabla f_j(z)| + C_n M \sup_{z \in K_\eta} \frac{|u(z)| |\mathfrak{R}\varphi(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right) \\ &\leq \|\mathfrak{R}u\|_{H_\omega^\infty(\mathbb{B}^n)} \sup_{\{z:|z|\leq\eta\}} |f_j(z)| + \sum_{j=1}^n L_j \sup_{\{z:|z|\leq\eta\}} |\nabla f_j(z)| + (C + C_n)M\varepsilon. \tag{29} \end{aligned}$$

It is easy to see that, if $\{f_j\}$ uniformly converges to zero on any compact subset of \mathbb{B}^n , then $\{\frac{\partial f_j}{\partial z_i}\}$ also does as $j \rightarrow \infty$ for each $i = 1, 2, \dots, n$. This shows that $\{|\nabla f_j|\}$ uniformly converges to zero on any compact subset of \mathbb{B}^n as $j \rightarrow \infty$. Since $\{z \in \mathbb{B}^n : |z| \leq \eta\}$ is compact subset of \mathbb{B}^n , by letting $j \rightarrow \infty$ in (29) we have

$$\lim_{j \rightarrow \infty} \|\mathfrak{R}W_{u,\varphi} f_j\|_{H_\omega^\infty(\mathbb{B}^n)} = 0.$$

This shows that $\mathfrak{R}W_{u,\varphi} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is compact. \square

4. Boundedness and compactness of $W_{u,\varphi}\mathfrak{R} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$

THEOREM 4.1. *Let $\alpha > -1$, $\varphi \in S(\mathbb{B}^n)$, $u \in H(\mathbb{B}^n)$, $\Phi \in \mathfrak{U} \cup \mathfrak{L}_p$, and ω a positive function defined on $(0, 1]$. Then the following statements are equivalent:*

- (i) *The operator $W_{u,\varphi}\mathfrak{R} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is bounded.*
- (ii) *The functions u and φ satisfy the following condition*

$$M_3 := \sup_{z \in \mathbb{B}^n} \frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right) < \infty, \tag{30}$$

where C is the positive constant in Remark 2.1.

Proof. (i) \Rightarrow (ii). Suppose that (i) holds. Then there exists a positive constant C such that for every $f \in A_\alpha^\Phi(\mathbb{B}^n)$,

$$\|W_{u,\varphi}\mathfrak{R}f\|_{H_\omega^\infty(\mathbb{B}^n)} \leq C \|f\|_{A_\alpha^\Phi(\mathbb{B}^n)}^{lux}.$$

Considering the function $f_w(z) = f_{\varphi(w),0}(z)$, we have

$$|\mathfrak{R}f_w(\varphi(w))| = c_{\alpha,n} \frac{|\varphi(w)|^2}{1-|\varphi(w)|^2} \Phi^{-1} \left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}} \right).$$

From this and the boundedness of $W_{u,\varphi}\mathfrak{R} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$, we obtain

$$\begin{aligned} \frac{1}{\omega(1-|w|)} |W_{u,\varphi}\mathfrak{R}f_w(w)| &= \frac{1}{\omega(1-|w|)} |u(w)\mathfrak{R}f_w(\varphi(w))| \\ &= c_{\alpha,n} \frac{|u(w)||\varphi(w)|^2}{\omega(1-|w|)(1-|\varphi(w)|^2)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(w)|^2)^{n+1+\alpha}} \right) \\ &\leq \|W_{u,\varphi}\mathfrak{R}f_w\|_{H_\omega^\infty(\mathbb{B}^n)} \leq C \|W_{u,\varphi}\mathfrak{R}\|, \end{aligned}$$

which shows

$$I := \sup_{z \in \mathbb{B}^n} \frac{|u(z)||\varphi(z)|^2}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right) < \infty.$$

Then for a fixed $\delta \in (0, 1)$, we have

$$\sup_{\{z:|\varphi(z)|>\delta\}} \frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}} \right) \leq \frac{I}{\delta^2} < \infty. \tag{31}$$

Also taking the function $f(z) = z_j$, we have

$$\|W_{u,\varphi}\mathfrak{R}f\|_{H_\omega^\infty(\mathbb{B}^n)} = \sup_{z \in \mathbb{B}^n} \frac{|u(z)||\varphi_j(z)|}{\omega(1-|z|)} < \infty.$$

From this and the boundedness of φ_j , we get

$$J := \sup_{z \in \mathbb{B}^n} \frac{|u(z)|}{\omega(1-|z|)} < \infty. \tag{32}$$

Hence,

$$\sup_{\{z:|\varphi(z)|\leq\delta\}} \frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) \leq d_\delta J < \infty, \tag{33}$$

where

$$d_\delta = \frac{1}{1-\delta^2} \Phi^{-1}\left(\frac{C}{(1-\delta^2)^{n+1+\alpha}}\right).$$

Consequently, (31) and (33) show that $M_3 < \infty$.

(ii) \Rightarrow (i). Suppose that (30) holds. Since

$$|\Re f(z)| = \left| \sum_{j=1}^n z_j \frac{\partial f(z)}{\partial z_j} \right| \leq \sum_{j=1}^n |z_j| \left| \frac{\partial f(z)}{\partial z_j} \right| \leq \sum_{j=1}^n \left| \frac{\partial f(z)}{\partial z_j} \right| = \sqrt{n} |\nabla f(z)|$$

for every $f \in A_\alpha^\Phi(\mathbb{B}^n)$, by Lemma 2.3 we have

$$\begin{aligned} \|W_{u,\varphi} \Re f\|_{H_\omega^\infty(\mathbb{B}^n)} &= \sup_{z \in \mathbb{B}^n} \frac{1}{\omega(1-|z|)} |u(z) \Re f(\varphi(z))| \\ &\leq \sqrt{n} C_n \sup_{z \in \mathbb{B}^n} \frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) \|f\|_{A_\alpha^\Phi(\mathbb{B}^n)}^{lux} \\ &= \sqrt{n} C_n M_3 \|f\|_{A_\alpha^\Phi(\mathbb{B}^n)}^{lux}, \end{aligned}$$

which shows that the operator $W_{u,\varphi} \Re : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is bounded. \square

The following result is proved similarly.

PROPOSITION 4.2. *Let $\alpha > -1$, $\varphi \in S(\mathbb{B}^n)$, $u \in H(\mathbb{B}^n)$, $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, and $\omega \in \Omega_2$. If u and φ satisfy the condition*

$$\sup_{z \in \mathbb{B}^n} \frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) < \infty,$$

where C is the positive constant in Remark 2.1, then the operator $W_{u,\varphi} \Re : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is bounded.

Now we prove the compactness criteria.

THEOREM 4.3. *Let $p \geq 1$, $\alpha > -1$, $\varphi \in S(\mathbb{B}^n)$, $u \in H(\mathbb{B}^n)$, $\Phi \in \mathfrak{U}^q \cup \mathfrak{L}_p$, and ω be a positive function defined on $(0, 1]$. Then the following statements are equivalent:*

- (i) *The operator $W_{u,\varphi} \Re : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is compact.*
- (ii) *The functions u and φ are such that $u \in H_\omega^\infty(\mathbb{B}^n)$ and*

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) = 0, \tag{34}$$

where C is the positive constant in Remark 2.1.

Proof. (i) \Rightarrow (ii). Suppose that (i) holds. Then $W_{u,\varphi} \Re : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is bounded, so for any $f \in A_\alpha^\Phi(\mathbb{B}^n)$,

$$\|W_{u,\varphi} \Re f\|_{H_\omega^\infty(\mathbb{B}^n)} = \sup_{z \in \mathbb{B}^n} \frac{|u(z) \Re f(\varphi(z))|}{\omega(1-|z|)} < \infty. \tag{35}$$

Taking $f(z) = z_j$ in (35), we obtain

$$\sup_{z \in \mathbb{B}^n} \frac{|u(z)| |\varphi_j(z)|}{\omega(1 - |z|)} < \infty. \tag{36}$$

From the boundedness of $\varphi_j(z)$ and (36), it follows that

$$\sup_{z \in \mathbb{B}^n} \frac{|u(z)|}{\omega(1 - |z|)} < \infty,$$

which means that $u \in H_\omega^\infty(\mathbb{B}^n)$.

Next consider a sequence $\{\varphi(z_j)\}$ in \mathbb{B}^n such that $|\varphi(z_j)| \rightarrow 1$ as $j \rightarrow \infty$. If such sequence does not exist, then (34) obviously holds. Using this sequence, we define the functions $f_j(z) = f_{\varphi(z_j), 0}(z)$. Then $\{f_j\}$ is uniformly bounded in $A_\alpha^\Phi(\mathbb{B}^n)$ and uniformly converges to zero on any compact subset of \mathbb{B}^n as $j \rightarrow \infty$. From some calculations, we obtain

$$|\Re f_j(\varphi(z_j))| = c_{\alpha, n} \frac{|\varphi(z_j)|^2}{1 - |\varphi(z_j)|^2} \Phi^{-1} \left(\frac{C}{(1 - |\varphi(z_j)|^2)^{n+1+\alpha}} \right).$$

Hence, the compactness of $W_{u, \varphi} \Re : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ implies that

$$\lim_{j \rightarrow \infty} \|W_{u, \varphi} \Re f_j\|_{H_\omega^\infty(\mathbb{B}^n)} = 0.$$

From this, we have

$$\lim_{j \rightarrow \infty} \frac{|u(z_j)|}{\omega(1 - |z_j|)(1 - |\varphi(z_j)|^2)} \Phi^{-1} \left(\frac{C}{(1 - |\varphi(z_j)|^2)^{n+1+\alpha}} \right) = 0.$$

This shows that (34) holds.

(ii) \Rightarrow (i). We first prove that $W_{u, \varphi} \Re : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is bounded. For this we observe that (34) implies that for every $\varepsilon > 0$, there exists an $\eta \in (0, 1)$ such that

$$\frac{|u(z)|}{\omega(1 - |z|)(1 - |\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1 - |\varphi(z)|^2)^{n+1+\alpha}} \right) < \varepsilon, \tag{37}$$

for any $z \in K_\eta = \{z \in \mathbb{B}^n : |\varphi(z)| > \eta\}$.

Let

$$K(z) := \frac{|u(z)|}{\omega(1 - |z|)(1 - |\varphi(z)|^2)} \Phi^{-1} \left(\frac{C}{(1 - |\varphi(z)|^2)^{n+1+\alpha}} \right).$$

Then we have

$$M_3 = \sup_{z \in \mathbb{B}^n \setminus K_\eta} K(z) + \sup_{z \in K_\eta} K(z) \leq \|u\|_{H_\omega^\infty(\mathbb{B}^n)} \frac{1}{1 - \eta^2} \Phi^{-1} \left(\frac{C}{(1 - \eta^2)^{n+1+\alpha}} \right) + \varepsilon, \tag{38}$$

which shows that $W_{u, \varphi} \Re : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is bounded.

To prove that $W_{u,\varphi}\mathfrak{R} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is compact, by Lemma 2.1 we just need to prove that, if $\{f_j\}$ is a sequence in $A_\alpha^\Phi(\mathbb{B}^n)$ such that $\|f_j\|_{A_\alpha^\Phi(\mathbb{B}^n)} \leq M$ and $\{f_j\}$ uniformly converges to zero on any compact subset of \mathbb{B}^n as $j \rightarrow \infty$, then

$$\lim_{j \rightarrow \infty} \|W_{u,\varphi}\mathfrak{R}f_j\|_{H_\omega^\infty(\mathbb{B}^n)} = 0.$$

For any $\varepsilon > 0$ and the associated η in (37), we have, by using again $u \in H_\omega^\infty(\mathbb{B}^n)$, (37) and Lemma 2.3,

$$\begin{aligned} \|W_{u,\varphi}\mathfrak{R}f_j\|_{H_\omega^\infty(\mathbb{B}^n)} &= \sup_{z \in \mathbb{B}^n} \frac{1}{\omega(1-|z|)} |W_{u,\varphi}\mathfrak{R}f_j(z)| = \sup_{z \in \mathbb{B}^n} \frac{1}{\omega(1-|z|)} |u(z)\mathfrak{R}f_j(\varphi(z))| \\ &\leq \sup_{z \in \mathbb{B}^n \setminus K_\eta} \frac{|u(z)|}{\omega(1-|z|)} |\mathfrak{R}f_j(\varphi(z))| + \sup_{z \in K_\eta} \frac{|u(z)|}{\omega(1-|z|)} |\mathfrak{R}f_j(\varphi(z))| \\ &\leq \|u\|_{H_\omega^\infty(\mathbb{B}^n)} \sup_{\{z \in \mathbb{B}^n : |z| \leq \eta\}} |\mathfrak{R}f_j(z)| \\ &\quad + C_n M \sup_{z \in K_\eta} \frac{|u(z)|}{\omega(1-|z|)(1-|\varphi(z)|^2)} \Phi^{-1}\left(\frac{C}{(1-|\varphi(z)|^2)^{n+1+\alpha}}\right) \\ &\leq \|u\|_{H_\omega^\infty(\mathbb{B}^n)} \sup_{\{z \in \mathbb{B}^n : |z| \leq \eta\}} |\mathfrak{R}f_j(z)| + C_n M \varepsilon. \end{aligned} \tag{39}$$

It is easy to see that, if $\{f_j\}$ uniformly converges to zero on any compact subset of \mathbb{B}^n , then $\{\mathfrak{R}f_j\}$ also does as $j \rightarrow \infty$. From this, and since $\{z \in \mathbb{B}^n : |z| \leq \eta\}$ is compact subset of \mathbb{B}^n , letting $j \rightarrow \infty$ in (39) gives

$$\lim_{j \rightarrow \infty} \|W_{u,\varphi}\mathfrak{R}f_j\|_{H_\omega^\infty(\mathbb{B}^n)} = 0.$$

This shows that the operator $W_{u,\varphi}\mathfrak{R} : A_\alpha^\Phi(\mathbb{B}^n) \rightarrow H_\omega^\infty(\mathbb{B}^n)$ is compact. \square

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