

ON THE CLASSES OF D-NORMAL OPERATORS AND D-QUASI-NORMAL OPERATORS ON HILBERT SPACE

M. DANA AND R. YOUSEFI

(Communicated by I. M. Spitkovsky)

Abstract. The purpose of this paper is to introduce some new classes of operators, called $[DN]$, $[nDN]$, $[DQN]$, $[nDQN]$, associated with a Drazin invertible operator using its Drazin inverse. $[nDN]$ and $[nDQN]$ operators extend the notion of $[DN]$ and $[DQN]$ operators, respectively. The relations between these classes and some basic properties of these operators are studied in this study.

1. Introduction

One important research field of the algebra of all bounded linear operators acting on Hilbert space is the class of normal operators ($TT^* = T^*T$). Normal operators comprise a broad class of interesting operators. The theory of these operators was investigated in [10] and [7]. There are other classes of interesting non-normal operators such as n -normal, quasi-normal and n -power quasi-normal operators. They have been of interest to many mathematicians and have been extensively investigated. There are several well-known relationships among these classes. See for instance [1], [2] and [4].

The purpose of this paper is to generalize this classes, in some sense, to the larger sets of so-called D-normal, n -power D-normal, D-quasi-normal, n -power D-quasi-normal, operators on Hilbert spaces.

Throughout, \mathcal{H} represents a complex Hilbert space, $\mathcal{B}(\mathcal{H})$ is the space of all bounded linear operators on a complex Hilbert space \mathcal{H} and $I = I_{\mathcal{H}}$ being the identity operator. For $T \in \mathcal{B}(\mathcal{H})$, denote by T^* , $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $W(T)$ the adjoint, the null space, the range and the numerical range of T , respectively.

A subspace $M \subset \mathcal{H}$ is said to be invariant for an operator $T \in \mathcal{B}(\mathcal{H})$ if $TM \subset M$, and in this situation we denote by $T|_M$ the restriction of T to M .

For any arbitrary operator $T \in \mathcal{B}(\mathcal{H})$, $|T| = (T^*T)^{1/2}$ and

$$[T^*, T] = T^*T - TT^* = |T|^2 - |T^*|^2$$

(the self-commutator of T).

Mathematics subject classification (2010): 47B15, 47B20, 47A15, 15A09.

Keywords and phrases: Drazin inverse, n -normal operators, quasi-normal operators, n -power quasi-normal operators.

Fully documented templates are available in the elsarticle package on <http://www.ctan.org/tex-archive/macros/latex/contrib/elsarticle>.

An operator $T \in \mathcal{B}(\mathcal{H})$ is called n -normal if $T^n T^* = T^* T^n$, quasi-normal if $TT^*T = T^*TT$, n -power quasi-normal if $T^n T^* T = T^* T T^n$. n -normal operators were introduced by Al-zuraiqi and Patel [2], quasi-normal by Brown [4], and n -power quasi-normal by Ahmed [1].

REMARK 1.1. Let $[N]$, $[nN]$, $[QN]$ and $[nQN]$ denote the classes constituting of normal, n -normal, quasi-normal and n -power quasi-normal operators. Then

1. $[N] \subset [QN] \subset [nQN]$,
2. $[N] \subset [nN] \subset [nQN]$.

An operator T is n -isometry if

$$T^{*n} T^n - \binom{n}{1} T^{*n-1} T^{n-1} + \binom{n}{2} T^{*n-2} T^{n-2} \dots + (-1)^n I = 0.$$

The Drazin inverse for bounded linear operators on complex Banach spaces was investigated by Caradus [6]. The Drazin inverse finds its applications in a number of areas such that differential and difference equations, Markov chains and control theory [3, 5].

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be Drazin invertible if there exists an operator $T^D \in \mathcal{B}(\mathcal{H})$ such that

$$TT^D = T^D T, \quad T^D T T^D = T^D, \quad T^{k+1} T^D = T^k$$

for some integer $k \geq 0$. The smallest integer $k \geq 0$, in the latter identity is called the index of T , denoted by $ind(T)$. Specifically, if $ind(T) = 1$, then T^D is called the group inverse of T and denoted by T^\sharp . Clearly, $ind(T) = 0$ if and only if T is invertible and in this case $T^D = T^{-1}$. If T is Drazin invertible, then the spectral idempotent T^π of T corresponding to $\{0\}$ is given by $T^\pi = I - TT^D$. We note that if T is nilpotent, then it is Drazin invertible, $T^D = 0$, and $ind(T) = r$, where r is the power of nilpotency of T .

For $T \in \mathcal{B}(\mathcal{H})$, the Drazin inverse T^D of T is unique if it exists and $(T^*)^D = (T^D)^*$. The Drazin invertibility of an operator in $\mathcal{B}(\mathcal{H})$ is similarly invariant, i.e. if T is Drazin invertible and $S \in \mathcal{B}(\mathcal{H})$ is an invertible operator, then $S^{-1}TS$ is Drazin invertible and $(S^{-1}TS)^D = S^{-1}T^D S$.

This article has been organized in five sections. In section 2, A new class of generalized normal operator, namely the D-normal, using the Drazin T^D inverse of T is given. One important motivation for this classification comes from the problem of finding operators that their Drazin inverses are normal. We give necessary and sufficient conditions for an operator to be D-normal. We also discuss some conditions on an D-normal operator implying normality.

In section 3, the class D-normal by considering operators in $\mathcal{B}(\mathcal{H})$ is enlarged, whose n -power are D-normal. Sufficient conditions implying D-normality for n -power D-normal operators are investigated.

In Section 4, the concept of D-quasi-normal is defined, and we show that the D-quasi-normal operators form a larger class than the D-normal operators. Also, we obtain necessary and sufficient conditions for an operator to be D-quasi-normal.

Finally, in Section 5, the class of n-power D-quasi-normal operators as a generalization of the class of D-quasi-normal operators is introduced and also some properties of such class are given.

Before our main results are presented, we state some auxiliary lemmas as follows.

LEMMA 1.2. ([5]) *Let $A, B \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. Then*

- (i) *AB is Drazin invertible if and only if BA is Drazin invertible, $ind(AB) \leq ind(BA) + 1$ and $(AB)^D = A[(BA)^D]^2B$.*
- (ii) *If A is idempotent, then $A^D = A^\# = A$.*
- (iii) *If $AB = BA$, then $(AB)^D = B^DA^D = A^DB^D$, $A^DB = BA^D$ and $AB^D = B^DA$.*
- (v) *If $AB = BA = 0$, then $(A + B)^D = A^D + B^D$.*

LEMMA 1.3. ([5]) *If $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ are Drazin invertible with $ind(A) = m$ and $ind(B) = n$. Then $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is also Drazin invertible and*

$$M^D = \begin{pmatrix} A^D & X \\ 0 & B^D \end{pmatrix},$$

where

$$X = \sum_{i=0}^{n-1} (A^D)^{i+2}CB^iB^\pi + A^\pi \sum_{i=0}^{m-1} A^iC(B^D)^{i+2} - A^DCB^D. \tag{1.1}$$

2. D-normal operators

This section is started by defining the class of D-normal operators on Hilbert spaces. In order to do this, we use the Drazin inverse (D) and therefore we name this new class of generalized normal operators as D-normal. Also, several properties of D-normal operators are studied.

DEFINITION 2.1. Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. T is called an D-normal operator if

$$T^DT^* = T^*T^D.$$

The class of all D-normal operators is denoted by $[DN]$.

PROPOSITION 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. Then T is D-normal if and only if T^D is normal.*

Proof. Let T be D-normal, $T^D T^* = T^* T^D$, by Lemma 1.2, $T^D (T^*)^D = (T^*)^D T^D$. Since $(T^*)^D = (T^D)^*$, T^D is normal. Now, let T^D be normal. Since $T^D T = T T^D$, by Fuglede theorem [7], $T^D T^* = T^* T^D$. Therefore T is D-normal. \square

D-normal operators provide a new class of generalized normal operators because in general the D-normal operator is different from normal operator.

EXAMPLE 2.3. Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible normal operator then it is clear that T is of class $[DN]$. But the converse is not true.

Let $\mathcal{H} = \mathbb{C}^4$ and let $T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^4)$, we have $T^D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, then by simple calculations we see that T is of class $[DN]$ which is not normal.

In the next remark we give a condition that the class of $[DN]$ operators coincide with class of $[N]$ operators.

REMARK 2.4. Let $T \in [DN]$. If $ind(T) \leq 1$ then $T \in [N]$.

PROPOSITION 2.5. *The set of all D-normal operators on \mathcal{H} is closed subset of $\mathcal{B}(\mathcal{H})$ which is closed under scalar multiplication.*

Proof. First if T is D-normal, and α is scalar, then it is easy to see that (αT) is D-normal. Now, suppose that (T_k) is sequence of D-normal operators converging to T in $\mathcal{B}(\mathcal{H})$. Now,

$$\|T^D T^* - T^* T^D\| \leq \|T^D T^* - T_k^D T_k^*\| + \|T_k^* T_k^D - T^* T^D\| \rightarrow 0.$$

as $k \rightarrow \infty$. Hence $T^D T^* = T^* T^D$. Thus T is D-normal. \square

In the following proposition some properties of the class $[DN]$ operators are collected.

PROPOSITION 2.6. *Let $T \in [DN]$. Then*

1. T^* is of class $[DN]$.
2. T^D is of class $[DN]$.
3. If $S \in \mathcal{B}(\mathcal{H})$ is Drazin invertible and unitary equivalent to T , then S is of class $[DN]$.
4. If M is a closed subspace of \mathcal{H} such that M reduces T , then $S = T|_M$ is of class $[DN]$.

Proof. The proofs of (1)–(2) are straightforward.

(3) Let $S \in \mathcal{B}(\mathcal{H})$, which is unitarily equivalent to T , then there is a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $S = U^*TU$, which implies that $S^D = U^*T^DU$. Since T^D is normal, S^D is normal. Therefore S is D-normal.

(4) Since T is D-normal, T^D is normal. So $T^D|M$ is normal. And since M is invariant under T , $T^D|M = (T|M)^D$. Thus $(T|M)^D$ is normal. So $(T|M)$ is D-normal. \square

PROPOSITION 2.7. *If $T \in [DN]$ is similar to an idempotent $S \in \mathcal{B}(\mathcal{H})$, then T is a projection.*

Proof. Since T is similar to S , there is an invertible operator $N \in \mathcal{B}(\mathcal{H})$ such that $T = N^{-1}SN$, which implies that $T^D = N^{-1}S^DN = N^{-1}SN$. Thus T is normal. The result now follows from ([10], p. 111). \square

THEOREM 2.8. *If T and S are of class $[DN]$ such that $[T, S] = 0$, then TS is of class $[DN]$.*

Proof. Since S, T are commuting D-normal operators, hence by Lemma 1.2 (iii), S^D and T^D are commuting normal operator. So T^DS^D is a normal operator. Since $(TS)^D = T^DS^D$, $(TS)^D$ is normal. Hence TS is of class $[DN]$. \square

The following example shows that Theorem 2.8 is not necessarily true if S, T are not commuting.

EXAMPLE 2.9. Let $S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ be operators on the Hilbert space \mathbb{C}^4 . Then $S^D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $T^D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Hence $S, T \in [DN]$. We note that $ST \neq TS$. But as $(ST)^D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is not normal, ST is not of class $[DN]$.

COROLLARY 2.10. *If T is of class $[DN]$, then T^m is of class $[DN]$ for any positive integer m .*

COROLLARY 2.11. *If T is of class $[DN]$, then T^DT^* and T^*T^D are of class $[DN]$.*

THEOREM 2.12. *If S and T are of class $[DN]$ such that $ST = TS = 0$, then $S + T$ is of class $[DN]$.*

Proof. First from Lemma 1.2, item (4), it follows that $(S + T)^D = S^D + T^D$. From $ST = TS = 0$ we get $S^D T = T S^D = 0$. Now, since S^D is normal by Fuglede theorem $S^D T^* = T^* S^D = 0$. Similarly $S^* T^D = T^D S^* = 0$. Thus

$$\begin{aligned} (S + T)^D (S + T)^* &= (S^D + T^D)(S^* + T^*) \\ &= S^D S^* + S^D T^* + T^D S^* + T^D T^* \\ &= S^* S^D + T^* T^D \\ &= (S + T)^* (S + T)^D. \end{aligned}$$

Which implies that $T + S$ is of class $[DN]$. \square

REMARK 2.13. It is well known that if T is normal and α is scalar, then $T + \alpha I$ is normal.

The following example shows that this need not be true in case of D-normal operator.

EXAMPLE 2.14. Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be operator on the Hilbert space \mathbb{C}^2 and $\alpha = 1$. After computation we get $(T + \alpha I)^D = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. So $(T + \alpha I)$ is not of class $[DN]$.

PROPOSITION 2.15. *The direct sum and the tensor product of two operators in $[DN]$ are in $[DN]$.*

Proof. Let $S, T \in [DN]$, then

$$\begin{aligned} (S \oplus T)^D (S \oplus T)^* &= (S^D \oplus T^D)(S^* \oplus T^*) \\ &= S^D S^* \oplus T^D T^* \\ &= S^* S^D \oplus T^* T^D \\ &= (S^* \oplus T^*)(S^D \oplus T^D) \\ &= (S \oplus T)^* (S \oplus T)^D. \end{aligned}$$

Then $(S \oplus T)$ is of class $[DN]$. Now, for $x_1, x_2 \in \mathcal{H}$

$$\begin{aligned} (S \otimes T)^D (S \otimes T)^* (x_1 \otimes x_2) &= (S^D \otimes T^D)(S^* \otimes T^*)(x_1 \otimes x_2) \\ &= S^D S^* x_1 \otimes T^D T^* x_2 \\ &= S^* S^D x_1 \otimes T^* T^D x_2 \\ &= (S^* \otimes T^*)(S^D \otimes T^D)(x_1 \otimes x_2) \\ &= (S \otimes T)^* (S \otimes T)^D (x_1 \otimes x_2). \end{aligned}$$

Then $(S \otimes T)$ is of class $[DN]$. \square

For $A \in \mathbb{C}^{n \times n}$, denote by A^t and \bar{A} the transpose and the conjugate of A , respectively.

PROPOSITION 2.16. *Let A be a square complex matrix. If $A \in [DN]$, then $A^D \bar{A} = \bar{A} A^D$ if and only if $A^D A^t = A^t A^D$.*

Proof. Suppose that $A^D \bar{A} = \bar{A} A^D$. Since $A \in [DN]$, hence A^D is normal. Then $A^D A^t = A^t A^D$ by Fuglede theorem. In a similar way, it can be seen that if $A^D A^t = A^t A^D$, then $A^D \bar{A} = \bar{A} A^D$, so these two statements are equivalent when A is D-normal. \square

PROPOSITION 2.17. *If $T \in [DN]$ such that T^D is unitarily equivalent to T , then T is normal.*

PROPOSITION 2.18. *Let $T \in [DN]$. Then $T^\pi T^* = T^* T^\pi$.*

Proof. Since $T \in [DN]$, we can easily get $T^D T^* T = T^* T T^D = T T^* T^D$. Hence the result follows. \square

LEMMA 2.19. *Let $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{M}_n(\mathbb{C})$ (A and B are square matrices). Then M is of class $[DN]$ if and only if A and B are of class $[DN]$ and $X = 0$, where X is defined by (1.1).*

Proof. Let M be of class $[DN]$, then M^D is normal. Using condition 8 in [9], we have A^D and B^D are of class $[N]$ and $X = 0$. In a similar way, it is obvious that if A and B are of class $[DN]$ and $X = 0$, then M^D is normal. \square

COROLLARY 2.20. *Let $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{M}_n(\mathbb{C})$ (A and B are square matrices) be of class $[DN]$. Then A and B are of class $[DN]$ and $A^D C B^D = 0$.*

COROLLARY 2.21. *Let $M = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ where $a, b, c \in \mathbb{C}$ and $a, b \neq 0$. Then M is of class $[DN]$ if and only if $c = 0$.*

Throughout this paper, some notations are needed. Let $T^D = U + iV$, where $U = ReT^D = \frac{T^D + T^{D*}}{2}$ and $V = ImT^D = \frac{T^D - T^{D*}}{2i}$ are the real and imaginary parts of T . Then write $B^2 = TT^*$ and $C^2 = T^*T$, where B and C are non-negative definite.

We give necessary and sufficient conditions for an operator to be D-normal.

PROPOSITION 2.22. *T is of class $[DN]$ if and only if T commutes with ReT^D .*

PROPOSITION 2.23. *T is of class $[DN]$ if and only if T commutes with ImT^D .*

PROPOSITION 2.24. *Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible and $F = T^D + T^*$, $G = T^D - T^*$. Then*

1. *T is of class $[DN]$ if and only if G commutes with F .*
2. *If T is of class $[DN]$, then $S = T^D T^*$ commutes with F and G .*
3. *T is of class $[DN]$ if and only if T^D commutes with F .*
4. *T is of class $[DN]$ if and only if T^D commutes with G .*

3. n-power D-normal operators

In this section, the class of n-power D-normal operators as a generalization of the classes of D-normal and n-normal operators is introduced. In addition, we make several observations about members from this class.

DEFINITION 3.1. Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. For $n \in \mathbb{N}$, T is said to be n-power D-normal operator if

$$(T^D)^n T^* = T^* (T^D)^n.$$

The class of all n-power D-normal operators is denoted by $[nDN]$.

PROPOSITION 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. Then T is n-power D-normal if and only if $(T^D)^n$ is normal.*

Proof. Let T be n-power D-normal, $(T^D)^n T^* = T^* (T^D)^n$ by Lemma 1.2 (iii), $(T^D)^n T^{*D} = T^{*D} (T^D)^n$. Then $(T^D)^n$ is normal. Now, let $(T^D)^n$ be normal. Since $(T^D)^n T = T (T^D)^n$, by Fuglede theorem, $(T^D)^n T^* = T^* (T^D)^n$. Therefore T is n-power D-normal. \square

It can be noted that the class of n-power D-normal operators properly includes classes of n-normal operators and D-normal operators, i.e., the following inclusions hold

$$[nN] \subset [nDN] \quad \text{and} \quad [DN] \subset [nDN].$$

REMARK 3.3.

1. A 1-power D-normal operator is D-normal.
2. Every D-normal operator is n-power D-normal for each n .
3. It is clear that a n-normal operator is also n-power D-normal. That the converse need not hold. Consider the operator $T = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ acting on \mathbb{C}^3 , then T is 2-power D-normal but T is not 2-normal.

In the next remark, a condition is presented where the class of $[nDN]$ operators coincide with class of $[nN]$ operators.

REMARK 3.4. Let $T \in [nDN]$. If $\text{ind}(T) \leq 1$ then $T \in [nN]$.

REMARK 3.5. All nonzero nilpotent operators are of class $[nDN]$, for any n . However they are not normal.

We record some elementary properties of $[nDN]$.

PROPOSITION 3.6. *The set of all n -power D -normal operators on \mathcal{H} is closed subset of $\mathcal{B}(\mathcal{H})$ which is closed under scalar multiplication.*

PROPOSITION 3.7. *Let $T \in [nDN]$. Then*

1. T is of class $[2nDN]$.
2. T^* is of class $[nDN]$.
3. T^D is of class $[nDN]$.
4. If $S \in \mathcal{B}(\mathcal{H})$ is Drazin invertible and unitary equivalent to T , then S is of class $[nDN]$.
5. If M is a closed subspace of \mathcal{H} such that M reduces T , then $S = T|_M$ is of class $[nDN]$.

PROPOSITION 3.8. *If $T \in [nDN]$ is similar to an idempotent $S \in \mathcal{B}(\mathcal{H})$, then T is a projection.*

THEOREM 3.9. *If T and S are of class $[nDN]$ such that $[T, S] = 0$, then TS is of class $[nDN]$.*

The following example shows that Theorem 3.9 is not necessarily true if S, T are not commuting.

EXAMPLE 3.10. Let $S = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ be operators on the Hilbert space \mathbb{C}^2 . Then $S^D = \begin{pmatrix} -i & -2 \\ 0 & i \end{pmatrix}$ and $T^D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$. Hence $S, T \in [2DN]$. We note that $ST \neq TS$. But as $((ST)^D)^2 = \begin{pmatrix} -1 & i \\ 0 & -1/4 \end{pmatrix}$ is not normal, ST is not of class $[2DN]$.

COROLLARY 3.11. *If T is of class $[nDN]$, then T^m is of class $[nDN]$ for any positive integer m .*

COROLLARY 3.12. *If T is of class $[nDN]$, then $(T^D)^n T^*$ and $T^*(T^D)^n$ are of class $[nDN]$.*

THEOREM 3.13. *If S and T are of class $[nDN]$ such that $ST = TS = 0$, then $S + T$ is of class $[nDN]$.*

LEMMA 3.14. *If S and T are of class $[2DN]$ such that $ST + TS = 0$, then ST is of class $[2DN]$.*

Proof. Since $ST + TS = 0$, $(ST)^2 = -S^2T^2 = -T^2S^2$. Hence by Lemma 1.2 (iii) $((ST)^2)^D = -(S^2)^D(T^2)^D = -(T^2)^D(S^2)^D$. Hence by Theorem 3.9 ST is of class $[2DN]$. \square

PROPOSITION 3.15. *The direct sum and the tensor product of two operators in $[nDN]$ are in $[nDN]$.*

PROPOSITION 3.16. *If $T \in [nDN]$ such that $(T^D)^n$ is unitarily equivalent to T , then T is normal.*

It is clear that if T is of class $[2DN]$ then it is of class $[2kDN]$ and if T is of class $[3DN]$ then it is of class $[3kDN]$. The following examples show that a 2-power D-normal operator need not be 3-power D-normal operator and vice versa.

EXAMPLE 3.17. Let $T = \begin{pmatrix} 3 & -2 \\ 0 & -3 \end{pmatrix}$ be operators on the Hilbert space \mathbb{C}^2 . Then $T^D = T^{-1} = \begin{pmatrix} 1/3 & -2/9 \\ 0 & -1/3 \end{pmatrix}$ hence $(T^D)^2 = \begin{pmatrix} 1/9 & 0 \\ 0 & 1/9 \end{pmatrix}$ is a normal operator. But $(T^D)^3 = \begin{pmatrix} 1/27 & -2/81 \\ 0 & -1/27 \end{pmatrix}$ is not normal. So T is of class $[2DN]$ but it is not of class $[3DN]$.

EXAMPLE 3.18. Let $T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ be operators on the Hilbert space \mathbb{C}^2 . Then $(T^D)^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is a normal operator. But $(T^D)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ is not normal. So T is of class $[3DN]$ but it is not of class $[2DN]$.

PROPOSITION 3.19. *Suppose that T is both in $[kDN]$ and in $[(k+1)DN]$ for some positive integer k . Then T is in $[(k+2)DN]$. In addition, T is in $[nDN]$ for all $n \geq k$.*

Proof. Since T is of class $[kDN]$,

$$(T^D)^k T^* = T^*(T^D)^k. \quad (3.1)$$

Multiplying (3.1) to the left and right by T^D we get

$$T^D(T^D)^k T^* T^D = T^D T^* (T^D)^k T^D.$$

So

$$(T^D)^{k+1} T^* T^D = T^D T^* (T^D)^{k+1}.$$

Since T is of class $[(k+1)DN]$,

$$T^* (T^D)^{k+2} = (T^D)^{k+2} T^*.$$

Thus T is of class $[(k+2)DN]$. \square

COROLLARY 3.20. *If T is both in $[2DN]$ and in $[3DN]$. Then T is in $[nDN]$ for all $n \geq 2$.*

PROPOSITION 3.21. *Suppose that T is in $[kDN]$ for some positive integer k and it is a partial isometry. Then T is in $[(k+1)DN]$. In addition, T is in $[nDN]$ for all $n \geq k$.*

Proof. Since T is partial isometry by [7],

$$T T^* T = T. \tag{3.2}$$

Multiplying (3.2) to the left by $(T^D)^2$ and to the right by $(T^D)^{k+1}$ we get

$$T^D T^* (T^D)^k = (T^D)^{k+2}. \tag{3.3}$$

Multiplying (3.2) to the left by $(T^D)^{k+1}$ and to the right by $(T^D)^2$ we get

$$(T^D)^k T^* T^D = (T^D)^{k+2}. \tag{3.4}$$

From $T \in [kDN]$ and (3.3), then,

$$T^* (T^D)^{k+1} = (T^D)^{k+2}. \tag{3.5}$$

Also, from $T \in [kDN]$ and (3.4), then,

$$(T^D)^{k+1} T^* = (T^D)^{k+2}. \tag{3.6}$$

In view of (3.5) and (3.6), we conclude $T \in [(k+1)DN]$. Finally, by Proposition 3.19, T is of class $[nDN]$ for all $n \geq k$. \square

COROLLARY 3.22. *If T is in $[2DN]$ and it is a partial isometry, then T is in $[nDN]$ for all $n \geq 2$.*

LEMMA 3.23. *Let T be both in $[kDN]$ and in $[(k+1)DN]$. If either T or T^* is injective, then T is of class $[DN]$.*

Proof. Since T is of class $[kDN]$,

$$T^*(T^D)^k = (T^D)^k T^*. \tag{3.7}$$

And since T is of class $[(k + 1)DN]$,

$$T^*(T^D)^{k+1} = (T^D)^{k+1} T^*. \tag{3.8}$$

From (3.7) and (3.8)

$$(T^D)^k (T^* T^D - T^D T^*) = 0.$$

Since T is injective,

$$(T^* T^D - T^D T^*) = 0.$$

Then T is of class $[DN]$. In the case that T^* is injective, since T^* is of class $[kDN]$, and $[(k + 1)DN]$, T^* is of class $[DN]$. Hence T is of class $[DN]$. \square

LEMMA 3.24. *If $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ are Drazin invertible. Then $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is of class $[nDN]$ if and only if A and B are of class $[nDN]$ and $\sum_{i=0}^n (A^D)^i X (B^D)^{n-i} = 0$, where X is defined by (1.1).*

PROPOSITION 3.25. *T is of class $[DN]$ if and only if T commutes with $Re(T^D)^n$.*

PROPOSITION 3.26. *T is of class $[DN]$ if and only if T commutes with $Im(T^D)^n$.*

PROPOSITION 3.27. *Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible and $F = (T^D)^n + T^*$, $G = (T^D)^n - T^*$. Then*

1. *T is of class $[nDN]$ if and only if G commutes with F .*
2. *If T is of class $[nDN]$, then $S = (T^D)^n T^*$ commutes with F and G .*
3. *T is of class $[nDN]$ if and only if $(T^D)^n$ commutes with F .*
4. *T is of class $[nDN]$ if and only if $(T^D)^n$ commutes with G .*

4. D-quasi-normal operators

In this section, a definition of D-quasi-normal operators is presented. We investigate some basic properties of such operators and study the relation among the D-quasi-normal operators and some other operators.

DEFINITION 4.1. Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. T is said D-quasi-normal if

$$T^D T^* T = T^* T T^D.$$

The class of all D-quasi-normal operators is denoted by $[DQN]$.

REMARK 4.2. Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. T is D-quasi-normal if and only if

$$[T^D, T^*T] = 0.$$

REMARK 4.3. Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. T is D-quasi-normal if and only if

$$T^D|T|^2 = |T|^2T^D.$$

Obviously, that the class of D-quasi-normal operators includes classes of quasi-normal operators and D-normal operators, i.e., the following inclusions holds

$$[N] \subset [QN] \subset [DQN] \quad \text{and} \quad [N] \subset [DN] \subset [DQN].$$

we give some sufficient conditions for a D-quasi-normal operator to be quasi-normal.

REMARK 4.4. Let $T \in [DQN]$. If $\text{ind}(T) < 1$ then $T \in [N]$.

REMARK 4.5. Let $T \in [DN]$. If $\text{ind}(T) = 1$ then $T \in [QN]$.

PROPOSITION 4.6. *The set of all D-quasi-normal operators on \mathcal{H} is closed subset of $\mathcal{B}(\mathcal{H})$ which is closed under scalar multiplication.*

Let us now examine some of the basic properties of the class of $[DQN]$.

THEOREM 4.7. *If $T \in [DQN]$, then*

1. *If $S \in \mathcal{B}(\mathcal{H})$ is Drazin invertible and unitary equivalent to T , then S is of class $[DQN]$.*
2. *If M is a closed subspace of \mathcal{H} such that M reduces T , then $S = T|M$ is of class $[DQN]$.*
3. *If T has a dense range in \mathcal{H} , T is of class $[DN]$.*
4. *If S is of class $[DQN]$ such that $[T, S] = [T, S^*] = 0$, then TS is of class $[DQN]$.*
5. *If S is of class $[N]$ such that $[T, S] = 0$, then TS is of class $[DQN]$.*
6. *If S is of class $[DQN]$ such that $ST = TS = T^*S = ST^* = 0$, then $S + T$ is of class $[DQN]$.*
7. *If S is of class $[N]$ such that $ST = TS = 0$, then $S + T$ is of class $[DQN]$.*

Proof. (1), (2) are trivial.

(3) Since T is of class $[DQN]$, we have for $y \in \mathcal{R}(T) : y = Tx, x \in H$,

$$\|(T^DT^* - T^*T^D)y\| = \|(T^DT^* - T^*T^D)Tx\| = \|(T^DT^*T - T^*TT^D)x\| = 0.$$

Thus, T is D-normal on $\mathcal{R}(T)$ and hence T is of class $[DN]$.

(4) Since $[T, S] = [T, S^*] = 0$, by Lemma 1.2 (iii), we have that $[T, S^D] = [T^D, S] = [T^D, S^*] = [T^*, S^D] = 0$. So

$$\begin{aligned}
 (ST)^D(ST)^*(ST) &= T^D S^D T^* S^* ST \\
 &= T^D T^* S^D S^* ST \\
 &= T^D T^* S^* S S^D T \\
 &= T^D T^* S^* S T S^D \\
 &= T^D T^* S^* T S S^D \\
 &= T^D T^* T S^* S S^D \\
 &= T^* T T^D S^* S S^D \\
 &= T^* T S^* T^D S S^D \\
 &= T^* T S^* S T^D S^D \\
 &= T^* S^* S T T^D S^D \\
 &= (ST)^*(ST)(ST)^D.
 \end{aligned}$$

Which implies that ST is of class $[DQN]$.

(5) Using the item (4) and the Fuglede theorem, the following results are obtained.
(6)

$$\begin{aligned}
 (S+T)^D(S+T)^*(S+T) &= (S^D+T^D)(T^*T+S^*S) \\
 &= S^D S^* S + T^D T^* T \\
 &= S^* S S^D + T^* T T^D \\
 &= (S+T)^*(S+T)(S+T)^D.
 \end{aligned}$$

Thus T is of class $[DQN]$.

(7) Immediate from the item (6) and the Fuglede theorem. \square

PROPOSITION 4.8. *The direct sum and the tensor product of two operators in $[DQN]$ are in $[DQN]$.*

The following example shows that Remark 2.13 need not be true in case of D-quasi-normal operator.

EXAMPLE 4.9. Let $T = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ be operator on the Hilbert space \mathbb{C}^2 and α is scalar. After computation we get $(T + \alpha I)^D = \begin{pmatrix} 1/\alpha & -2/\alpha \\ 0 & 1/\alpha \end{pmatrix}$. So $(T + \alpha I)$ is not of class $[DQN]$.

PROPOSITION 4.10. *If $T \in [DQN]$ is similar to an idempotent $S \in \mathcal{B}(\mathcal{H})$, then T is a projection.*

Proof. Since T is similar to S , there is an invertible operator $N \in \mathcal{B}(\mathcal{H})$ such that $T = N^{-1}SN$, which implies that $T^D = N^{-1}S^D N = N^{-1}SN$. Thus T is quasi-normal. The result now follows from ([10], p. 111). \square

Necessary and sufficient conditions for an operator to be D-quasi-normal are provided.

THEOREM 4.11. T is of class $[DQN]$ if and only if C commutes with ReT^D and ImT^D .

Proof. It is easy to see that $C^2 ReT^D = ReT^D C^2$. Since C is non-negative definite it follows that $C ReT^D = ReT^D C$. Similarly $C ImT^D = ImT^D C$.

Conversely, let $C ReT^D = ReT^D C$ and $C ImT^D = ImT^D C$. Then $C^2 ReT^D = ReT^D C^2$ and $C^2 ImT^D = ImT^D C^2$. Hence $C^2(ReT^D + iImT^D) = (ReT^D + iImT^D)C^2$ and then, $C^2 T^D = T^D C^2$. Therefore $T^* T T^D = T^D T^* T$. \square

THEOREM 4.12. Let T is of class $[DQN]$ and $C^2 T^D = T^D B^2$. Then B commutes with ReT^D and ImT^D .

Proof. Since $C^2 T^D = T^D B^2$ we have $T^* T T^D = T^D T T^*$. Hence

$$(T^*)^D T^* T = T T^* (T^*)^D.$$

Now

$$\begin{aligned} B^2 ReT^D &= 1/2 [T T^* (T^D + T^{D*})] \\ &= 1/2 [T T^* T (T^D)^2 + T T^{D*} T^*] \\ &= 1/2 [T (T^D)^2 T^* T + T T^* (T^{D*})^2 T^*] \\ &= 1/2 [T^* T T^D + T^{D*} T^* T T^{D*} T^*] \\ &= 1/2 [T^D T T^* + (T^{D*})^2 T^* T T^*] \\ &= 1/2 [T^D T T^* + T^{D*} T T^*] \\ &= ReT^D B^2. \end{aligned}$$

Hence $B ReT^D = ReT^D B$. Similarly $B ImT^D = ImT^D B$. \square

PROPOSITION 4.13. Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible and $F = T^D + T^* T$, $G = T^D - T^* T$. Then

1. T is of class $[DQN]$ if and only if G commutes with F .
2. If T is of class $[DQN]$, then $B = T^D T^* T$ commutes with F and G .
3. T is of class $[DQN]$ if and only if T^D commutes with F .
4. T is of class $[DQN]$ if and only if T^D commutes with G .

PROPOSITION 4.14. *If T is of class $[DQN]$ such that $\mathcal{N}(T^{*D}) \subset \mathcal{N}(T^D)$ then T is of class $[DN]$.*

Proof. Since T is of $[DQN]$, $[T^{*}T^D - T^DT^{*}]T = 0$, i.e. $[T^{*}T^D - T^DT^{*}] = 0$, on $cl\mathcal{R}(T)$. Also the fact that $\mathcal{N}(T^{*D})$ is a subset of $\mathcal{N}(T^D)$ gives $[T^{*}T^D - T^DT^{*}] = 0$, on $\mathcal{N}(T^{*})$. Hence the result follows. \square

LEMMA 4.15. *If T is of class $[DQN]$, then $\mathcal{N}(T^n) \subset \mathcal{N}(T^{*D})$ for every $n \in \mathbb{N}$.*

Proof. If $n = 1$ then we show that $\mathcal{N}(T) \subset \mathcal{N}(T^{*D})$. Suppose that $Tx = 0$. Then

$$T^{*D}T^{*}Tx = 0.$$

By hypotheses,

$$T^{*}TT^{*D}x = 0,$$

which implies

$$TT^{*D}x = 0.$$

Hence

$$T^DT^{*D}x = 0.$$

We deduce that

$$T^{*D}x = 0.$$

Now, taking $n \geq 2$. Let $T^n x = 0$. Then

$$T^{*D}T^{*}TT^{n-1}x = 0.$$

By hypotheses,

$$T^{*}TT^{*D}T^{n-1}x = 0.$$

Which implies

$$TT^{*D}T^{n-1}x = 0. \tag{4.1}$$

Multiplying (4.1) to the left by $(T^D)^2$, then,

$$T^DT^{*D}T^{n-1}x = 0.$$

Hence

$$T^{*D}T^{n-1}x = 0.$$

Under the condition on T , then,

$$T^{*}TT^{*D}T^{n-2}x = 0.$$

Hence

$$T^{*D}T^{n-2}x = 0.$$

By repeating this process,

$$T^{*D}x = 0. \quad \square$$

COROLLARY 4.16. *If T is of class $[DQN]$, then $\mathcal{N}(T^D) \subset \mathcal{N}(T^{*D})$.*

Proof. If T has index k then it is easy to see that $\mathcal{N}(T^D) = \mathcal{N}(T^l)$ for all $l \geq k$. Now the conclusion follows from Lemma 4.15. \square

THEOREM 4.17. *If T and T^* are of class $[DQN]$, then T^D is normal.*

Proof. By hypotheses and Lemma 4.15

$$\mathcal{N}(T^n) \subset \mathcal{N}(T^{*D}), \quad \mathcal{N}(T^{*n}) \subset \mathcal{N}(T^D).$$

for every $n \in \mathbb{N}$. Since T is of $[DQN]$, $[T^*T^D - T^DT^*]T = 0$, i.e. $[T^*T^D - T^DT^*] = 0$, on $cl\mathcal{R}(T)$. Also the fact that $\mathcal{N}(T^*)$ is a subset of $\mathcal{N}(T^D)$ gives $[T^*T^D - T^DT^*] = 0$, on $\mathcal{N}(T^*)$. Hence the result follows. \square

5. n-power D-quasi-normal operators

As an extension of the class of D-quasi-normal operators, the following definition describes the class of operators.

DEFINITION 5.1. Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. For $n \in \mathbb{N}$, T is said to be n-power D-quasi-normal operator if

$$(T^D)^n T^* T = T^* T (T^D)^n.$$

The class of all n-power D-quasi-normal operators is denoted by $[nDQN]$.

REMARK 5.2.

1. A 1-power D-quasi-normal operator is D-quasi-normal.
2. Every D-quasi-normal operator is n-power D-quasi-normal for each n .
3. It is clear that a n-power D-normal operator is also n-power D-quasi-normal. That the converse need not hold. Consider the operator $T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ acting on \mathbb{C}^2 , then T is 2-power D-quasi-normal but T is not 2-power D-normal.

REMARK 5.3. Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. T is n-power D-quasi-normal if and only if

$$[(T^D)^n, T^* T] = 0.$$

REMARK 5.4. Let $T \in \mathcal{B}(\mathcal{H})$ be Drazin invertible. T is n-power D-quasi-normal if and only if

$$(T^D)^n |T|^2 = |T|^2 (T^D)^n.$$

Some sufficient conditions for a n -power D -quasi-normal operator to be n -power quasi-normal are presented.

REMARK 5.5. Let $T \in [nDQN]$. If $ind(T) \leq 1$ then $T \in [nQN]$.

PROPOSITION 5.6. *The set of all n -power D -quasi-normal operators on \mathcal{H} is closed subset of $\mathcal{B}(\mathcal{H})$ which is closed under scalar multiplication.*

Next results state some properties of the class $[nDQN]$.

THEOREM 5.7. *If $T \in [nDQN]$, then*

1. T is of class $[2nDQN]$.
2. If $S \in \mathcal{B}(\mathcal{H})$ is Drazin invertible and unitary equivalent to T , then S is of class $[nDQN]$.
3. If M is a closed subspace of \mathcal{H} such that M reduces T , then $S = T|M$ is of class $[nDQN]$.
4. If T has a dense range in \mathcal{H} , T is of class $[nDN]$.
5. If T and S are of class $[nDQN]$ such that $[T, S] = [T, S^*] = 0$, then TS is of class $[nDQN]$.
6. If S is normal such that $[T, S] = 0$, then TS is of class $[nDQN]$.
7. If S and T are of class $[nDQN]$ such that $ST = TS = T^*S = ST^* = 0$, then $S + T$ is of class $[nDQN]$.
8. If S is normal such that $ST = TS = 0$, then $S + T$ is of class $[nDQN]$.

Proof. (1) Since T is of $[nDQN]$, then

$$(T^D)^n T^* T = T^* T (T^D)^n. \tag{5.1}$$

Multiplying (5.1) to the left by $(T^D)^n$, then $(T^D)^{2n} T^* T = T^* T (T^D)^{2n}$. Thus T is of class $[2nDQN]$.

Using similar methods as in Theorem 4.7 the statement (2)-(8) of the theorem is true. \square

PROPOSITION 5.8. *If $T \in [nDQN]$ is similar to an idempotent $S \in \mathcal{B}(\mathcal{H})$, then T is a projection.*

PROPOSITION 5.9. *The direct sum and the tensor product of two operators in $[nDQN]$ are in $[nDQN]$.*

THEOREM 5.10. *T is of class [nDQN] if and only if C commutes with Re(T^D)ⁿ and Im(T^D)ⁿ.*

THEOREM 5.11. *Let T is of class [nDQN] and C²(T^D)ⁿ = (T^D)ⁿB². Then B commutes with Re(T^D)ⁿ and Im(T^D)ⁿ.*

PROPOSITION 5.12. *Let T ∈ B(H) be Drazin invertible and F = (T^D)ⁿ + T^{*}T, G = (T^D)ⁿ - T^{*}T. Then*

1. *T is of class [nDQN] if and only if G commutes with F.*
2. *If T is of class [nDQN], then S = (T^D)ⁿT^{*}T commutes with F and G.*
3. *T is of class [nDQN] if and only if (T^D)ⁿ commutes with F.*
4. *T is of class [nDQN] if and only if (T^D)ⁿ commutes with G.*

It is clear that if T is of class [2DQN] then it is of class [2kDQN] if T is of class [3DQN] then it is of class [3kDQN]. The following examples show that the two classes [2DQN] and [3DQN] are not the same.

EXAMPLE 5.13. Let $T = \begin{pmatrix} 2 & -2 \\ 0 & -2 \end{pmatrix}$ be operator on the Hilbert space \mathbb{C}^2 . Then $(T^D)^2 = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}$ and $(T^D)^3 = \begin{pmatrix} 1/8 & -1/8 \\ 0 & -1/8 \end{pmatrix}$. Hence by simple calculations we see that T is not of class [3DQN] but of class [2DQN].

EXAMPLE 5.14. Let $T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ be operators on the Hilbert space \mathbb{C}^2 . Then $T^D = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, so by simple calculations we see that T is not of class [2DQN] but of class [3DQN].

PROPOSITION 5.15. *Let T ∈ B(H) such that T is of class [kDQN] ∩ [(k + 1)DQN], for some positive integer k. Then T is in [(k + 2)DN]. In addition, T is in [nDN] for all n ≥ k.*

Proof. Since T is of class [kDQN],

$$(T^D)^k T^* T = T^* T (T^D)^k. \tag{5.2}$$

Multiplying (5.2) to the left and right by T^D we get

$$T^D (T^D)^k T^* T T^D = T^D T^* T (T^D)^k T^D.$$

So

$$(T^D)^{k+1} T^* T T^D = T^D T^* T (T^D)^{k+1}.$$

Since T is of class $[(k + 1)DQN]$,

$$T^*T(T^D)^{k+2} = (T^D)^{k+2}T^*T.$$

Thus T is of class $[(k + 2)DQN]$. \square

COROLLARY 5.16. *Let $T \in \mathcal{B}(\mathcal{H})$ such that T is of class $[2DQN] \cap [3DQN]$, then T is of class $[nDQN]$ for all positive integer $n \geq 2$.*

COROLLARY 5.17. *If T is in the class $[2DQN]$ and T is an 2-isometry, then T^2 is in the class $[nDQN]$ for all positive integer $n \geq 2$.*

Proof. From Proposition 5.15, it suffices to prove that T^2 is in the class $[2DQN]$ and T^2 is in the class $[3DQN]$.

Since T is in the class $[2DQN]$ and T is an 2-isometry, then,

$$\begin{aligned} (T^D)^4(T^{*2}T^2) &= (T^D)^4(2T^*T - I) \\ &= 2T^*T(T^D)^4 - (T^D)^4 \\ &= (2T^*T - I)(T^D)^4 \\ &= (T^{*2}T^2)(T^D)^4. \end{aligned}$$

Thus, T^2 is in the class $[2DQN]$.

On the other hand,

$$\begin{aligned} (T^D)^6(T^{*2}T^2) &= (T^D)^6(2T^*T - I) \\ &= 2T^*T(T^D)^6 - (T^D)^6 \\ &= (2T^*T - I)(T^D)^6 \\ &= (T^{*2}T^2)(T^D)^6. \end{aligned}$$

Thus, T^2 is in the class $[3DQN]$. \square

THEOREM 5.18. *Suppose that T is in $[kDQN]$ for some positive integer k and it is a partial isometry. Then T is in $[(k + 1)DQN]$. In addition, T is in $[nDQN]$ for all $n \geq k$.*

Proof. Since T is a partial isometry by [7],

$$TT^*T = T. \tag{5.3}$$

Multiplying (5.3) to the left by $(T^D)^{k+1}$ and to the right by T^D , then,

$$(T^D)^{k+1}TT^*TT^D = (T^D)^{k+1}. \tag{5.4}$$

Also, multiplying (5.3) to the left by $(T^D)^2$ and to the right by $(T^D)^k$,

$$(T^D)^2TT^*T(T^D)^k = (T^D)^{k+1}. \tag{5.5}$$

In view of (5.4) and (5.5), we conclude

$$(T^D)^{k+1}TT^*TT^D = (T^D)^2TT^*T(T^D)^k. \tag{5.6}$$

Using the fact that T is of class $[kDQN]$, we get that T is of class $[(k+1)DQN]$. \square

LEMMA 5.19. *Let T be both in $[kDQN]$ and in $[(k+1)DQN]$. If T is injective, then T is of class $[DQN]$.*

Proof. Since T is of class $[kDQN]$,

$$T^*T(T^D)^k = (T^D)^kT^*T. \tag{5.7}$$

And since T is of class $[(k+1)DN]$,

$$T^*T(T^D)^{k+1} = (T^D)^{k+1}T^*T. \tag{5.8}$$

From (5.7) and (5.8)

$$(T^D)^k(T^*TT^D - T^DT^*T) = 0.$$

Since T is injective,

$$(T^*TT^D - T^DT^*T) = 0.$$

Then T is of class $[DQN]$. \square

PROPOSITION 5.20. *If T is of class $[nDQN]$ such that $N(T^{*D}) \subset N(T^D)$ then T is of class $[nDN]$.*

LEMMA 5.21. *If T is of class $[nDQN]$, then $N(T^n) \subset N(T^{*D})$ for every $n \in \mathbb{N}$.*

COROLLARY 5.22. *If T is of class $[nDQN]$, then $\mathcal{N}(T^D) \subset \mathcal{N}(T^{*D})$.*

THEOREM 5.23. *If T and T^* are of class $[nDQN]$, then $(T^D)^n$ is normal.*

THEOREM 5.24. *If T is of class $[2DQN]$ and T^2 is of class $[3DQN]$, then T^2 is D -quasi-normal.*

Proof. By the condition that T^2 is of class $[3DQN]$, we have

$$\begin{aligned} (T^{*2}T^2)(T^{*D})^6 &= (T^{*D})^6(T^{*2}T^2) \\ &= T^*(T^{*D})^6(T^*T)T \\ &= T^*(T^*T)(T^{*D})^6T && (T \in [2DQN]) \\ &= T^*(T^*T)T^{*D}(T^{*D})^6(T^*T) \\ &= T^*(T^*T)T^{*D}(T^*T)(T^{*D})^6 && (T \in [2DQN]) \\ &= T^*(T^*T)T^*(T^{*D})^2(T^*T)(T^{*D})^6 \\ &= T^*(T^*T)T^*(T^*T)(T^{*D})^8 && (T \in [2DQN]) \\ &= [T^*(T^*T)]^2(T^{*D})^8. \end{aligned}$$

Thus we have

$$\{(T^{*2}T^2)(T^{*D})^2 - [T^*(T^*T)]^2(T^{*D})^4\}(T^{*D})^4 = 0$$

or

$$(T^4)^D\{(T^D)^2(T^{*2}T^2) - (T^D)^4[(T^*T)T]^2\} = 0.$$

Then

$$T^D\{(T^D)^2(T^{*2}T^2) - (T^D)^4[(T^*T)T]^2\} = 0.$$

or

$$\{(T^{*2}T^2)(T^{*D})^2 - [T^*(T^*T)]^2(T^{*D})^4\}x = 0 \quad \text{for } x \in clR(T^{*D}).$$

Hence from Corollary 5.22,

$$\{(T^{*2}T^2)(T^{*D})^2 - [T^*(T^*T)]^2(T^{*D})^4\}y = 0 \quad \text{for } y \in N(T^D).$$

Thus

$$\{(T^{*2}T^2)(T^{*D})^2 - [T^*(T^*T)]^2(T^{*D})^4\} = 0$$

or

$$\begin{aligned} (T^D)^2(T^{*2}T^2) &= (T^D)^4[(T^*T)T]^2 \\ &= (T^D)^4(T^*T^2)(T^*T^2) \\ &= T^*T(T^D)^4T(T^*T^2) && (T \in [2DQN]) \\ &= T^*(T^D)^2(T^*T^2) && (T \in [2DQN]) \\ &= T^*T^*T(T^D)^2T && (T \in [2DQN]) \\ &= (T^{*2}T^2)(T^D)^2, \end{aligned}$$

and the results are proven. \square

COROLLARY 5.25. *If T is of class $[2DQN]$ and $0 \notin W(T^D)$, then T is of class $[DN]$*

Proof. Since $0 \notin W(T^D)$, gives $\mathcal{N}(T^D) = \mathcal{N}(T^{*D}) = 0$ and so by our Proposition 5.20, T is of class $[2DN]$. Then $[T^{*D}T^D, T^DT^{*D}] = 0$. Now the conclusion follows from [8]. \square

Finally, it seems to be natural to enquire Fuglede-Putnam theorem for these generalized normal operators. This problem will be addressed in a forthcoming paper.

Acknowledgement. The authors would like to thank the referees for their valuable comments and suggestions, which considerably helped improve the paper.

REFERENCES

- [1] O. M. AHMED, *On The Class of n -Power Quasi-Normal Operators on Hilbert Space*, Bulletin of Mathematical Analysis and Applications **3** (2011), no. 2, 213–228, .
- [2] S. A. AL-ZURAIQI AND A. B. PATEL, *On n -normal Operators*, General Mathematics Notes **2** (2010), no. 1, 61–73.
- [3] A. BEN-ISRAEL, T. N. E. GREVILLE, *Generalized Inverses: Theory and Applications*, second ed., Springer Verlag, New York, 2003.
- [4] A. BROWN, *On a class of operators*, Proc. Amer. Math. Soc **4** (1953), 723–728.
- [5] S. L. CAMPBELL, C. D. MEYER, *Generalized Inverse of Linear Transformations*, Pitman, London, 1979 (Dover, New York, 1991).
- [6] S. R. CARADUS, *Operator Theory of the Generalized Inverse*, Science Press, New York, 2004.
- [7] J. B. CONWAY, *A Course in Functional Analysis*, Second Edition, Springer Verlag, Berlin Heidelberg – New York 1990.
- [8] M. R. EMBRY, *Conditions implying normality in Hilbert space*, Pacific. J. Math. **18** (1966), 457–460.
- [9] R. GRONE, C. R. JOHNSON, E. M. SA, H. WOLKOWICZ, *Normal matrices*, Linear Algebra Appl. **87** (1987), 213–225.
- [10] P. R. HALMOS, *A Hilbert Space Book*, Springer-Verlag, New York, 1982.

(Received October 21, 2017)

M. Dana
Faculty of Mathematics
University of Kurdistan
Sanandaj, P. O. Box 416, Islamic Republic of Iran
e-mail: mdana@uok.ac.ir

R. Yousefi
Faculty of Mathematics
University of Kurdistan
Sanandaj, P. O. Box 416, Islamic Republic of Iran
e-mail: Ramesh.yousefi@yahoo.com