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# NONADDITIVE STRONG COMMUTATIVITY PRESERVING MAPS ON RANK–$k$ MATRICES OVER DIVISION RINGS

CHENG-KAI LIU, PAO-KUEI LIAU AND YUAN-TSUNG TSAI

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*Abstract.* Let $M_n(\mathbb{D})$ be the ring of all $n\times n$ matrices over a division ring $\mathbb{D}$, where $n\geqslant 2$ is an integer and let $\mathscr{S}$ be the set of all rank-$k$ matrices in $M_n(\mathbb{D})$, where $k$ is an integer with $1\leqslant k\leqslant n$. We characterize maps $f:\mathscr{S}\to M_n(\mathbb{D})$ such that $[f(x),f(y)]=[x,y]$ for all $x,y\in\mathscr{S}$.

## 1. Introduction and results

Let $\mathscr{A}$ be a ring with center $Z(\mathscr{A})$. For $x,y\in\mathscr{A}$, let $[x,y]$ denote the commutator of $x$ and $y$, that is, $[x,y]=xy-yx$. We say that a map $f:\mathscr{A}\to\mathscr{A}$ preserves commutativity if $[f(x),f(y)]=0$ whenever $[x,y]=0$ for $x,y\in\mathscr{A}$. The study of describing maps that preserve commutativity has become an active research area in matrix theory, operator theory and ring theory. In [5] Bell and Mason introduced the notion of a certain kind of commutativity preserving maps as follows: For a subset $\mathscr{S}$ of $\mathscr{A}$, a map $f:\mathscr{S}\to\mathscr{A}$ is said to be strong commutativity preserving on $\mathscr{S}$ if $[f(x),f(y)]=[x,y]$ for all $x,y\in\mathscr{S}$. Bell and Daif [4] proved that if a semiprime ring $\mathscr{A}$ admits a derivation $d$ satisfying $[d(x),d(y)]=[x,y]$ for all $x,y\in\mathscr{A}$, then $\mathscr{A}$ is commutative. Brešar and Miers [9] characterized additive maps $f:\mathscr{A}\to\mathscr{A}$ which is strong commutativity preserving on the entire semiprime ring $\mathscr{A}$ and showed that $f$ must be of the form $f(x)=\lambda x+\mu(x)$ for all $x\in\mathscr{A}$, where $\lambda$ is an element in the extended centroid $\mathscr{C}$ of $\mathscr{A}$, $\lambda^2=1$ and $\mu:\mathscr{A}\to\mathscr{C}$ is an additive map. In [39] Qi and Hou studied nonadditive strong commutativity preserving maps and proved that if $\mathscr{A}$ is a unital prime ring containing nontrivial idempotents, then every nonadditive surjective strong commutativity preserving maps $f$ on $\mathscr{A}$ must be of the form $f(x)=\lambda x+\mu(x)$ for all $x\in\mathscr{A}$, where $\lambda\in\{1,-1\}$ and $\mu:\mathscr{A}\to Z(\mathscr{A})$ is a map. Later Lee and Wong [24] generalized the result of Qi and Hou by removing the surjection of the map $f$ and the assumption on the existence of nontrivial idempotents in $\mathscr{A}$. These results have been now generalized in various directions [2, 3, 11, 15, 26, 27, 29, 31, 37, 38, 40, 41, 45]. Many important results on linear preserver problems treat subsets of matrices that are not closed under addition such as invertible matrices, singular matrices, nilpotent matrices, matrices of rank one, etc (see the survey paper [25] for details). In [16, 17, 21]

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Franca initiated the study of functional identities on invertible matrices, singular matrices or rank-$k$ matrices. Precisely, he successfully described the structure of additive maps $f: M_n(\mathbb{K}) \to M_n(\mathbb{K})$ such that $[f(x), x] = 0$ for every invertible matrix (singular matrix, rank-$k$ matrix) $x \in M_n(\mathbb{K})$, where $M_n(\mathbb{K})$ denotes the ring of all $n \times n$ matrices over a field $\mathbb{K}$. Since then, several related results had been obtained in the literature [18–21, 30, 33, 34, 42–44]. Recently, Liu [34] characterized nonadditive maps which are strong commutativity preserving on the set of invertible matrices or singular matrices as follows:

THEOREM L. ([34, Theorem 2.2 and Theorem 2.4]) *Let $M_n(\mathbb{D})$ be the ring of all $n \times n$ matrices over a division ring $\mathbb{D}$, where $n \geqslant 2$ is an integer and let $\mathscr{S}$ be a subset of $M_n(\mathbb{D})$ containing all invertible (singular) matrices in $M_n(\mathbb{D})$. Suppose that $f: \mathscr{S} \to M_n(\mathbb{D})$ is a map satisfying $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathscr{S}$. Then there exists a map $\mu: \mathscr{S} \to Z(\mathbb{D})I_n$ such that $f(x) = x + \mu(x)$ for all $x \in \mathscr{S}$ or $f(x) = -x + \mu(x)$ for all $x \in \mathscr{S}$, where $I_n$ denotes the identity matrix of $M_n(\mathbb{D})$.*

Let $\mathbb{D}$ be a division ring and let $\mathbb{D}^n = \mathbb{D} \oplus \cdots \oplus \mathbb{D}$ be the left $\mathbb{D}$-vector space consisting of all $1 \times n$ row vectors over $\mathbb{D}$. Given a matrix $x \in M_n(\mathbb{D})$, the row space of $x$ over $\mathbb{D}$ is the $\mathbb{D}$-subspace of $\mathbb{D}^n$ generated by the rows of $x$ and the rank of $x$, denoted by $\operatorname{rank} x$, is the dimension of the row space of $x$ over $\mathbb{D}$. As usual, for a matrix $x \in M_n(\mathbb{D})$, there are three kinds of elementary row operations on $x$: (i) interchange two rows of $x$; (ii) left multiply a row of $x$ by a nonzero $\alpha \in \mathbb{D}$; (iii) for $\alpha \in \mathbb{D}$ and $i \neq j$, add $\alpha$ times row $j$ to row $i$. It is known that every matrix $x \in M_n(\mathbb{D})$ can be changed to a matrix $y$ in reduced row echelon form by a finite sequence of elementary row operations. In particular, $\operatorname{rank} x = \operatorname{rank} y$, $\operatorname{rank} y$ is the number of nonzero rows in $y$ and $x$ is invertible in $M_n(\mathbb{D})$ iff $\operatorname{rank} x = n$ (see [23, Chapter VII]). Thus, writing Theorem L in terms of rank, we can say that every strong commutativity preserving maps $f$ on the set $\mathscr{S}$ of all rank-$n$ matrices in $M_n(\mathbb{D})$ is of the form $f(x) = \lambda x + \mu(x)$ for all $x \in \mathscr{S}$, where $\lambda \in \{1, -1\}$ and $\mu: \mathscr{S} \to Z(\mathbb{D})I_n$ is a map. So it gives rise to a natural question: For an integer $k$ with $1 \leqslant k \leqslant n-1$, does every strong commutativity preserving maps on the set of all rank-$k$ matrices in $M_n(\mathbb{D})$ have the standard form described in Theorem L? The goal of this paper is to give a positive answer to this question. Our main result is as follows:

THEOREM 1.1. *Let $M_n(\mathbb{D})$ be the ring of all $n \times n$ matrices over a division ring $\mathbb{D}$, where $n \geqslant 2$ is an integer and let $\mathscr{S}$ be a subset of $M_n(\mathbb{D})$ containing all rank-$k$ matrices in $M_n(\mathbb{D})$, where $k$ is an integer such that $1 \leqslant k \leqslant n$. If $f: \mathscr{S} \to M_n(\mathbb{D})$ is a map satisfying $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathscr{S}$, then there exists a map $\mu: \mathscr{S} \to Z(\mathbb{D})I_n$ such that $f(x) = x + \mu(x)$ for all $x \in \mathscr{S}$ or $f(x) = -x + \mu(x)$ for all $x \in \mathscr{S}$, where $I_n$ denotes the identity matrix of $M_n(\mathbb{D})$.*

Let $\sigma$ be an automorphism of $\mathscr{A}$ and let $1_{\mathscr{A}}$ denote the identity automorphism of $\mathscr{A}$. An additive map $\delta: \mathscr{A} \to \mathscr{A}$ is called a $\sigma$-derivation if $\delta(xy) = \sigma(x)\delta(y) + \delta(x)y$ for all $x, y \in \mathscr{A}$. Clearly, $1_{\mathscr{A}}$-derivations are just ordinary derivations. An additive map $g: \mathscr{A} \to \mathscr{A}$ is called a generalized $\sigma$-derivation if there exists a $\sigma$-derivation $\delta: \mathscr{A} \to \mathscr{A}$ such that $g(xy) = g(x)y + \sigma(x)\delta(y)$ for all $x, y \in \mathscr{A}$ (see [1, 6, 12–14, 28, 32, 35]). Generally, we call generalized $\sigma$-derivations *generalized skew deriva-*

*tions*. Generalized $1_{\mathscr{A}}$-derivations are just generalized derivations. A map $g:\mathscr{A}\to\mathscr{A}$ is called a multiplicative generalized $\sigma$-derivation if there exists a map $\delta:\mathscr{A}\to\mathscr{A}$ such that $g(xy)=g(x)y+\sigma(x)\delta(y)$ for all $x,y\in\mathscr{A}$. Recently, strong commutativity preserving generalized derivations and skew derivations had been widely studied in the literature [2, 11, 27, 31, 38, 45]. As an application of Theorem 1.1, we obtain the analogous result for multiplicative generalized $\sigma$-derivations on rank-$k$ matrices over division rings.

COROLLARY 1.2. *Let $M_n(\mathbb{D})$ be the ring of all $n\times n$ matrices over a division ring $\mathbb{D}$, where $n\geqslant 2$ is an integer and let $\sigma$ be an automorphism of $M_n(\mathbb{D})$. Suppose that $g:M_n(\mathbb{D})\to M_n(\mathbb{D})$ is a multiplicative generalized $\sigma$-derivation satisfying $[g(x),g(y)]=[x,y]$ for all rank-$k$ matrices $x,y\in M_n(\mathbb{D})$, where $k$ is an integer such that $1\leqslant k\leqslant n$. Then $g(x)=x$ for all $x\in M_n(\mathbb{D})$ or $g(x)=-x$ for all $x\in M_n(\mathbb{D})$.*

## 2. The Jacobi type identity

As usual, let $M_n(\mathbb{D})$ denote the ring of all $n\times n$ matrices over a division ring $\mathbb{D}$, let $I_n$ be the identity matrix of $M_n(\mathbb{D})$ and let $\{e_{ij}\mid 1\leqslant i,j\leqslant n\}$ be the set of matrix units in $M_n(\mathbb{D})$. Note that the center of $M_n(\mathbb{D})$ is $Z(M_n(\mathbb{D}))=Z(\mathbb{D})I_n$.

The following two lemmas are obvious.

LEMMA 2.1. *Let $\mathbb{D}$ be a division ring and $\alpha,\beta\in\mathbb{D}$. If $\alpha u=u\beta$ for all $0\neq u\in\mathbb{D}$, then $\alpha=\beta\in Z(\mathbb{D})$.*

LEMMA 2.2. *Let $M_n(\mathbb{D})$ be the ring of all $n\times n$ matrices over a division ring $\mathbb{D}$, where $n\geqslant 2$ is an integer. If $a\in M_n(\mathbb{D})$ and $[a,de_{ij}]=0$ for all distinct integers $i,j$ with $1\leqslant i,j\leqslant n$ and $d\in\mathbb{D}$, then $a\in Z(\mathbb{D})I_n$.*

In [34], Liu proved the following result:

PROPOSITION 2.3. ([34, Proposition 2.1]) *Let $M_n(\mathbb{D})$ be the ring of all $n\times n$ matrices over a division ring $\mathbb{D}$, where $n\geqslant 3$ is an integer and let $\mathscr{S}$ be a subset of $M_n(\mathbb{D})$. Suppose that for every $d\in\mathbb{D}$ and every distinct integers $i,j$ with $1\leqslant i,j\leqslant n$, there exists at least one element $\lambda_{ij,d}\in Z(\mathbb{D})$ such that $de_{ij}+\lambda_{ij,d}I_n\in\mathscr{S}$. If $f:\mathscr{S}\to M_n(\mathbb{D})$ is a map satisfying $[f(x),[y,z]]+[f(y),[z,x]]+[f(z),[x,y]]=0$ for all $x,y,z\in\mathscr{S}$, then there exist $\lambda\in Z(\mathbb{D})$ and a map $\mu:\mathscr{S}\to Z(\mathbb{D})I_n$ such that $f(x)=\lambda x+\mu(x)$ for all $x\in\mathscr{S}$.*

As an immediate consequence of Proposition 2.3, we have:

LEMMA 2.4. *Let $M_n(\mathbb{D})$ be the ring of all $n\times n$ matrices over a division ring $\mathbb{D}$, where $n\geqslant 3$ is an integer and let $\mathscr{S}$ be a subset of $M_n(\mathbb{D})$ containing all rank-1 matrices in $M_n(\mathbb{D})$. If $f:\mathscr{S}\to M_n(\mathbb{D})$ is a map satisfying $[f(x),[y,z]]+[f(y),[z,x]]+[f(z),[x,y]]=0$ for all $x,y,z\in\mathscr{S}$, then there exist $\lambda\in Z(\mathbb{D})$ and a map $\mu:\mathscr{S}\to Z(\mathbb{D})I_n$ such that $f(x)=\lambda x+\mu(x)$ for all $x\in\mathscr{S}$.*

*Proof.* Let $\mathscr{S}'=\mathscr{S}\cup\{0\}$. Clearly, for every $d\in\mathbb{D}$ and every distinct integers $i,j$ with $1\leqslant i,j\leqslant n$, $de_{ij}+0I_n\in\mathscr{S}'$. Let $g:\mathscr{S}'\to M_n(\mathbb{D})$ be the map defined by the rules as follows: (a) $g(s)=f(s)$ if $s\in\mathscr{S}\setminus\{0\}$; (b) $g(0)=f(0)$ if $0\in\mathscr{S}$ and $g(0)=0$

if $0 \notin \mathscr{S}$. Then by assumption, we have $[g(x),[y,z]]+[g(y),[z,x]]+[g(z),[x,y]]=0$ for all $x,y,z \in \mathscr{S}'$. By Proposition 2.3, there exist $\lambda \in Z(\mathbb{D})$ and a map $\mu : \mathscr{S}' \to Z(\mathbb{D})I_n$ such that $g(x)=\lambda x+\mu(x)$ for all $x \in \mathscr{S}'$. Hence $f(x)=\lambda x+\mu(x)$ for all $x \in \mathscr{S}$, as desired. $\square$

PROPOSITION 2.5. *Let $M_n(\mathbb{D})$ be the ring of all $n\times n$ matrices over a division ring $\mathbb{D}$, where $n \geqslant 3$ is an integer and let $\mathscr{S}$ be a subset of $M_n(\mathbb{D})$ containing all rank-$k$ matrices in $M_n(\mathbb{D})$, where $k$ is an integer such that $1 \leqslant k \leqslant n$. If $f : \mathscr{S} \to M_n(\mathbb{D})$ is a map satisfying*

$$[f(x),[y,z]]+[f(y),[z,x]]+[f(z),[x,y]]=0 \tag{2.1}$$

*for all $x,y,z \in \mathscr{S}$, then there exist $\lambda \in Z(\mathbb{D})$ and a map $\mu : \mathscr{S} \to Z(\mathbb{D})I_n$ such that $f(x)=\lambda x+\mu(x)$ for all $x \in \mathscr{S}$.*

*Proof.* Note that if $k=n$, then for every $d \in \mathbb{D}$ and every distinct integers $i,j$ with $1 \leqslant i,j \leqslant n$, $de_{ij}+I_n \in \mathscr{S}$ and hence we are done by Proposition 2.3. Clearly, when $k=1$, the proposition follows directly from Lemma 2.4. So from now on we assume $2 \leqslant k \leqslant n-1$. For simplicity, if $U=\{s_1,\dots,s_t\} \subseteq \{1,2,\dots,n\}$, then we let $\sum_{\ell\in U} e_{\ell\ell}$ denote the matrix $\sum_{i=1}^{t} e_{s_is_i}$.

Let $U$ be a set such that $U \subseteq \{1,2,\dots,n\}$ and $|U|=k$. Then $\sum_{\ell\in U} e_{\ell\ell} \in \mathscr{S}$. Fix this $U$ and write $f(\sum_{\ell\in U} e_{\ell\ell})=\sum_{i,j=1}^{n}\alpha_{ij}e_{ij}$, where $\alpha_{ij} \in \mathbb{D}$. Let $i,j,t$ be three distinct integers such that $1 \leqslant i,j,t \leqslant n$, $i,j \in U$ and $t \notin U$. Let $V=U\setminus\{i,j\}$. Note that $i,j,t \notin V$. Then $\sum_{\ell\in V} e_{\ell\ell}+u(e_{ij}+e_{ji}), \sum_{\ell\in V} e_{\ell\ell}+e_{ii}+e_{tt} \in \mathscr{S}$ for all $0 \neq u \in \mathbb{D}$. Setting $x=\sum_{\ell\in U} e_{\ell\ell}$, $y=\sum_{\ell\in V} e_{\ell\ell}+u(e_{ij}+e_{ji})$, $z=\sum_{\ell\in V} e_{\ell\ell}+e_{ii}+e_{tt}$ in (2.1), where $0 \neq u \in \mathbb{D}$, since $[y,z]=-u(e_{ij}-e_{ji})$, $[z,x]=0$ and $[x,y]=0$, we obtain

$$0=[f(\sum_{\ell\in U} e_{\ell\ell}),-u(e_{ij}-e_{ji})]=-f(\sum_{\ell\in U} e_{\ell\ell})u(e_{ij}-e_{ji})+u(e_{ij}-e_{ji})f(\sum_{\ell\in U} e_{\ell\ell}) \tag{2.2}$$

for all $0 \neq u \in \mathbb{D}$. Multiplying (2.2) by $e_{ii}$ from the left and by $e_{jj}$ from the right, we obtain $-e_{ii}f(\sum_{\ell\in U} e_{\ell\ell})ue_{ij}+ue_{ij}f(\sum_{\ell\in U} e_{\ell\ell})e_{jj}=0$. This implies $\alpha_{ii}u=u\alpha_{jj}$ for all $0 \neq u \in \mathbb{D}$. By Lemma 2.1, $\alpha_{ii}=\alpha_{jj} \in Z(\mathbb{D})$. Let $s$ be an integer with $s \neq i,j$ and $1 \leqslant s \leqslant n$. Multiplying (2.2) by $e_{ss}$ from the left and by $e_{jj}$ from the right, we obtain $-e_{ss}f(\sum_{\ell\in U} e_{\ell\ell})ue_{ij}=0$. This implies $\alpha_{si}u=0$ for all $0 \neq u \in \mathbb{D}$. Thus $\alpha_{si}=0$. Similarly, multiplying (2.2) by $e_{ss}$ from the right and by $e_{jj}$ from the left, we obtain $\alpha_{is}=0$. Since $i,j$ can be chosen arbitrary from $U$ and $s$ can be chosen arbitrary from $U\setminus\{i,j\}$, now we have

$$\alpha_{ii}=\alpha_{jj} \in Z(\mathbb{D}) \text{ for all } i,j \in U \tag{2.3}$$

and

$$\alpha_{si}=\alpha_{is}=0 \text{ for all } i \in U \text{ and } s \notin U \text{ with } 1 \leqslant s \leqslant n. \tag{2.4}$$

Let $i,t$ be two distinct integers such that $1 \leqslant i,t \leqslant n$, $i \in U$ and $t \notin U$. Let $V=(U\setminus\{i\})\cup\{t\}$. Then $\sum_{\ell\in V} e_{\ell\ell}=(\sum_{\ell\in U\setminus\{i\}} e_{\ell\ell})+e_{tt}, (\sum_{\ell\in U\setminus\{i\}} e_{\ell\ell})+ue_{it} \in \mathscr{S}$ for all $0 \neq u \in \mathbb{D}$. Setting $x=\sum_{\ell\in U} e_{\ell\ell}$, $y=\sum_{\ell\in V} e_{\ell\ell}$, $z=(\sum_{\ell\in U\setminus\{i\}} e_{\ell\ell})+ue_{it}$ in (2.1), where $0 \neq u \in \mathbb{D}$, since $[y,z] = -ue_{it}$, $[z,x] = -ue_{it}$ and $[x,y] = 0$, we obtain

$$\begin{aligned}
0 &= [f(\sum_{\ell\in U} e_{\ell\ell}), -ue_{it}] + [f(\sum_{\ell\in V} e_{\ell\ell}), -ue_{it}] \\
&= -f(\sum_{\ell\in U} e_{\ell\ell})ue_{it} + ue_{it}f(\sum_{\ell\in U} e_{\ell\ell}) - f(\sum_{\ell\in V} e_{\ell\ell})ue_{it} + ue_{it}f(\sum_{\ell\in V} e_{\ell\ell})
\end{aligned} \tag{2.5}$$

for all $0 \neq u \in \mathbb{D}$. Write $f(\sum_{\ell\in V} e_{\ell\ell}) = \sum_{i,j=1}^{n} \beta_{ij}e_{ij}$, where $\beta_{ij} \in \mathbb{D}$. In view of (2.3) and (2.4), we have

$$\beta_{tt} = \beta_{\ell\ell} \in Z(\mathbb{D}) \ \text{ for all } \ \ell \in V \tag{2.6}$$

as $t \in V$ and

$$\beta_{\ell i} = \beta_{i\ell} = 0 \ \text{ for all } \ \ell \in V \tag{2.7}$$

as $i \notin V$. Let $j \in U$ and $j \neq i$. Multiplying (2.5) by $e_{jj}$ from the left and by $e_{tt}$ from the right, we obtain $-e_{jj}f(\sum_{\ell\in U} e_{\ell\ell})ue_{it} - e_{jj}f(\sum_{\ell\in V} e_{\ell\ell})ue_{it} = 0$. This implies $-\alpha_{ji}u - \beta_{ji}u = 0$ for all $0 \neq u \in \mathbb{D}$. By (2.7) $\beta_{ji} = 0$ as $j \in V$ and $i \notin V$. Thus $\alpha_{ji} = 0$. Since $i, j$ can be chosen arbitrary from $U$, we obtain

$$\alpha_{pq} = 0 \ \text{ for all } \ p, q \in U \ \text{ with } \ p \neq q. \tag{2.8}$$

Next multiplying (2.5) by $e_{ii}$ from the left and by $e_{tt}$ from the right, we obtain $-e_{ii}f(\sum_{\ell\in U} e_{\ell\ell})ue_{it} + ue_{it}f(\sum_{\ell\in U} e_{\ell\ell})e_{tt} - e_{ii}f(\sum_{\ell\in V} e_{\ell\ell})ue_{it} + ue_{it}f(\sum_{\ell\in V} e_{\ell\ell})e_{tt} = 0$. This implies $-\alpha_{ii}u + u\alpha_{tt} - \beta_{ii}u + u\beta_{tt} = 0$ for all $0 \neq u \in \mathbb{D}$. By (2.3) and (2.6), $\alpha_{ii}, \beta_{tt} \in Z(\mathbb{D})$. Thus $(\beta_{tt} - \beta_{ii})u = u(\alpha_{ii} - \alpha_{tt})$ for all $0 \neq u \in \mathbb{D}$. So by Lemma 2.1

$$\beta_{tt} - \beta_{ii} = \alpha_{ii} - \alpha_{tt} \in Z(\mathbb{D}). \tag{2.9}$$

Moreover,

$$\alpha_{tt}, \beta_{ii} \in Z(\mathbb{D}) \tag{2.10}$$

as $\alpha_{ii}, \beta_{tt} \in Z(\mathbb{D})$. Let $p$ be an integer such that $p \notin U$, $p \neq t$ and $1 \leqslant p \leqslant n$ and let $W = (U \setminus \{i\}) \cup \{p\}$. Note that $p, t \notin U$, $t \notin W$ and $p \notin V = (U \setminus \{i\}) \cup \{t\}$. Then $(\sum_{\ell\in W} e_{\ell\ell}) + e_{tp} \in \mathscr{S}$. Setting $x = \sum_{\ell\in U} e_{\ell\ell}$, $y = \sum_{\ell\in V} e_{\ell\ell}$, $z = (\sum_{\ell\in W} e_{\ell\ell}) + e_{tp}$ in (2.1), since $[y,z] = e_{tp}$, $[z,x] = 0$ and $[x,y] = 0$, we obtain

$$0 = [f(\sum_{\ell\in U} e_{\ell\ell}), e_{tp}] = f(\sum_{\ell\in U} e_{\ell\ell})e_{tp} - e_{tp}f(\sum_{\ell\in U} e_{\ell\ell}). \tag{2.11}$$

Now multiplying (2.11) by $e_{pp}$ on both sides, we obtain $e_{pp}f(\sum_{\ell\in U} e_{\ell\ell})e_{tp} = 0$. This implies $\alpha_{pt} = 0$. Since $p, t$ can be chosen arbitrary from $\{1, 2, \ldots, n\} \setminus U$ with $p \neq t$, we have

$$\alpha_{pq} = 0 \ \text{ for all } \ p, q \notin U, \ \ p \neq q \ \text{ and } \ 1 \leqslant p, q \leqslant n. \tag{2.12}$$

Next multiplying (2.11) by $e_{tt}$ from the left and by $e_{pp}$ from the right, we obtain $e_{tt}f(\sum_{\ell\in U} e_{\ell\ell})e_{tp} - e_{tp}f(\sum_{\ell\in U} e_{\ell\ell})e_{pp} = 0$. This implies $\alpha_{tt} = \alpha_{pp}$. Since $p$ can be chosen arbitrary from $\{1, 2, \ldots, n\} \setminus (U \cup \{t\})$, we get

$$\alpha_{tt} = \alpha_{pp} \ \text{ for all } \ p \notin U, \ \ p \neq t \ \text{ and } \ 1 \leqslant p \leqslant n. \tag{2.13}$$

In view of (2.4), (2.8) and (2.12), $f(\sum_{\ell\in U} e_{\ell\ell})$ is a diagonal matrix. Moreover, using (2.3) and (2.13), we obtain

$$\begin{aligned} f(\sum_{\ell\in U} e_{\ell\ell}) &= \sum_{\ell\in U} \alpha_{\ell\ell}e_{\ell\ell} + \sum_{\ell\in\{1,2,\ldots,n\}\setminus U} \alpha_{\ell\ell}e_{\ell\ell} \\ &= \alpha_{ii}\sum_{\ell\in U} e_{\ell\ell} + \alpha_{tt}\sum_{\ell\in\{1,2,\ldots,n\}\setminus U} e_{\ell\ell} \\ &= (\alpha_{ii}-\alpha_{tt})\sum_{\ell\in U} e_{\ell\ell} + \alpha_{tt}I_n. \end{aligned} \tag{2.14}$$

Recall that $\alpha_{ii},\alpha_{tt}\in Z(\mathbb{D})$ by (2.3) and (2.10). Let $\lambda=\alpha_{ii}-\alpha_{tt}\in Z(\mathbb{D})$. From (2.14), it follows that $f(\sum_{\ell\in U} e_{\ell\ell})-\lambda(\sum_{\ell\in U} e_{\ell\ell})\in Z(\mathbb{D})I_n$. By symmetry, $f(\sum_{\ell\in V} e_{\ell\ell})=(\beta_{tt}-\beta_{ii})\sum_{\ell\in V} e_{\ell\ell}+\beta_{ii}I_n$ and $\beta_{ii},\beta_{tt}\in Z(\mathbb{D})$. In view of (2.9), similarly, we obtain $f(\sum_{\ell\in V} e_{\ell\ell}) -\lambda(\sum_{\ell\in V} e_{\ell\ell})\in Z(\mathbb{D})I_n$. Now we conclude that there exists $\lambda\in Z(\mathbb{D})$ such that for every set $U$ with $U\subseteq\{1,2,\ldots,n\}$ and $|U|=k$,

$$f(\sum_{\ell\in U} e_{\ell\ell})-\lambda(\sum_{\ell\in U} e_{\ell\ell})\in Z(\mathbb{D})I_n. \tag{2.15}$$

Let $U$ be a set such that $U\subseteq\{1,2,\ldots,n\}$ with $|U|=k$, let $i,j\in U$ with $i\neq j$ and let $u\in\mathbb{D}$. Clearly, $\sum_{\ell\in U} e_{\ell\ell}+ue_{ij}\in\mathscr{S}$. Fix these $U$, $i,j$ and $u$. Write $f(\sum_{\ell\in U} e_{\ell\ell}+ue_{ij})=\sum_{i,j=1}^n \alpha_{ij}e_{ij}$, where $\alpha_{ij}\in\mathbb{D}$. Let $t$ be an integer such that $t\notin U$, $1\leqslant t\leqslant n$ and let $V$ be a set such that $V\subseteq\{1,2,\ldots,n\}$, $|V|=k$, $i,t\in V$ and $j\notin V$. Then $\sum_{\ell\in U} e_{\ell\ell},\sum_{\ell\in U} e_{\ell\ell}+ue_{ij},\sum_{\ell\in V} e_{\ell\ell}+ve_{ti}\in\mathscr{S}$ for $v\in\mathbb{D}$. Setting $x=\sum_{\ell\in U} e_{\ell\ell}+ue_{ij}$, $y=\sum_{\ell\in U} e_{\ell\ell}$, $z=\sum_{\ell\in V} e_{\ell\ell}+ve_{ti}$ in (2.1), where $v\in\mathbb{D}$, since $[y,z]=-ve_{ti}$, $[z,x]=ue_{ij}+ve_{ti}+vue_{tj}$, $[x,y]=0$ and by (2.15) $f(y)=\lambda y+\gamma I_n$ for some $\gamma\in Z(\mathbb{D})$, we have

$$\begin{aligned} 0 &= [f(\sum_{\ell\in U} e_{\ell\ell}+ue_{ij}),-ve_{ti}]+[f(\sum_{\ell\in U} e_{\ell\ell}),ue_{ij}+ve_{ti}+vue_{tj}] \\ &= [f(\sum_{\ell\in U} e_{\ell\ell}+ue_{ij}),-ve_{ti}]+[\lambda(\sum_{\ell\in U} e_{\ell\ell})+\gamma I_n,ue_{ij}+ve_{ti}+vue_{tj}] \\ &= -f(\sum_{\ell\in U} e_{\ell\ell}+ue_{ij})ve_{ti}+ve_{ti}f(\sum_{\ell\in U} e_{\ell\ell}+ue_{ij}) \\ &\quad +\lambda(\sum_{\ell\in U} e_{\ell\ell})(ue_{ij}+ve_{ti}+vue_{tj})-(ue_{ij}+ve_{ti}+vue_{tj})\lambda(\sum_{\ell\in U} e_{\ell\ell}) \\ &= -f(\sum_{\ell\in U} e_{\ell\ell}+ue_{ij})ve_{ti}+ve_{ti}f(\sum_{\ell\in U} e_{\ell\ell}+ue_{ij})-\lambda ve_{ti}-\lambda vue_{tj} \end{aligned} \tag{2.16}$$

for all $v\in\mathbb{D}$. Let $s$ be an integer such that $s\neq t$ and $1\leqslant s\leqslant n$. Multiplying (2.16) by $e_{ss}$ from the left, we obtain $-e_{ss}f(\sum_{\ell\in U} e_{\ell\ell}+ue_{ij})ve_{ti}=0$. Thus $\alpha_{st}=0$. Since $t$ can be chosen arbitrary from $\{1,2,\ldots,n\}\setminus U$, we get

$$\alpha_{pq}=0 \text{ for all } p\neq q,\ q\notin U \text{ and } 1\leqslant p,q\leqslant n. \tag{2.17}$$

Now multiplying (2.16) by $e_{tt}$ from the left and by $e_{jj}$ from the right, we obtain $ve_{ti}f(\sum_{\ell\in U} e_{\ell\ell}+ue_{ij})e_{jj}-\lambda vue_{tj}=0$. Thus $v\alpha_{ij}-\lambda vu=0$ for all $v\in\mathbb{D}$, implying

$$\alpha_{ij}=\lambda u. \tag{2.18}$$

Next multiplying (2.16) by $e_{tt}$ from the left and by $e_{ii}$ from the right, we obtain $-e_{tt}f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij})ve_{ti}+ve_{ti}f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij})e_{ii}-\lambda ve_{ti}=0$. Thus $(\alpha_{tt}+\lambda)v=v\alpha_{ii}$ for all $v\in\mathbb{D}$. By Lemma 2.1, $\alpha_{ii}=\alpha_{tt}+\lambda\in Z(\mathbb{D})$. In particular, $\alpha_{tt}\in Z(\mathbb{D})$ as $\lambda\in Z(\mathbb{D})$. Since $t$ can be chosen arbitrary from $\{1,2,\ldots,n\}\setminus U$, we have

$$\alpha_{pp}=\alpha_{ii}-\lambda\in Z(\mathbb{D})\ \text{ for all }\ p\notin U\ \text{ and }\ 1\leqslant p\leqslant n. \tag{2.19}$$

Let $t$ be an integer such that $t\notin U$, $1\leqslant t\leqslant n$ and let $V'$ be a set such that $V'\subseteq\{1,2,\ldots,n\}$, $|V'|=k$, $j,t\in V'$ and $i\notin V'$. Then $\sum_{\ell\in V'}e_{\ell\ell}+ve_{jt}\in\mathscr{S}$ for $v\in\mathbb{D}$. Setting $x=\sum_{\ell\in U}e_{\ell\ell}+ue_{ij}$, $y=\sum_{\ell\in U}e_{\ell\ell}$, $z=\sum_{\ell\in V'}e_{\ell\ell}+ve_{jt}$ in (2.1), where $v\in\mathbb{D}$, since $[y,z]=ve_{jt}$, $[z,x]=-ue_{ij}-ve_{jt}-uve_{it}$, $[x,y]=0$ and by (2.15) $f(y)=\lambda y+\gamma I_n$ for some $\gamma\in Z(\mathbb{D})$, we have

$$\begin{aligned}
0&=[f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij}),ve_{jt}]+[f(\sum_{\ell\in U}e_{\ell\ell}),-ue_{ij}-ve_{jt}-uve_{it}]\\
&=[f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij}),ve_{jt}]+[\lambda(\sum_{\ell\in U}e_{\ell\ell})+\gamma I_n,-ue_{ij}-ve_{jt}-uve_{it}]\\
&=f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij})ve_{jt}-ve_{jt}f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij})\\
&\quad+\lambda(\sum_{\ell\in U}e_{\ell\ell})(-ue_{ij}-ve_{jt}-uve_{it})-(-ue_{ij}-ve_{jt}-uve_{it})\lambda(\sum_{\ell\in U}e_{\ell\ell})\\
&=f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij})ve_{jt}-ve_{jt}f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij})-\lambda ve_{jt}-\lambda uve_{it}
\end{aligned}\tag{2.20}$$

for all $v\in\mathbb{D}$. Multiplying (2.20) by $e_{jj}$ from the left and by $e_{tt}$ from the right, we obtain $e_{jj}f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij})ve_{jt}-ve_{jt}f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij})e_{tt}-\lambda ve_{jt}=0$. Thus $\alpha_{jj}v=v(\alpha_{tt}+\lambda)$ for all $v\in\mathbb{D}$. By Lemma 2.1, $\alpha_{jj}=\alpha_{tt}+\lambda\in Z(\mathbb{D})$ and hence by (2.19)

$$\alpha_{ii}=\alpha_{jj}\in Z(\mathbb{D}). \tag{2.21}$$

Let $s$ be an integer such that $s\neq t$ and $1\leqslant s\leqslant t$. Multiplying (2.20) by $e_{ss}$ from the right, we obtain $-ve_{jt}f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij})e_{ss}=0$. Thus $\alpha_{ts}=0$. Since $t$ can be chosen arbitrary from $\{1,2,\ldots,n\}\setminus U$, we get

$$\alpha_{pq}=0\ \text{ for all }\ p\neq q,\ \ p\notin U\ \text{ and }\ 1\leqslant p,q\leqslant n. \tag{2.22}$$

Let $p$ be an integer such that $p\in U$ and $p\neq i,j$ and let $W$ be a set such that $W\subseteq\{1,2,\ldots,n\}$ with $|W|=k$, $i,j\in W$ and $p\notin W$. Then $\sum_{\ell\in W}e_{\ell\ell},\sum_{\ell\in W}e_{\ell\ell}+e_{ip}\in\mathscr{S}$. Setting $x=\sum_{\ell\in U}e_{\ell\ell}+ue_{ij}$, $y=\sum_{\ell\in W}e_{\ell\ell}$, $z=\sum_{\ell\in W}e_{\ell\ell}+e_{ip}$ in (2.1), since $[y,z]=e_{ip}$, $[z,x]=0$ and $[x,y]=0$, we obtain

$$0=[f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij}),e_{ip}]=f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij})e_{ip}-e_{ip}f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij}). \tag{2.23}$$

Now multiplying (2.23) by $e_{ii}$ from the left and by $e_{pp}$ from the right, we obtain $e_{ii}f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij})e_{ip}-e_{ip}f(\sum_{\ell\in U}e_{\ell\ell}+ue_{ij})e_{pp}=0$. Thus $\alpha_{ii}-\alpha_{pp}=0$. This implies

$$\alpha_{pp}=\alpha_{ii}\ \text{ for all }\ p\in U\ \text{ with }\ p\neq i,j. \tag{2.24}$$

Let $q$ be an integer such that $q \neq p$ and $1 \leqslant q \leqslant n$. Multiplying (2.23) by $e_{qq}$ from the right, we get $-e_{ip}f(\sum_{\ell\in U} e_{\ell\ell} + ue_{ij})e_{qq} = 0$. Thus $\alpha_{pq} = 0$. Since $p$ can be chosen arbitrary from $p \in U \setminus \{i,j\}$, we have

$$\alpha_{pq} = 0 \text{ for all } p \neq q, \ p \in U \text{ with } p \neq i,j \text{ and } 1 \leqslant q \leqslant n. \tag{2.25}$$

Multiplying (2.23) by $e_{jj}$ from the left, we get $e_{jj}f(\sum_{\ell\in U} e_{\ell\ell} + ue_{ij})e_{ip} = 0$. Thus

$$\alpha_{ji} = 0. \tag{2.26}$$

Clearly, $\sum_{\ell\in W} e_{\ell\ell} + e_{pj} \in \mathscr{S}$. Setting $x = \sum_{\ell\in U} e_{\ell\ell} + ue_{ij}$, $y = \sum_{\ell\in W} e_{\ell\ell}$, $z = \sum_{\ell\in W} e_{\ell\ell} + e_{pj}$ in (2.1), since $[y,z] = -e_{pj}$, $[z,x] = 0$ and $[x,y] = 0$, we obtain

$$0 = [f(\sum_{\ell\in U} e_{\ell\ell} + ue_{ij}), -e_{pj}] = -f(\sum_{\ell\in U} e_{\ell\ell} + ue_{ij})e_{pj} + e_{pj}f(\sum_{\ell\in U} e_{\ell\ell} + ue_{ij}). \tag{2.27}$$

Let $q$ be an integer such that $q \neq p$ and $1 \leqslant q \leqslant n$. Multiplying (2.27) by $e_{qq}$ from the left, we get $-e_{qq}f(\sum_{\ell\in U} e_{\ell\ell} + ue_{ij})e_{pj} = 0$. Thus $\alpha_{qp} = 0$. Since $p$ can be chosen arbitrary from $p \in U \setminus \{i,j\}$, we have

$$\alpha_{qp} = 0 \text{ for all } p \neq q, \ p \in U \text{ with } p \neq i,j \text{ and } 1 \leqslant q \leqslant n. \tag{2.28}$$

By (2.17), (2.18), (2.19), (2.21), (2.22), (2.24), (2.25), (2.26) and (2.28), we have

$$\begin{aligned}
f(\sum_{\ell\in U} e_{\ell\ell} + ue_{ij}) &= \lambda ue_{ij} + \sum_{\ell\in U} \alpha_{\ell\ell}e_{\ell\ell} + \sum_{\ell\in\{1,2,\ldots,n\}\setminus U} \alpha_{\ell\ell}e_{\ell\ell} \\
&= \lambda ue_{ij} + \alpha_{ii}\sum_{\ell\in U} e_{\ell\ell} + (\alpha_{ii} - \lambda)\sum_{\ell\in\{1,2,\ldots,n\}\setminus U} e_{\ell\ell} \\
&= \lambda ue_{ij} + \lambda(\sum_{\ell\in U} e_{\ell\ell}) + (\alpha_{ii} - \lambda)I_n.
\end{aligned}$$

Hence we conclude that for every set $U$ with $U \subseteq \{1,2,\ldots,n\}$ and $|U| = k$, for every $i,j \in U$ with $i \neq j$ and for every $u \in \mathbb{D}$,

$$f(\sum_{\ell\in U} e_{\ell\ell} + ue_{ij}) - \lambda(\sum_{\ell\in U} e_{\ell\ell} + ue_{ij}) \in Z(\mathbb{D})I_n.$$

Let $x \in \mathscr{S}$, let $i,j$ be two distinct integers with $1 \leqslant i,j \leqslant n$ and let $u \in \mathbb{D}$. Choose a set $U$ such that $U \subseteq \{1,2,\ldots,n\}$, $|U| = k$ and $i,j \in U$ and choose a set $V$ such that $V \subseteq \{1,2,\ldots,n\}$ such that $|V| = k$, $j \in V$ and $i \notin V$. Clearly, $\sum_{\ell\in U} e_{\ell\ell} + ue_{ij}, \sum_{\ell\in V} e_{\ell\ell} \in \mathscr{S}$. Setting $y = \sum_{\ell\in U} e_{\ell\ell} + ue_{ij}$, $z = \sum_{\ell\in V} e_{\ell\ell}$, in (2.1), since $[y,z] = ue_{ij}$, $[y,[z,x]] + [z,[x,y]] = -[x,[y,z]]$ by Jacobi identity and there exist $\gamma_y, \gamma_z \in Z(\mathbb{D})$ such that $f(y) = \lambda y + \gamma_y I_n$ and $f(z) = \lambda z + \gamma_z I_n$, we obtain

$$\begin{aligned}
0 &= [f(x),[y,z]] + [f(y),[z,x]] + [f(z),[x,y]] \\
&= [f(x),[y,z]] + [\lambda y + \gamma_y I_n,[z,x]] + [\lambda z + \gamma_z I_n,[x,y]] \\
&= [f(x),[y,z]] + [\lambda y,[z,x]] + [\lambda z,[x,y]] \\
&= [f(x),[y,z]] + \lambda([y,[z,x]] + [z,[x,y]]) \\
&= [f(x),[y,z]] - \lambda[x,[y,z]] \\
&= [f(x) - \lambda x,[y,z]] \\
&= [f(x) - \lambda x, ue_{ij}].
\end{aligned}$$

This implies $[f(x)-\lambda x, ue_{ij}]=0$ for all distinct integers $i,j$ with $1\leqslant i,j\leqslant n$ and $u\in\mathbb{D}$. By Lemma 2.2, $f(x)-\lambda x\in Z(\mathbb{D})I_n$ for all $x\in\mathscr{S}$. Let $\mu:\mathscr{S}\to Z(\mathbb{D})I_n$ be the map defined by $\mu(x)=f(x)-\lambda x$ for $x\in\mathscr{S}$. Then $f(x)=\lambda x+\mu(x)$ for all $x\in\mathscr{S}$. This proves the proposition. $\square$

LEMMA 2.6. ([10, p. 239, Theorem A7]) *Let $\mathbb{D}$ be a division ring and $a_i,b_i,c_j,d_j\in\mathbb{D}$. Suppose that $\sum_{i=1}^m a_iub_i+\sum_{j=1}^n c_jud_j=0$ for all $u\in\mathbb{D}$. If $a_1,\ldots,a_m$ are $Z(\mathbb{D})$-independent, then each $b_i$ is a $Z(\mathbb{D})$-linear combination of $d_1,\ldots,d_n$. If $b_1,\ldots,b_m$ are $Z(\mathbb{D})$-independent, then each $a_i$ is a $Z(\mathbb{D})$-linear combination of $c_1,\ldots,c_n$.*

PROPOSITION 2.7. *Let $M_2(\mathbb{D})$ be the ring of all $2\times 2$ matrices over a non-commutative division ring $\mathbb{D}$ and let $\mathscr{S}$ be a subset of $M_2(\mathbb{D})$ such that for every $d\in\mathbb{D}$ and every distinct integers $i,j$ with $1\leqslant i,j\leqslant 2$, there exists at least one element $\lambda_{ij,d}\in Z(\mathbb{D})$ such that $de_{ij}+\lambda_{ij,d}I_2\in\mathscr{S}$ and for every $d\in\mathbb{D}\setminus Z(\mathbb{D})$ and every integer $i$ with $1\leqslant i\leqslant 2$, there exists at least one element $\lambda_{ii,d}\in Z(\mathbb{D})$ such that $de_{ii}+\lambda_{ii,d}I_2\in\mathscr{S}$. If $f:\mathscr{S}\to M_2(\mathbb{D})$ is a map satisfying $[f(x),[y,z]]+[f(y),[z,x]]+[f(z),[x,y]]=0$ for all $x,y,z\in\mathscr{S}$, then there exist $\lambda\in Z(\mathbb{D})$ and a map $\mu:\mathscr{S}\to Z(\mathbb{D})I_2$ such that $f(x)=\lambda x+\mu(x)$ for all $x\in\mathscr{S}$.*

*Proof.* For every $d\in\mathbb{D}$ and every distinct integers $i,j$ with $1\leqslant i,j\leqslant 2$, we choose and fix $\lambda_{ij,d}\in Z(\mathbb{D})$ such that $de_{ij}+\lambda_{ij,d}I_2\in\mathscr{S}$. Also, for every $d\in\mathbb{D}\setminus Z(\mathbb{D})$ and every integer $i$ with $1\leqslant i\leqslant 2$, we choose and fix $\lambda_{ii,d}\in Z(\mathbb{D})$ such that $de_{ii}+\lambda_{ii,d}I_2\in\mathscr{S}$. Let $a_{st},b_{st}:\mathbb{D}\to\mathbb{D}$ and $c_{st}:\mathbb{D}\setminus Z(\mathbb{D})\to\mathbb{D}$ be maps such that $f(ue_{12}+\lambda_{12,u}I_2)=\begin{pmatrix}a_{11}(u) & a_{12}(u)\\ a_{21}(u) & a_{22}(u)\end{pmatrix}$, $f(ve_{21}+\lambda_{21,v}I_2)=\begin{pmatrix}b_{11}(v) & b_{12}(v)\\ b_{21}(v) & b_{22}(v)\end{pmatrix}$ and $f(we_{11}+\lambda_{11,w}I_2)=\begin{pmatrix}c_{11}(w) & c_{12}(w)\\ c_{21}(w) & c_{22}(w)\end{pmatrix}$ for $u,v\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$.

Setting $x=we_{11}+\lambda_{11,w}I_2$, $y=ue_{12}+\lambda_{12,u}I_2$, $z=ve_{21}+\lambda_{21,v}I_2$ in (2.1), where $u,v\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$, since $[y,z]=uve_{11}-vue_{22}$, $[z,x]=vwe_{21}$ and $[x,y]=wue_{12}$, we obtain

$$\begin{aligned}0=&\left[\begin{pmatrix}c_{11}(w) & c_{12}(w)\\ c_{21}(w) & c_{22}(w)\end{pmatrix},\begin{pmatrix}uv & 0\\ 0 & -vu\end{pmatrix}\right]+\left[\begin{pmatrix}a_{11}(u) & a_{12}(u)\\ a_{21}(u) & a_{22}(u)\end{pmatrix},\begin{pmatrix}0 & 0\\ vw & 0\end{pmatrix}\right]\\ &+\left[\begin{pmatrix}b_{11}(v) & b_{12}(v)\\ b_{21}(v) & b_{22}(v)\end{pmatrix},\begin{pmatrix}0 & wu\\ 0 & 0\end{pmatrix}\right]\end{aligned}$$

for all $u,v\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$. In view of corresponding entries, we obtain that

$$a_{12}(u)vw-wub_{21}(v)+c_{11}(w)uv-uvc_{11}(w)=0, \tag{2.29}$$

$$-c_{12}(w)vu-uvc_{12}(w)+b_{11}(v)wu-wub_{22}(v)=0, \tag{2.30}$$

$$a_{22}(u)vw-vwa_{11}(u)+c_{21}(w)uv+vuc_{21}(w)=0, \tag{2.31}$$

and

$$b_{21}(v)wu-vwa_{12}(u)-c_{22}(w)vu+vuc_{22}(w)=0 \tag{2.32}$$

for all $u,v\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Replacing $w$ with $w+1$ in (2.29) for $w\in\mathbb{D}\setminus Z(\mathbb{D})$ and using (2.29), we obtain $a_{12}(u)v-ub_{21}(v)+(c_{11}(w+1)-c_{11}(w))uv-uv(c_{11}(w+$

$1)-c_{11}(w))=0$ for all $u,v\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Let $w_0\in\mathbb{D}\setminus Z(\mathbb{D})$ and let $\alpha=c_{11}(w_0+1)-c_{11}(w_0)$. Then we have $(a_{12}(u)+\alpha u)v-u(b_{21}(v)+v\alpha)=0$ for all $u,v\in\mathbb{D}$. Setting $u=v=1$, we obtain $a_{12}(1)=b_{21}(1)$. Next setting $u=1$, we get

$$b_{21}(v)=(a_{12}(1)+\alpha)v-v\alpha=(b_{21}(1)+\alpha)v-v\alpha \tag{2.33}$$

and setting $v=1$, we get

$$a_{12}(u)=u(b_{21}(1)+\alpha)-\alpha u \tag{2.34}$$

for all $u,v\in\mathbb{D}$. Using (2.34) and letting $v=1$ in (2.29), we have

$$1\cdot u((b_{21}(1)+\alpha)w-c_{11}(w))-w\cdot ub_{21}(1)-\alpha u\cdot w+c_{11}(w)u\cdot 1=0 \tag{2.35}$$

for all $u\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Clearly, $1,w$ are $Z(\mathbb{D})$-independent for every $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Applying Lemma 2.6 to (2.35), we obtain $\alpha\in Z(\mathbb{D})1+Z(\mathbb{D})w$. Thus $[\alpha,w]=0$ for every $w\in\mathbb{D}\setminus Z(\mathbb{D})$ and hence $[\alpha,w]=0$ for all $w\in\mathbb{D}$, implying $\alpha\in Z(\mathbb{D})$. By (2.33) and (2.34), $a_{12}(u)=ub_{21}(1)$ and $b_{21}(v)=b_{21}(1)v$ for all $u,v\in\mathbb{D}$. Similarly, using (2.32), we obtain $a_{12}(u)=b_{21}(1)u$. Thus $a_{12}(u)=ub_{21}(1)=b_{21}(1)u$ for all $u\in\mathbb{D}$. This implies $b_{21}(1)\in Z(\mathbb{D})$. Hence $a_{12}(u)=\lambda u$ and $b_{21}(v)=\lambda v$ for $u,v\in\mathbb{D}$, where $\lambda=b_{21}(1)\in Z(\mathbb{D})$. With these, (2.29) and (2.32) can be reduced to $[c_{11}(w)-\lambda w,uv]=0$ and $[c_{22}(w),vu]=0$ for all $u,v\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Thus $c_{11}(w)=\lambda w+c'_{11}(w)$ and $c_{22}(w)\in Z(\mathbb{D})$ for all $w\in\mathbb{D}\setminus Z(\mathbb{D})$, where $c'_{11}:\mathbb{D}\setminus Z(\mathbb{D})\to Z(\mathbb{D})$ is a map.

Replacing $w$ with $w+1$ in (2.31) for $w\in\mathbb{D}\setminus Z(\mathbb{D})$ and using (2.31), we obtain $a_{22}(u)v-va_{11}(u)+(c_{21}(w+1)-c_{21}(w))uv+vu(c_{21}(w+1)-c_{21}(w))=0$ for all $u,v\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Let $w_0\in\mathbb{D}\setminus Z(\mathbb{D})$ and let $\beta=c_{21}(w_0+1)-c_{21}(w_0)$. Then we have $(a_{22}(u)+\beta u)v-v(a_{11}(u)-u\beta)=0$ for all $u,v\in\mathbb{D}$. By Lemma 2.1, $a_{22}(u)+\beta u=a_{11}(u)-u\beta\in Z(\mathbb{D})$ for all $u\in\mathbb{D}$. Thus there exists a map $\gamma:\mathbb{D}\to Z(\mathbb{D})$ such that $a_{22}(u)+\beta u=a_{11}(u)-u\beta=\gamma(u)$ for all $u\in\mathbb{D}$. Then $a_{22}(u)=-\beta u+\gamma(u)$ and $a_{11}(u)=u\beta+\gamma(u)$ for all $u\in\mathbb{D}$. With these, (2.31) can be reduced to

$$-\beta uv\cdot w+c_{21}(w)uv\cdot 1+1\cdot v(uc_{21}(w)-wu\beta)=0 \tag{2.36}$$

for all $u,v\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Clearly, $1,w$ are $Z(\mathbb{D})$-independent for every $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Applying Lemma 2.6 to (2.36), we obtain $\beta u\in Z(\mathbb{D})1$ and $c_{21}(w)u\in Z(\mathbb{D})1$ for all $u\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Note that $\beta=0$; otherwise $u\in Z(\mathbb{D})\beta^{-1}$ for all $u\in\mathbb{D}$, implying $\mathbb{D}$ is commutative, a contradiction. Similarly, $c_{21}(w)=0$ for all $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Hence $a_{22}(u)=a_{11}(u)=\gamma(u)\in Z(\mathbb{D})$ for all $u\in\mathbb{D}$ and $c_{21}=0$. Similarly, using (2.30), we obtain $b_{11}(v)=b_{22}(v)\in Z(\mathbb{D})$ for all $v\in\mathbb{D}$ and $c_{12}=0$. Now we have $f(ue_{12}+\lambda_{12,u}I_2)=\begin{pmatrix}0 & \lambda u\\ a_{21}(u) & 0\end{pmatrix}+a_{11}(u)I_2$, $f(ve_{21}+\lambda_{21,v}I_2)=\begin{pmatrix}0 & b_{12}(v)\\ \lambda v & 0\end{pmatrix}+b_{11}(v)I_2$, and $f(we_{11}+\lambda_{11,w}I_2)=\begin{pmatrix}\lambda w+\xi(w) & 0\\ 0 & 0\end{pmatrix}+c_{22}(w)I_2$ for $u,v\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$, where $a_{11}(u),b_{11}(v),c_{22}(w),\xi(w)=c'_{11}(w)-c_{22}(w)\in Z(\mathbb{D})$ for all $u,v\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$. By symmetry, setting $x=we_{22}+\lambda_{22,w}I_2$, $y=ue_{12}+\lambda_{12,u}I_2$, $z=ve_{21}+\lambda_{21,v}I_2$ in (2.1), where $u,v\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$, we obtain $f(we_{22}+\lambda_{22,w}I_2)=$

$\begin{pmatrix} 0 & 0 \\ 0 & \lambda w+\zeta(w) \end{pmatrix}+d_{11}(w)I_2$ for $w\in\mathbb{D}\setminus Z(\mathbb{D})$, where $\zeta, d_{22}:\mathbb{D}\setminus Z(\mathbb{D})\to Z(\mathbb{D})$ are maps. Next, setting $x=se_{11}+\lambda_{11,s}I_2$, $y=te_{22}+\lambda_{22,t}I_2$, $z=ue_{12}+\lambda_{12,u}I_2$ in (2.1), where $s,t\in\mathbb{D}\setminus Z(\mathbb{D})$ and $u\in\mathbb{D}$, since $[y,z]=-ute_{12}$, $[z,x]=-sue_{12}$ and $[x,y]=0$, we obtain

$$\begin{aligned} 0&=[f(se_{11}+\lambda_{11,s}I_2),-ute_{12}]+[f(te_{22}+\lambda_{22,t}I_2),-sue_{12}]\\ &=[(\lambda s+\xi(s))e_{11}+c_{22}(s)I_2,-ute_{12}]+[(\lambda t+\zeta(t))e_{22}+d_{11}(w)I_2,-sue_{12}]\\ &=\big(-(\lambda s+\xi(s))ut+su(\lambda t+\zeta(t))\big)e_{12}. \end{aligned}$$

Thus

$$0=-(\lambda s+\xi(s))ut+su(\lambda t+\zeta(t))=-\xi(s)ut+\zeta(t)su \tag{2.37}$$

for all $s,t\in\mathbb{D}\setminus Z(\mathbb{D})$ and $u\in\mathbb{D}$. Let $t_0\in\mathbb{D}\setminus Z(\mathbb{D})$. Thus $1,t_0$ are $Z(\mathbb{D})$-independent. By (2.37), $-\xi(s)u\cdot t_0+\zeta(t_0)su\cdot 1=0$ for all $u\in\mathbb{D}$ and $s\in\mathbb{D}\setminus Z(\mathbb{D})$. So by Lemma 2.6, we have $\xi(s)=0$ for all $s\in\mathbb{D}\setminus Z(\mathbb{D})$. Then (2.37) implies that $\zeta(t)su=0$ for all $s,t\in\mathbb{D}\setminus Z(\mathbb{D})$ and $u\in\mathbb{D}$. Thus $\zeta(t)=0$ for all $t\in\mathbb{D}\setminus Z(\mathbb{D})$. From $\xi=\zeta=0$ it follows that $f(we_{11}+\lambda_{11,w}I_2)=\lambda we_{11}+c_{22}(w)I_2$ and $f(we_{22}+\lambda_{22,w}I_2)=\lambda we_{22}+d_{11}(w)I_2$ for $w\in\mathbb{D}\setminus Z(\mathbb{D})$.

Setting $x=ue_{12}+\lambda_{12,u}I_2$, $y=we_{11}+\lambda_{11,w}I_2$, $z=e_{12}+\lambda_{12,1}I_2$ in (2.1), where $u\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$, since $[y,z]=we_{12}$, $[z,x]=0$ and $[x,y]=-wue_{12}$, we obtain

$$\begin{aligned} 0&=[f(ue_{12}+\lambda_{12,u}I_2),we_{12}]+[f(e_{12}+\lambda_{12,1}I_2),-wue_{12}]\\ &=\left[\begin{pmatrix} 0 & \lambda u\\ a_{21}(u) & 0\end{pmatrix}+a_{11}(u)I_2,\begin{pmatrix} 0 & w\\ 0 & 0\end{pmatrix}\right]+\left[\begin{pmatrix} 0 & \lambda\\ a_{21}(1) & 0\end{pmatrix}+a_{11}(1)I_2,\begin{pmatrix} 0 & -wu\\ 0 & 0\end{pmatrix}\right]\\ &=\begin{pmatrix} * & *\\ * & a_{21}(u)w-a_{21}(1)wu\end{pmatrix}. \end{aligned}$$

Thus $a_{21}(u)w-a_{21}(1)wu=0$ for all $u\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Replacing $w$ with $w+1$, we get $a_{21}(u)=a_{21}(1)u$ and then $0=a_{21}(u)w-a_{21}(1)wu=a_{21}(1)[u,w]$ for all $u\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Clearly, $[u,w]\neq 0$ for some $u\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$. Thus $a_{21}(1)=0$ and hence $a_{21}(u)=0$ for all $u\in\mathbb{D}$, that is, $a_{21}=0$. By symmetry, setting $x=ve_{21}+\lambda_{21,v}I_2$, $y=we_{22}+\lambda_{22,w}I_2$, $z=e_{21}+\lambda_{21,1}I_2$ in (2.1), where $v\in\mathbb{D}$ and $w\in\mathbb{D}\setminus Z(\mathbb{D})$, we obtain $b_{12}=0$. Hence $f(ue_{12}+\lambda_{12,u}I_2)=\lambda ue_{12}+a_{11}(u)I_2$ and $f(ve_{21}+\lambda_{21,v}I_2)=\lambda ve_{21}+b_{11}(v)I_2$ for $u,v\in\mathbb{D}$.

Let $x\in\mathscr{S}$, let $i,j$ be distinct integers with $1\leqslant i,j\leqslant 2$, let $w\in\mathbb{D}\setminus Z(\mathbb{D})$ and let $u\in\mathbb{D}$. Setting $y=we_{ii}+\lambda_{ii,1}I_2$ and $z=w^{-1}ue_{ij}+\lambda_{ij,w^{-1}u}I_2$ in (2.1) and recalling that there exist $\gamma_y,\gamma_z\in Z(\mathbb{D})$ such that $f(y)=\lambda y+\gamma_yI_2$ and $f(z)=\lambda z+\gamma_zI_2$, by the same proof of Proposition 2.5, we obtain $[f(x)-\lambda x,ue_{ij}]=0$. By Lemma 2.2, $f(x)-\lambda x\in Z(\mathbb{D})I_n$ for all $x\in\mathscr{S}$. This proves the proposition. $\square$

COROLLARY 2.8. *Let $M_2(\mathbb{D})$ be the ring of all $2\times 2$ matrices over a noncommutative division ring $\mathbb{D}$ and let $\mathscr{S}$ be a subset of $M_2(\mathbb{D})$ containing all rank-$k$ matrices in $M_2(\mathbb{D})$, where $k$ is an integer such that $1\leqslant k\leqslant 2$. If $f:\mathscr{S}\to M_2(\mathbb{D})$ is a map satisfying $[f(x),[y,z]]+[f(y),[z,x]]+[f(z),[x,y]]=0$ for all $x,y,z\in\mathscr{S}$, then there exist $\lambda\in Z(\mathbb{D})$ and a map $\mu:\mathscr{S}\to Z(\mathbb{D})I_2$ such that $f(x)=\lambda x+\mu(x)$ for all $x\in\mathscr{S}$.*

*Proof.* Let $\mathscr{S}' = \mathscr{S} \cup \{0\}$. Suppose first $k = 1$. Then $de_{ij} + 0I_2 \in \mathscr{S}'$ for every $d \in \mathbb{D}$ and every integers $i, j$ with $1 \leqslant i, j \leqslant 2$. Suppose next $k = 2$. Then $de_{ij} + I_2 \in \mathscr{S}'$ for every $d \in \mathbb{D}$ and every distinct integers $i, j$ with $1 \leqslant i, j \leqslant 2$ and $de_{ii} + I_2 \in \mathscr{S}'$ for every $d \in \mathbb{D} \setminus Z(\mathbb{D})$ and every integer $i$ with $1 \leqslant i \leqslant 2$. Using Proposition 2.7 and by a similar proof of Lemma 2.4, we are done. $\square$

The conclusion of Corollary 2.8 is false if $\mathbb{D}$ is commutative.

EXAMPLE. Let $\mathbb{K}$ be a field and let $f : M_2(\mathbb{K}) \to M_2(\mathbb{K})$ be the $\mathbb{K}$-linear map defined by $f(e_{11}) = e_{11} + e_{12}$, $f(e_{22}) = -e_{11} - e_{12}$, $f(e_{12}) = e_{12}$ and $f(e_{21}) = e_{11} - e_{22} + e_{21}$. Then $[f(x), [y, z]] + [f(y), [z, x]] + [f(z), [x, y]] = 0$ for all $x, y, z \in M_2(\mathbb{K})$ and hence for all rank-$k$ matrices $x, y, z \in M_2(\mathbb{K})$, where $k$ is an integer such that $1 \leqslant k \leqslant 2$. However, $f$ is not of the form described in Corollary 2.8 as $f(e_{11}) - \lambda e_{11} = (1-\lambda)e_{11} + e_{12} \notin \mathbb{K}I_2$ and $f(e_{11} + e_{12} + e_{21}) - \lambda(e_{11} + e_{12} + e_{21}) = (2-\lambda)e_{11} - e_{22} + (2-\lambda)e_{12} + (1-\lambda)e_{21} \notin \mathbb{K}I_2$ for all $\lambda \in \mathbb{K}$.

## 3. Proofs of main results

LEMMA 3.1. *Let $M_2(\mathbb{D})$ be the ring of all $2 \times 2$ matrices over a division ring $\mathbb{D}$ and let $\mathscr{S}$ be a subset of $M_2(\mathbb{D})$ containing all rank-$k$ matrices in $M_2(\mathbb{D})$, where $k$ is an integer such that $1 \leqslant k \leqslant 2$. If $f : \mathscr{S} \to M_2(\mathbb{D})$ is a map satisfying $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathscr{S}$, then there exists a map $\mu : \mathscr{S} \to Z(\mathbb{D})I_2$ such that $f(x) = x + \mu(x)$ for all $x \in \mathscr{S}$ or $f(x) = -x + \mu(x)$ for all $x \in \mathscr{S}$.*

*Proof.* If $k = 2$, then $\mathscr{S}$ contains all invertible matrices in $M_2(\mathbb{D})$ and hence we are done by Theorem L. Suppose now $k = 1$. Let $\mathscr{S}' = \mathscr{S} \cup \{0\}$ and let $g : \mathscr{S}' \to M_2(\mathbb{D})$ be the map defined by the rules as follows: (a) $g(s) = f(s)$ if $s \in \mathscr{S} \setminus \{0\}$; (b) $g(0) = f(0)$ if $0 \in \mathscr{S}$ and $g(0) = 0$ if $0 \notin \mathscr{S}$. Then by assumption, we have $[g(x), g(y)] = [x, y]$ for all $x, y \in \mathscr{S}'$. Clearly, $\mathscr{S}'$ contains all singular matrices in $M_2(\mathbb{D})$. By Theorem L, there exists a map $\mu : \mathscr{S}' \to Z(\mathbb{D})I_2$ such that $g(x) = x + \mu(x)$ for all $x \in \mathscr{S}'$ or $g(x) = -x + \mu(x)$ for all $x \in \mathscr{S}'$. So $f(x) = x + \mu(x)$ for all $x \in \mathscr{S}$ or $f(x) = -x + \mu(x)$ for all $x \in \mathscr{S}$. This proves the lemma. $\square$

LEMMA 3.2. *Let $M_n(\mathbb{D})$ be the ring of all $n \times n$ matrices over a division ring $\mathbb{D}$, where $n \geqslant 2$ is an integer and let $k$ be an integer such that $1 \leqslant k \leqslant n$. Then for every integers $i, j$ with $1 \leqslant i, j \leqslant n$ and every $d \in \mathbb{D}$, there exist rank-$k$ matrices $y, z \in M_n(\mathbb{D})$ such that $de_{ij} = y - z$.*

*Proof.* Let $i, j$ be distinct integers such that $1 \leqslant i, j \leqslant n$ and let $d \in \mathbb{D}$. If $d = 0$, then $de_{ii} = y - y$ and $de_{ij} = y - y$ for every rank-$k$ matrix $y \in M_n(\mathbb{D})$. So we may assume $d \neq 0$. Suppose first that $k = 1$. Clearly, $de_{ii}$, $d(e_{ii} + e_{ij})$ and $de_{ij}$ are all rank-1 matrices in $M_n(\mathbb{D})$. In view of $de_{ii} = d(e_{ii} + e_{ij}) - de_{ij}$ and $de_{ij} = d(e_{ii} + e_{ij}) - de_{ii}$, we are done. Suppose next that $k \geqslant 2$. Let $U$ be a set such that $U \subseteq \{1, 2, \ldots, n\}$, $|U| = k - 2$ and $U \cap \{i, j\} = \emptyset$. Clearly, $d(e_{ii} + e_{ij} + e_{ji}) + \sum_{\ell \in U} e_{\ell\ell}$, $d(e_{ij} + e_{ji}) + \sum_{\ell \in U} e_{\ell\ell}$, $d(e_{ii} + e_{ij} + e_{jj}) + \sum_{\ell \in U} e_{\ell\ell}$ and $d(e_{ii} + e_{jj}) + \sum_{\ell \in U} e_{\ell\ell}$ are all rank-$k$ matrices in $M_n(\mathbb{D})$. In view of $de_{ii} = (d(e_{ii} + e_{ij} + e_{ji}) + \sum_{\ell \in U} e_{\ell\ell}) - (d(e_{ij} + e_{ji}) + \sum_{\ell \in U} e_{\ell\ell})$ and $de_{ij} = (d(e_{ii} + e_{ij} + e_{jj}) + \sum_{\ell \in U} e_{\ell\ell}) - (d(e_{ii} + e_{jj}) + \sum_{\ell \in U} e_{\ell\ell})$, we are done. $\square$

Deng and Ashraf [15] proved that if $\mathscr{A}$ is a prime ring of characteristic not 2 and there exists a non-identity endomorphism T of $\mathscr{A}$ such that $[T(x),T(y)]-[x,y]\in Z(\mathscr{A})$ for all $x,y$ in some essential right ideal of $\mathscr{A}$, then $\mathscr{A}$ is commutative. Lee and Wong [24] proved that if $f:\mathscr{R}\to\mathscr{A}$ is a map, where $\mathscr{R}$ is a noncentral Lie ideal of a prime ring $\mathscr{A}$, satisfying $[f(x),f(y)]-[x,y]\in Z(\mathscr{A})$ for all $x,y\in\mathscr{R}$, then $f$ is of the form $f(x)=\lambda x+\mu(x)$ for all $x\in\mathscr{R}$, where $\lambda^2=1$ and $\mu:\mathscr{R}\to Z(\mathscr{A})$ is a map, unless $\operatorname{char}\mathscr{A}=2$ and $\mathscr{A}\subseteq M_2(\mathbb{K})$ for a field $\mathbb{K}$. Now we prove a more general version of Theorem 1.1 as follows.

THEOREM 3.3. *Let $\mathbb{D}$ be a division ring, let $M_n(\mathbb{D})$ be the ring of all $n\times n$ matrices over $\mathbb{D}$ with center $\mathscr{Z}$, where $n\geqslant 2$ is an integer and let $\mathscr{S}$ be a subset of $M_n(\mathbb{D})$ containing all rank-$k$ matrices in $M_n(\mathbb{D})$, where $k$ is an integer such that $1\leqslant k\leqslant n$. If $f:\mathscr{S}\to M_n(\mathbb{D})$ is a map satisfying $[f(x),f(y)]-[x,y]\in\mathscr{Z}$ for all $x,y\in\mathscr{S}$, then there exists a map $\mu:\mathscr{S}\to Z(\mathbb{D})I_n$ such that $f(x)=x+\mu(x)$ for all $x\in\mathscr{S}$ or $f(x)=-x+\mu(x)$ for all $x\in\mathscr{S}$ unless $n=2$, $\operatorname{char}\mathbb{D}=2$ and $\mathbb{D}$ is commutative.*

*Proof.* Suppose first that $n\geqslant 3$ or $n=2$ and $\mathbb{D}$ is noncommutative. From the Jacobi identity, it follows that

$$[f(x),[f(y),f(z)]]+[f(y),[f(z),f(x)]]+[f(z),[f(x),f(y)]]=0$$

for all $x,y,z\in\mathscr{S}$. Since $[f(x),f(y)]-[x,y]\in\mathscr{Z}$ for all $x,y\in\mathscr{S}$, we have

$$[f(x),[y,z]]+[f(y),[z,x]]+[f(z),[x,y]]=0$$

for all $x,y,z\in\mathscr{S}$. By Proposition 2.5 and Corollary 2.8, there exist $\lambda\in Z(\mathbb{D})$ and a map $\mu:\mathscr{S}\to Z(\mathbb{D})I_n$ such that $f(x)=\lambda x+\mu(x)$ for all $x\in\mathscr{S}$. Then for $x,y\in\mathscr{S}$,

$$\mathscr{Z}\ni[f(x),f(y)]-[x,y]=[\lambda x+\mu(x),\lambda y+\mu(y)]-[x,y]=(\lambda^2-1)[x,y].$$

That is, $(\lambda^2-1)[x,y]\in Z(\mathbb{D})I_n$ for all $x,y\in\mathscr{S}$. Let $\mathscr{R}$ be the additive subgroup of $M_n(\mathbb{D})$ generated by $\mathscr{S}$. Then $(\lambda^2-1)[x,y]\in Z(\mathbb{D})I_n$ for all $x,y\in\mathscr{R}$. By Lemma 3.2, $e_{11},e_{12}\in\mathscr{R}$ and from $(\lambda^2-1)[e_{11},e_{12}]=(\lambda^2-1)e_{12}\in Z(\mathbb{D})I_n$, it follows that $\lambda^2=1$. So $\lambda=1$ or $-1$, proving the theorem.

Suppose now that $n=2$, $\mathbb{D}$ is commutative and $\operatorname{char}\mathbb{D}\neq 2$. By assumption, for every $x,y\in\mathscr{S}$, there exists $\alpha_{x,y}\in\mathbb{D}$ such that $[f(x),f(y)]-[x,y]=\alpha_{x,y}I_2\in\mathbb{D}I_2=\mathscr{Z}$. From $0=\operatorname{tr}([f(x),f(y)]-[x,y])=\operatorname{tr}(\alpha_{x,y}I_2)=2\alpha_{x,y}$ it follows that $\alpha_{x,y}=0$. Thus $[f(x),f(y)]-[x,y]=0$ for all $x,y\in\mathscr{S}$. By Lemma 3.1, we are done. $\square$

The conclusion of Theorem 3.2 is false if $n=2$, $\operatorname{char}\mathbb{D}=2$ and $\mathbb{D}$ is commutative.

EXAMPLE. Let $\mathbb{K}$ be a field with $\operatorname{char}\mathbb{K}=2$ and let $f:M_2(\mathbb{K})\to M_2(\mathbb{K})$ be the $\mathbb{K}$-linear map defined by $f(e_{11})=e_{11}+e_{12}$, $f(e_{22})=e_{11}+e_{12}$, $f(e_{12})=e_{12}$ and $f(e_{21})=e_{21}$. Then $[f(x),f(y)]-[x,y]\in\mathbb{K}I_2$ for all $x,y\in M_2(\mathbb{K})$. In particular, $[f(x),f(y)]-[x,y]\in\mathbb{K}I_2$ for all rank-$k$ matrices $x,y\in M_2(\mathbb{K})$, where $k$ is an integer such that $1\leqslant k\leqslant 2$. However, $f$ is not of the form described in Theorem 3.2 as

$f(e_{11}) - \lambda e_{11} = (1-\lambda)e_{11} + e_{12} \notin \mathbb{K}I_2$ and $f(e_{11}+e_{12}+e_{21}) - \lambda(e_{11}+e_{12}+e_{21}) = (1-\lambda)e_{11} - \lambda e_{12} + (1-\lambda)e_{21} \notin \mathbb{K}I_2$ for all $\lambda \in \mathbb{K}$.

Clearly, Theorem 1.1 follows directly from Lemma 3.1 and Theorem 3.3.

*Proof of Corollary 1.2.* By assumption, there exists a map $\delta : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ such that $g(xy) = g(x)y + \sigma(x)\delta(y)$ for all $x, y \in M_n(\mathbb{D})$. Then

$$g(y) = g(I_n y) = g(I_n)y + \sigma(I_n)\delta(y) = g(I_n)y + \delta(y) \tag{3.1}$$

for all $y \in M_n(\mathbb{D})$. We claim that $\delta$ is a $\sigma$-derivation. Let $x, y, z \in M_n(\mathbb{D})$. Then

$$g(x(yz)) = g(x)yz + \sigma(x)\delta(yz) \tag{3.2}$$

and

$$g((xy)z) = g(xy)z + \sigma(xy)\delta(z) = (g(x)y + \sigma(x)\delta(y))z + \sigma(xy)\delta(z). \tag{3.3}$$

The difference of (3.2) and (3.3) yields $\sigma(x)(\delta(yz) - \sigma(y)\delta(z) - \delta(y)z) = 0$ for all $x, y, z \in M_n(\mathbb{D})$. Thus $\delta(yz) = \sigma(y)\delta(z) + \delta(y)z$ for all $y, z \in M_n(\mathbb{D})$. By [22, Theorem 1], $\delta$ is additive and hence $\delta$ is a $\sigma$-derivation, as claimed.

Let $\mathscr{S}$ be the set of all rank-$k$ matrices in $M_n(\mathbb{D})$ and let $\mathscr{R}$ be the additive subgroup of $M_n(\mathbb{D})$ generated by $\mathscr{S}$. By Theorem 1.1, $g(x) - \lambda x \in Z(\mathbb{D})I_n$ for all $x \in \mathscr{S}$, where $\lambda \in \{1, -1\}$. In view of (3.1), $g(x) = g(I_n)x + \delta(x)$ for all $x \in M_n(\mathbb{D})$. Thus $g(I_n)x + \delta(x) - \lambda x \in Z(\mathbb{D})I_n$ for all $x \in \mathscr{S}$. By the additivity of $\delta$, we see that $g(I_n)x + \delta(x) - \lambda x \in Z(\mathbb{D})I_n$ for all $x \in \mathscr{R}$. In view of Lemma 3.2, $\mathscr{R} = M_n(\mathbb{D})$. Thus

$$\delta(x) + (g(I_n) - \lambda I_n)x \in Z(\mathbb{D})I_n \tag{3.4}$$

for all $x \in M_n(\mathbb{D})$. Setting $x = I_n$ in (3.4), we obtain $\delta(I_n) + (g(I_n) - \lambda I_n) \in Z(\mathbb{D})I_n$. Note that $\delta(I_n) = \delta(I_n I_n) = \sigma(I_n)\delta(I_n) + \delta(I_n)I_n = 2\delta(I_n)$, implying $\delta(I_n) = 0$. So $g(I_n) - \lambda I_n \in Z(\mathbb{D})I_n$ and then by (3.4) we have $[\delta(x), x] = 0$ for all $x \in M_n(\mathbb{D})$. By [36, Theorem 1.2], $\delta = 0$. Then (3.4) is reduced to $(g(I_n) - \lambda I_n)x \in Z(\mathbb{D})I_n$ for all $x \in M_n(\mathbb{D})$. Since $g(I_n) - \lambda I_n \in Z(\mathbb{D})I_n$, we see that $g(I_n) - \lambda I_n = 0$. Now from (3.1) it follows that $g(x) = \lambda x$ for all $x \in M_n(\mathbb{D})$. This proves the corollary. □

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*Cheng-Kai Liu*
*Department of Mathematics*
*National Changhua University of Education*
*Changhua 500, Taiwan*
*e-mail:* ckliu@cc.ncue.edu.tw

*Pao-Kuei Liau*
*Department of Mathematics*
*National Changhua University of Education*
*Changhua 500, Taiwan*
*e-mail:* paokuei.liau@gmail.com

*Yuan-Tsung Tsai*
*General Education Center*
*Tatung University*
*Taipei 104, Taiwan*
*e-mail:* yttsai@ttu.edu.tw