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SEMIGROUPS OF TRUNCATED TOEPLITZ OPERATORS

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Abstract. We characterise the one parameter families of truncated Toeplitz operators that are semigroups, uniformly continuous semigroups and C_0-semigroups of contractions. We also study their generators.

1. Introduction

Let L^2 denote the Lebesgue space of square integrable functions on the unit circle $\mathbb{T}$ and let H^2 be the classical Hardy space on the unit disk $\mathbb{D}$. The model spaces are the closed invariant subspaces for the backward shift operator S^* on H^2. These spaces are of the form $\mathscr{K}_u := H^2 \ominus uH^2$, where u is an inner function. Truncated Toeplitz operators are compressions of multiplication operators to the spaces $\mathscr{K}_u$. Let P_u be the orthogonal projection from L^2 onto the subspace $\mathscr{K}_u$. The truncated Toeplitz operator with symbol $\varphi \in L^2$ is defined by $A^u_\varphi(f) = P_u(\varphi f)$, on the dense subspace $\mathscr{K}_u \cap L^\infty$ of $\mathscr{K}_u$. The symbol φ is never unique. In [10] Sarason explored truncated Toeplitz operators, thus generating a huge interest in this class of operators (see [1, 7, 13, 14]).

The purpose of this paper is to investigate the semigroups of truncated Toeplitz operators and their generators. Our interest in this subject comes from the works of Suárez ([18]), Seubert ([15, 16, 17]) and Sarason ([11]).

Let S_u denote the compressed shift on $\mathscr{K}_u$ defined by $S_u(f) = P_u(zf)$. In [18], Suárez characterised the closed densely defined operators on $\mathscr{K}_u$ that commute with S_u^*. Sarason completed the Suárez result in [11], by showing that the closed densely defined operators on $\mathscr{K}_u$ commuting with S_u are the truncated Toeplitz operators with symbols in a certain local Smirnov class related to u (see section 6).

Seubert characterised in [17] the dissipative closed and densely defined operators on $\mathscr{K}_u$ that commute with S_u^*. Such operators are the generators of semigroups of contractions commuting with S_u^* which are also described by Seubert in [17]. It is shown that all these operators are TTOs.

In the present paper we establish a necessary and sufficient condition for a family of truncated Toeplitz operators to be a semigroup, then a uniformly continuous semigroup, and finally a C_0-semigroup of contractions. The proofs involve the Sedlock

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classes introduced in [13, 14], the Seubert result cited above and the Crofoot transform (a unitary operator between different model spaces).

The paper is organised as follows. The next section contains preliminary material concerning model spaces and truncated Toeplitz operators. In section 3 we introduce the Sedlock classes. Section 4 and section 5 are respectively devoted to semigroups and to uniformly continuous semigroups of truncated Toeplitz operators. The last section is dedicated to C_0-semigroups of contractions.

2. Preliminaries

Let $\mathbb{D}$ be the open unit disk, $\mathbb{T} = \partial\mathbb{D}$ the unit circle in $\mathbb{C}$, $m = d\theta/2\pi$ the normalized Lebesgue measure on $\mathbb{T}$, $L^2 := L^2(\mathbb{T}, m)$ the standard Lebesgue space on $\mathbb{T}$ and $\widehat{\mathbb{C}}$ the extended complex plane $\mathbb{C} \cup \{\infty\}$. Let H^2 denote the Hardy space on $\mathbb{D}$ and H^∞ the space of bounded analytic functions on $\mathbb{D}$. The unilateral shift operator on H^2 is defined by $Sf(z) = zf(z)$. Its adjoint, the backward shift S^*, is given by $S^*f(z) = (f(z) - f(0))/z$. To each non-constant inner function u we associate the model space

$$\mathscr{K}_u := H^2 \ominus uH^2.$$

The model space $\mathscr{K}_u$ is a reproducing kernel Hilbert space of holomorphic functions and the reproducing kernel at $\lambda \in \mathbb{D}$ is given by

$$K_\lambda^u(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \overline{\lambda}z}, \quad z \in \mathbb{D}. \tag{2.1}$$

If u has an angular derivative at $\zeta \in \mathbb{T}$ in the sense of Caratheodory then each function f in $\mathscr{K}_u$ admits a nontangential limit $f(\zeta)$ at ζ. In this case the function

$$K_\zeta^u(z) = \frac{1 - \overline{u(\zeta)}u(z)}{1 - \overline{\zeta}z}, \quad z \in \mathbb{D} \tag{2.2}$$

belongs to $\mathscr{K}_u$ and is a reproducing kernel at ζ.

The space $\mathscr{K}_u$ carries the natural conjugation

$$Cf = \widetilde{f} := u\overline{zf} \tag{2.3}$$

which is a bijection from $\mathscr{K}_u$ to itself. And a computation shows that

$$\widetilde{K_\lambda^u}(z) = \frac{u(z) - u(\lambda)}{z - \lambda}, \quad z \in \mathbb{D}, \ \lambda \in \overline{\mathbb{D}}. \tag{2.4}$$

The compression of S to $\mathscr{K}_u$ will be denoted by S_u. Its adjoint, S_u^*, is the restriction of S^* to $\mathscr{K}_u$. Let P_u be the orthogonal projection $P_u : L^2 \to \mathscr{K}_u$. For each function φ in L^2 the corresponding truncated Toeplitz operator (TTO) A_φ^u is the densely defined operator on $\mathscr{K}_u$ given by the formula

$$A_\varphi^u f = P_u(\varphi f), \quad f \in \mathscr{K}_u^\infty := \mathscr{K}_u \cap H^\infty.$$

Let $\mathscr{T}_u$ denote the set of all bounded TTOs on $\mathscr{K}_u$. Much is known about these operators (see the Sarason paper [10] for a detailed discussion) but we list a few interesting and useful facts below:

1. The operators S_u and S_u^* are the TTOs with symbols z and $\bar{z}$, respectively.

2. The operator $C : f \to \widetilde{f}$ defines an isometric, anti-linear, involution on $\mathscr{K}_u$ for which $CA^u_\varphi C = (A^u_\varphi)^* = A^u_{\overline{\varphi}}$, whenever $\varphi \in L^2$ and $A^u_\varphi \in \mathscr{T}_u$. This makes $\mathscr{T}_u$ a collection of complex symmetric operators [6, 10].

3. If $\lambda \in \mathbb{D}$, then
$$S_u\widetilde{K^u_\lambda} = \lambda\widetilde{K^u_\lambda} - u(\lambda)K^u_0 \text{ and } S_u^*K^u_\lambda = \overline{\lambda}K^u_\lambda - \overline{u(\lambda)}\widetilde{K^u_0}$$
and if $\lambda \in \mathbb{D}\setminus\{0\}$, then
$$S_uK^u_\lambda = \frac{1}{\lambda}(K^u_\lambda - K^u_0) \text{ and } S_u^*\widetilde{K^u_\lambda} = \frac{1}{\lambda}(\widetilde{K^u_\lambda} - \widetilde{K^u_0}).$$

4. We have $I - S_uS_u^* = K^u_0 \otimes K^u_0$ and $I - S_u^*S_u = \widetilde{K^u_0} \otimes \widetilde{K^u_0}$.

Note that for $f, g \in \mathscr{K}_u$, we let $f \otimes g$ be the rank one operator defined by $(f \otimes g)(h) = \langle h, g\rangle f$, for $h \in \mathscr{K}_u$.

By [10], Theorem 5.1, $\mathscr{T}_u$ is not an algebra of operators but it contains some algebras of interest. For example $\{A^u_\Phi : \Phi \in H^\infty\}$, the set of holomorphic TTOs on $\mathscr{K}_u$, and $\{A^u_{\overline{\Phi}} : \Phi \in H^\infty\}$, the corresponding set of antiholomorphic TTOs. By Sarason [10], the algebra of holomorphic TTOs is the commutant of the compressed shift S_u on $\mathscr{K}_u$, and the algebra of antiholomorphic TTOs is the commutant of S_u^*. In the next section we consider other interesting algebras contained in $\mathscr{T}_u$.

3. Sedlock algebras

Studying the product of TTOs, Sedlock introduced the following classes depending on the parameter $\alpha \in \widehat{\mathbb{C}}$ by
$$\mathscr{B}^\alpha_u = \{A^u_{\varphi + \alpha\overline{S_u\widetilde{\varphi}} + c} \in \mathscr{T}_u, \varphi \in \mathscr{K}_u, c \in \mathbb{C}\}, \text{ if } \alpha \in \mathbb{C},$$
and
$$\mathscr{B}^\infty_u = \{A^u_{\overline{\Phi}} : \Phi \in H^\infty\}.$$
These classes are closely linked to the modified compressed shift defined for $\alpha \in \overline{\mathbb{D}}$ by:
$$S^\alpha_u = S_u + \frac{\alpha}{1 - \alpha\overline{u(0)}} K^u_0 \otimes \widetilde{K^u_0}. \tag{3.1}$$
For $\alpha \in \mathbb{T}$, S^α_u is the so-called Clark unitary operator [2, 10] and its spectral measure is the Clark measure μ_α defined by the following formula
$$\Re\left(\frac{\alpha + u(z)}{\alpha - u(z)}\right) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu_\alpha(\zeta), \quad z \in \mathbb{D}.$$

For $A \in \mathscr{T}_u$, we denote by $\{A\}'$ the commutant of A. In the following theorem, we summarize the main properties of the classes $\mathscr{B}_u^\alpha$ and their links to S_u^α.

THEOREM 3.2. ([14]) *Let u be an inner function.*

1. *If $\alpha \in \mathbb{T}$, S_u^α is a unitary operator and*
$$\mathscr{B}_u^\alpha = \{S_u^\alpha\}' = \{\Phi(S_u^\alpha), \Phi \in L^\infty(\mu_\alpha)\},$$
where μ_α is the Clark measure. Each operator in $\mathscr{B}_u^\alpha$ is unitarily equivalent to a multiplication operator M_Φ on $L^2(\mu_\alpha)$ induced by a function $\Phi \in L^\infty(\mu_\alpha)$.

2. *If $\alpha \in \mathbb{D}$, S_u^α is a completely non-unitary operator and*
$$\mathscr{B}_u^\alpha = \{S_u^\alpha\}' = \left\{\Psi(S_u^\alpha) = A^u_{\frac{\Psi}{1-\alpha\overline{u}}}, \Psi \in H^\infty\right\}.$$
Moreover the characteristic function of S_u^α is $u_\alpha = \frac{\alpha-u}{1-\overline{\alpha}u}$.

3. *If $\alpha \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, then $\mathscr{B}_u^\alpha = \{(S_u^{1/\overline{\alpha}})^*\}'$ and the elements of $\mathscr{B}_u^\alpha$ may be described as*
$$\Psi^*(S_u^{1/\overline{\alpha}})^* = A^u_{\frac{\alpha\overline{\Psi}}{\alpha-u}}, \quad \Psi \in H^\infty,$$
where $\Psi^*(z) = \overline{\Psi(\overline{z})}$.

4. *If $A, B \in \mathscr{T}_u$, then $AB \in \mathscr{T}_u$ if and only if one of two cases holds:*
   (a) A or B is equal to cI for some $c \in \mathbb{C}$.
   (b) $A, B \in \mathscr{B}_u^\alpha$ for some $\alpha \in \widehat{\mathbb{C}}$.

   In the last case we also have $AB \in \mathscr{B}_u^\alpha$.

The following proposition gives us a symbol of operators in classes $\mathscr{B}_u^\alpha$.

PROPOSITION 3.3. *Let $A \in \mathscr{B}_u^\alpha$. Then*

1. *If $|\alpha| \leqslant 1$, then $\varphi + \alpha\overline{S_u\widetilde{\varphi}}$ is a symbol of A, where $\varphi = (1-\alpha\overline{u(0)})^{-1}AK_0^u$.*

2. *If $|\alpha| > 1$, then $\varphi + \alpha\overline{S_u\widetilde{\varphi}} + c$ is a symbol of A, where $\varphi = \frac{1}{(\alpha-u(0))}S_uA\widetilde{K_0^u}$ and $c = \frac{\alpha}{(\alpha-u(0))}\langle A\widetilde{K_0^u}, \widetilde{K_0^u}\rangle$.*

3. *If $\alpha = \infty$, then $\overline{S_u\widetilde{\varphi}} + c$ is a symbol of A, where $\varphi = S_uA\widetilde{K_0^u}$ and $c = \langle A\widetilde{K_0^u}, \widetilde{K_0^u}\rangle$.*

Proof. (1) This is Proposition 3.2 in [13].

(2) By Theorem 3.2 (1), $A^* \in \mathscr{B}_u^{1/\overline{\alpha}}$ and therefore has symbol $\psi + (1/\overline{\alpha})\overline{S_u\widetilde{\psi}}$, where $\psi = (1-(1/\overline{\alpha})\overline{u(0)})^{-1}A^*K_0^u = (1-(1/\overline{\alpha})\overline{u(0)})^{-1}\widetilde{A\widetilde{K_0^u}}$. Thus $(1/\alpha)S_u\widetilde{\psi} + \overline{\psi}$

is a symbol of A. Define $\varphi = (1/\alpha) S_u \widetilde{\psi} = (\alpha - u(0))^{-1} A\widetilde{K_0^u}$. Since $CS_uC = S_u^*$ and $S_u S_u^* = I - K_0^u \otimes K_0^u$, we get

$$S_u \widetilde{\varphi} = (1/\overline{\alpha}) S_u \widetilde{S_u \widetilde{\psi}} = (1/\overline{\alpha}) S_u S_u^* \psi = (1/\overline{\alpha})(\psi - \langle \psi, K_0^u \rangle K_0^u).$$

Thus $\overline{\psi} = \alpha \overline{S_u \widetilde{\varphi}} + \langle K_0^u, \psi \rangle \overline{K_0^u}$ and then $\varphi + \alpha \overline{S_u \widetilde{\varphi}} + \langle K_0^u, \psi \rangle$ is a symbol for A.

(3) We have $A^* \in \mathscr{B}_u^0$, so $A = (A^u_{A^* K_0^u})^* = A^u_{\overline{A^* K_0^u}}$. Again, since $CS_uC = S_u^*$ and $I = S_u S_u^* + K_0^u \otimes K_0^u$ we obtain $A^* = S_u C S_u C A^* + (K_0^u \otimes K_0^u) A^*$. Therefore

$$A^* K_0^u = S_u C S_u C A^* K_0^u + \langle A^* K_0^u, K_0^u \rangle K_0^u.$$

Using that $CA^* = AC$, we get

$$A^* K_0^u = S_u C S_u A \widetilde{K_0^u} + \langle A \widetilde{K_0^u}, \widetilde{K_0^u} \rangle K_0^u$$

and

$$\overline{A^* K_0^u} = \overline{S_u \widetilde{\varphi}} + c K_0^u,$$

which finishes the proof. $\square$

The next corollary follows immediately from Proposition 3.3. It gives a necessary and sufficient condition for two TTOs in Sedlock classes to be equal.

COROLLARY 3.4. *Let* $A, B \in \mathscr{B}_u^\alpha, \alpha \in \widehat{\mathbb{C}}$. *Then*

1. *If* $\alpha \in \overline{\mathbb{D}}$, *then* $A = B$ *if and only if* $AK_0^u = BK_0^u$.
2. *If* $\alpha \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, *then* $A = B$ *if and only if* $A\widetilde{K_0^u} = B\widetilde{K_0^u}$.

4. Semigroups of truncated Toeplitz operators.

For the definitions and properties about semigroups we refer to [9]. Using functional calculus we give a first characterization of semigroups.

PROPOSITION 4.1. *Let* $(T_t)_{t \geqslant 0} \subset \mathscr{T}_u$. *Then* $(T_t)_{t \geqslant 0}$ *is a semigroup if and only if there exists* $\alpha \in \widehat{\mathbb{C}}$ *such that for every* $t \geqslant 0$, $T_t \in \mathscr{B}_u^\alpha$ *and one of the following conditions is satisfied:*

1. $|\alpha| = 1$ *and for every* $t \geqslant 0$, *there exists* $\Phi_t \in L^\infty(\mu_\alpha)$ *such that* $T_t = \Phi_t(S_u^\alpha)$, $t \geqslant 0$, $\Phi_0 = 1$ μ_α*-a.e and for all* $t, s \geqslant 0$, $\Phi_t \Phi_s = \Phi_{t+s}$ μ_α*-a.e.*
2. $|\alpha| < 1$ *and for every* $t \geqslant 0$, *there exists* $\Phi_t \in H^\infty$ *such that* $T_t = \Phi_t(S_u^\alpha)$, $t \geqslant 0$ *and for all* $t, s \geqslant 0$, *the inner function* u_α *divides* $\Phi_t \Phi_s - \Phi_{t+s}$ *and* $\Phi_0 - 1$.
3. $|\alpha| > 1$ *and for every* $t \geqslant 0$, *there exists* $\Phi_t \in H^\infty$ *such that* $T_t = \Phi_t((S_u^{1/\overline{\alpha}})^*)$, $t \geqslant 0$, *with* $\Phi_t \in H^\infty$ *and the inner function* $u_{1/\overline{\alpha}}$ *divides* $\Phi_t \Phi_s - \Phi_{t+s}$ *and* $\Phi_0 - 1$.

Proof. Since for all $t,s \geqslant 0$, $T_t T_s = T_{t+s} \in \mathscr{T}_u$, by Theorem 3.2 (4) there exists an $\alpha \in \widehat{\mathbb{C}}$ such that for each $t \geqslant 0$, $T_t \in \mathscr{B}_u^\alpha$. Then we have the following cases:

(1) If $|\alpha| = 1$, then $T_t = \Phi_t(S_u^\alpha), t \geqslant 0$, with $\Phi_t \in L^\infty(\mu_\alpha)$ and $\Phi_t \Phi_s(S_u^\alpha) = \Phi_t(S_u^\alpha)\Phi_s(S_u^\alpha) = \Phi_{t+s}(S_u^\alpha)$ which implies that $\Phi_t \Phi_s = \Phi_{t+s}$ μ_α-a.e. Also $T_0 = \Phi_0(S_u^\alpha) = I$ implies that $\Phi_0 = 1$ μ_α-a.e.

(2) If $|\alpha| < 1$, one should note that if $(\Phi_t \Phi_s - \Phi_{t+s}) \in H^\infty$ then $(\Phi_t \Phi_s - \Phi_{t+s})(S_u^\alpha) = 0$ if and only if u_α divides $\Phi_t \Phi_s - \Phi_{t+s}$. This follows from the fact, noted above, that the characteristic function of S_u^α is u_α.

(3) If $|\alpha| > 1$, then the result follows as in (2) by considering $(T_t^*)_{t \geqslant 0}$. $\square$

We have also the following characterization of semigroups.

THEOREM 4.2. *Let $(T_t)_{t \geqslant 0} \subset \mathscr{T}_u$. Then $(T_t)_{t \geqslant 0}$ is a semigroup if and only if there is $\alpha \in \widehat{\mathbb{C}}$ such that for all $t \geqslant 0$, $T_t \in \mathscr{B}_u^\alpha$ and one of the following condition is satisfied*

1. $|\alpha| \leqslant 1$ *and for all* $t,s \geqslant 0$,
$$T_{t+s} K_0^u = T_t T_s K_0^u \text{ and } T_0 K_0^u = K_0^u. \tag{4.3}$$

2. $|\alpha| > 1$ *and for all* $t,s \geqslant 0$,
$$T_{t+s} \widetilde{K_0^u} = T_t T_s \widetilde{K_0^u} \text{ and } T_0 \widetilde{K_0^u} = \widetilde{K_0^u}. \tag{4.4}$$

Proof. Let $(T_t)_{t \geqslant 0} \subset \mathscr{T}_u$. Suppose that $(T_t)_{t \geqslant 0}$ is a semigroup. Then there exists $\alpha \in \widehat{\mathbb{C}}$ such that for every $t \geqslant 0$, $T_t \in \mathscr{B}_u^\alpha$. The conditions (4.3) and (4.4) are clearly satisfied.

For the converse suppose that for every $t \geqslant 0$, $T_t \in \mathscr{B}_u^\alpha$ with $|\alpha| \leqslant 1$ and that condition (4.3) is satisfied. Then by Theorem 3.2 (4), $T_t T_s$ is also in $\mathscr{B}_u^\alpha$, and by Corollary 3.4 (1), $T_{t+s} = T_t T_s$. Similarly if $|\alpha| > 1$ and condition (4.4) holds, we obtain that $(T_t)_{t \geqslant 0}$ is a semigroup. $\square$

EXAMPLE 4.5. It is well known that the model space $\mathscr{K}_u$ is finite dimensional if and only if u is a finite Blaschke product. If n is a positive integer and $u(z) = z^n$, then
$$\mathscr{K}_{z^n} = \operatorname{span}\{1, z, z^2, \ldots, z^{n-1}\},$$
$K_0^{z^n} = 1$ and $\widetilde{K_0^{z^n}} = z^{n-1}$.

Note that for $\varphi(z) = \sum_{k \geqslant 0}^{n-1} \widehat{\varphi}(k) z^k \in \mathscr{K}_{z^n}$ we have $S_{z^n} \widetilde{\varphi} = \sum_{k=1}^{n-1} \overline{\widehat{\varphi}(n-k)} z^k$. Then it follows from Theorem 4.2 that a family of operators $(T_t)_{t \geqslant 0} \subset \mathscr{T}_u$ is a semigroup if and only if there exists $\alpha \in \widehat{\mathbb{C}}$ such that for every $t \geqslant 0$, $T_t \in \mathscr{B}_u^\alpha$ and one of the two cases holds :

(1) $\alpha \in \mathbb{C}$ and T_t has a symbol of the form
$$\sum_{k \geqslant 0}^{n-1} \widehat{\varphi_t}(k) z^k + \alpha \sum_{k \geqslant 1}^{n-1} \widehat{\varphi_t}(n-k) z^{-k}$$

such that
$$\widehat{\varphi}_0(0)=1,\ \widehat{\varphi}_0(1)=\ldots=\widehat{\varphi}_0(n-1)=0$$
and for all $t,s\geqslant 0$, $0\leqslant k\leqslant n-1$,
$$\widehat{\varphi}_{t+s}(k)=\sum_{m=0}^{k}\widehat{\varphi}_t(m)\widehat{\varphi}_s(k-m)+\alpha\sum_{m=k+1}^{n-1}\widehat{\varphi}_t(m)\widehat{\varphi}_s(n-m+k).$$

(2) $\alpha=\infty$ and T_t has a symbol of the form
$$\sum_{k=0}^{n-1}\overline{\widehat{\varphi}_t(k)}z^{-k}$$
such that
$$\widehat{\varphi}_0(0)=1,\quad \widehat{\varphi}_0(1)=\ldots=\widehat{\varphi}_0(n-1)=0,$$
and for all $t,s\geqslant 0$, $0\leqslant k\leqslant n-1$,
$$\widehat{\varphi}_{t+s}(k)=\sum_{m=0}^{k}\widehat{\varphi}_t(m)\widehat{\varphi}_s(k-m).$$

5. Uniformly continuous semigroups

We start this section with an elementary result which characterizes the generator of a semigroup of TTOs.

PROPOSITION 5.1. *A bounded operator A on $\mathscr{K}_u$ is a generator of a uniformly continuous semigroup of TTOs if and only if $A\in\mathscr{B}_u^{\alpha}$ for some $\alpha\in\widehat{\mathbb{C}}$.*

Proof. Let $(T_t)_{t\geqslant 0}$ be a uniformly continuous semigroup of TTOs and A its generator. Then there exists $\alpha\in\widehat{\mathbb{C}}$ such that for each $t\geqslant 0$, $T_t\in\mathscr{B}_u^{\alpha}$. Since $\mathscr{B}_u^{\alpha}$ is a closed algebra and $A=\lim_{t\to 0^+}\frac{T_t-1}{t}$ in the operator norm, we get that $A\in\mathscr{B}_u^{\alpha}$.

For the converse suppose that $A\in\mathscr{B}_u^{\alpha}$ for some $\alpha\in\widehat{\mathbb{C}}$. Again, since $\mathscr{B}_u^{\alpha}$ is a closed algebra, for every $t\geqslant 0$, $e^{tA}\in\mathscr{B}_u^{\alpha}$. So A is the generator of the TTOs $(e^{tA})_{t\geqslant 0}$. $\square$

The following theorem characterizes the uniformly continuous semigroups of TTOs. It gives also the relationship between the symbols of the elements of the semigroup and the symbol of its generator.

THEOREM 5.2. *Let $(T_t)_{t\geqslant 0}\subset\mathscr{T}_u$ be a semigroup of bounded TTOs. Then $(T_t)_{t\geqslant 0}$ is uniformly continuous if and only if there exists $\alpha\in\widehat{\mathbb{C}}$ such that for every $t\geqslant 0$, $T_t\in\mathscr{B}_u^{\alpha}$ and one of the following conditions is satisfied:*

1. *$|\alpha|\leqslant 1$, $\Psi:=\frac{1}{1-\alpha\overline{u(0)}}\lim_{t\to 0^+}\frac{T_tK_0^u-K_0^u}{t}$ exists in the norm of $\mathscr{K}_u$ and the operator $A=A^u_{\Psi+\alpha\overline{S_u\widetilde{\Psi}}}$ is bounded.*

2. $|\alpha| > 1$, $\Psi_0 := \frac{1}{\alpha - u(0)} \lim_{t\to 0^+} \frac{T_t\widetilde{K_0^u} - \widetilde{K_0^u}}{t}$ *exists in the norm of* $\mathscr{K}_u$ *and the operator* $A = A^u_{\Psi+\alpha\overline{S_u\widetilde{\Psi}}+c}$ *is bounded, where*

$$\Psi = S_u\Psi_0 \text{ and } c = \frac{\alpha}{(\alpha - u(0))} \lim_{t\to 0^+} \left\langle \frac{T_t\widetilde{K_0^u} - \widetilde{K_0^u}}{t}, \widetilde{K_0^u} \right\rangle.$$

3. $\alpha = \infty$, $\Psi_0 := \lim_{t\to 0^+} \frac{T_t\widetilde{K_0^u} - \widetilde{K_0^u}}{t}$ *exists in the norm of* $\mathscr{K}_u$ *and the operator* $A = A^u_{\overline{S_u\widetilde{\Psi}}+c}$ *is bounded, where*

$$\Psi = S_u\Psi_0 \text{ and } c = \lim_{t\to 0^+} \left\langle \frac{T_t\widetilde{K_0^u} - \widetilde{K_0^u}}{t}, \widetilde{K_0^u} \right\rangle.$$

In all cases A *is the generator of* $(T_t)_{t\geqslant 0}$.

Proof. Let $(T_t)_{t\geqslant 0}$ be a semigroup of bounded TTOs. Then there exists $\alpha \in \widehat{\mathbb{C}}$ such that for every $t \geqslant 0$, $T_t \in \mathscr{B}_u^\alpha$. Suppose that $(T_t)_{t\geqslant 0}$ is uniformly continuous with generator $B = \lim_{t\to 0^+} \frac{T_t - I}{t}$, which is a bounded operator. As $\frac{T_t - I}{t} \in \mathscr{B}_u^\alpha$, $t \geqslant 0$, and $\mathscr{B}_u^\alpha$ is closed for the operator norm, $B \in \mathscr{B}_u^\alpha$. Then one of the three cases holds:

(1) $|\alpha| \leqslant 1$. In this case we have

$$BK_0^u = \lim_{t\to 0^+} \frac{T_tK_0^u - K_0^u}{t} = (1 - \alpha\overline{u(0)})\Psi.$$

Since $B \in \mathscr{B}_u^\alpha$, it follows from Proposition 3.3 (1) that $B = A^u_{\Psi+\alpha\overline{S_u\widetilde{\Psi}}}$. This implies in particular that the operator $A^u_{\Psi+\alpha\overline{S_u\widetilde{\Psi}}}$ is bounded.

(2) $|\alpha| > 1$. As in the above case, $B \in \mathscr{B}_u^\alpha$ and by Proposition 3.3 (2) $\varphi + \alpha\overline{S_u\widetilde{\varphi}} + c'$ is a symbol of B, where $\varphi = \frac{1}{(\alpha - u(0))} S_uB\widetilde{K_0^u}$ and $c' = \frac{\alpha}{(\alpha - u(0))}\langle B\widetilde{K_0^u}, \widetilde{K_0^u}\rangle$. Since $B\widetilde{K_0^u} = \lim_{t\to 0^+} \frac{T_t\widetilde{K_0^u} - \widetilde{K_0^u}}{t}$ we see that $\varphi = \Psi$, $c' = c$ and $B = A$.

(3) $|\alpha| = \infty$. The proof of this case is similar to the above one.

For the converse suppose that $(T_t)_{t\geqslant 0} \subset \mathscr{B}_u^\alpha$ and that (1) holds. We set $\varphi_t = \frac{1}{1-\alpha\overline{u(0)}} \frac{T_tK_0^u - K_0^u}{t}$. By Proposition 3.3 (1), the function $\varphi_t + \alpha\overline{S_u\widetilde{\varphi_t}}$ is a symbol of $\frac{T_t - I}{t}$. Thus for every function $f \in \mathscr{K}_u^\infty$,

$$\begin{aligned}\lim_{t\to 0+} \frac{T_t - I}{t} f &= \lim_{t\to 0+} P_u((\varphi_t + \alpha\overline{S_u\widetilde{\varphi_t}})f) \\ &= P_u((\Psi + \alpha\overline{S_u\widetilde{\Psi}})f) = A^u_{\Psi+\alpha\overline{S_u\widetilde{\Psi}}} f.\end{aligned}$$

As $\mathscr{K}_u^\infty$ is dense in $\mathscr{K}_u$, $A^u_{\Psi+\alpha\overline{S_u\widetilde{\Psi}}}$ is the generator of the semigroup $(T_t)_{t\geqslant 0}$. Since the generator is bounded $(T_t)_{t\geqslant 0}$ is uniformly continuous.

The proofs of cases (2) and (3) are similar to case (1). $\square$

EXAMPLES 5.3.

(1) Let $\lambda \in \mathbb{D}$ and consider the rank one operator $A = \widetilde{K_\lambda^u} \otimes K_\lambda^u$. By [14], Example 4.2.12, A is a TTO of type $u(\lambda)$, $A = A^u_{\widetilde{K_\lambda^u}+u(\lambda)\overline{S_u K_\lambda^u}}$. Then by Proposition 5.1, A generates a uniformly continuous semigroup of TTOs $(e^{tA})_{t\geqslant 0}$. We have

$$\begin{aligned} e^{tA} &= I + t(\widetilde{K_\lambda^u} \otimes K_\lambda^u) + \frac{t^2}{2!}(\widetilde{K_\lambda^u} \otimes K_\lambda^u)^2 + \frac{t^3}{3!}(\widetilde{K_\lambda^u} \otimes K_\lambda^u)^3 + \dots \\ &= I + t(\widetilde{K_\lambda^u} \otimes K_\lambda^u) + \frac{t^2}{2!}u'(\lambda)(\widetilde{K_\lambda^u} \otimes K_\lambda^u) + \frac{t^3}{3!}u'(\lambda)^2(\widetilde{K_\lambda^u} \otimes K_\lambda^u) + \dots \end{aligned}$$

If $u'(\lambda) = 0$, then

$$e^{tA} = I + t(\widetilde{K_\lambda^u} \otimes K_\lambda^u),$$

and if $u'(\lambda) \neq 0$, then

$$e^{tA} = I + \frac{e^{tu'(\lambda)} - 1}{u'(\lambda)}(\widetilde{K_\lambda^u} \otimes K_\lambda^u).$$

Notice that $K_\lambda^u \otimes \widetilde{K_\lambda^u}$ is the adjoint operator of $\widetilde{K_\lambda^u} \otimes K_\lambda^u$. So $K_\lambda^u \otimes \widetilde{K_\lambda^u}$ is the generator of the uniformly continuous semigroup of TTOs

$$\begin{aligned} T_t &= I + t(K_\lambda^u \otimes \widetilde{K_\lambda^u}) && \textit{if } u'(\lambda) = 0, \\ T_t &= I + \frac{e^{t\overline{u'(\lambda)}} - 1}{\overline{u'(\lambda)}}(K_\lambda^u \otimes \widetilde{K_\lambda^u}) && \textit{if } u'(\lambda) \neq 0. \end{aligned}$$

(2) Suppose u has an angular derivative in the sense of Caratheodory at $\zeta \in \mathbb{T}$, that is, the nontangential limit of u and u' exist in ζ with the limit of u in ζ of module 1. Consider $A = K_\zeta^u \otimes K_\zeta^u$. By [14], Example 4.2.12, A is a TTO of type $u(\zeta)$, with symbol $K_\zeta^u + u(\zeta)\overline{S_u \widetilde{K_\zeta^u}}$. We have $A^2 = K_\zeta^u(\zeta)A$. Since $|u'(\zeta)| = K_\zeta^u(\zeta) = \|K_\zeta^u\|^2 \neq 0$, we see as in (1) that

$$e^{tA} = I + \frac{e^{t|u'(\zeta)|} - 1}{|u'(\zeta)|}(K_\zeta^u \otimes K_\zeta^u).$$

(3) Let $\alpha \in \overline{\mathbb{D}}$ and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be n distinct solutions of the equation $u(\lambda) = \alpha$. Let $\varphi = \sum_{j=1}^n a_j \widetilde{K_{\lambda_j}^u}$, where the a_j are complex numbers. Notice that if u is a finite Blaschke product of order n then $\{\widetilde{K_{\lambda_j}^u},\ 1 \leqslant j \leqslant n\}$ is a basis of $\mathscr{K}_u$ and every function φ in $\mathscr{K}_u$ has the above form.

We have

$$A = A^u_{\varphi+\alpha\overline{S_u\widetilde{\varphi}}} = \sum_{j=1}^n a_j A^u_{\widetilde{K_{\lambda_j}^u}+u(\lambda_j)\overline{S_u K_{\lambda_j}^u}} = \sum_{j=1}^n a_j \widetilde{K_{\lambda_j}^u} \otimes K_{\lambda_j}^u.$$

The operators $\widetilde{K^u_{\lambda_j}} \otimes K^u_{\lambda_j}$ are of type $u(\lambda_j) = \alpha$ and for every j,k,

$$(\widetilde{K^u_{\lambda_j}} \otimes K^u_{\lambda_j})(\widetilde{K^u_{\lambda_k}} \otimes K^u_{\lambda_k}) = \delta_{jk} u'(\lambda_j) \widetilde{K^u_{\lambda_j}} \otimes K^u_{\lambda_j}.$$

Then A generates the semigroup of TTOs

$$\begin{aligned} T_t = e^{tA} &= \prod_{j=1}^{n} e^{ta_j \widetilde{K^u_{\lambda_j}} \otimes K^u_{\lambda_j}} \\ &= \prod_{j\in M} \left[I + ta_j (\widetilde{K^u_{\lambda_j}} \otimes K^u_{\lambda_j}) \right] \prod_{j\in N} \left[I + \frac{e^{ta_j u'(\lambda_j)} - 1}{u'(\lambda_j)} (\widetilde{K_{\lambda^u_j}} \otimes K^u_{\lambda_j}) \right] \\ &= I + t \sum_{j\in M} a_j \widetilde{K^u_{\lambda_j}} \otimes K^u_{\lambda_j} + \sum_{j\in N} \left[\frac{e^{ta_j u'(\lambda_j)} - 1}{u'(\lambda_j)} (\widetilde{K_{\lambda^u_j}} \otimes K^u_{\lambda_j}) \right]. \end{aligned}$$

where $M = \{j,\ u'(\lambda_j) = 0\}$ and $N = \{j,\ u'(\lambda_j) \neq 0\}$.

(4) Let u be a finite Blaschke product of order n and $|\alpha| = 1$. Using the fact that u' never vanishes on $\mathbb{T}$, the equation $u(\zeta) = \alpha$ has n distinct solutions $\zeta_1, \zeta_2, \ldots, \zeta_n$ which are in the unit circle. The family $\left\{K^u_{\zeta_j},\ 1 \leqslant j \leqslant n\right\}$ is an orthogonal basis for $\mathscr{K}_u$. For every $\varphi \in \mathscr{K}_u$, $\varphi = \sum_{j=1}^{n} a_j K^u_{\zeta_j}$, where $a_j = \frac{\varphi(\zeta_j)}{|u'(\zeta_j)|}$, and we have

$$A = A^u_{\varphi + \alpha \overline{S_u \widetilde{\varphi}}} = \sum_{j=1}^{n} a_j A^u_{K^u_{\zeta_j} + u(\zeta_j)\overline{S_u \widetilde{K^u_{\zeta_j}}}} = \sum_{j=1}^{n} a_j K^u_{\zeta_j} \otimes K^u_{\zeta_j}.$$

Since the operators $K^u_{\zeta_j} \otimes K^u_{\zeta_j}$ are of type α and for every j,k,

$$(K^u_{\zeta_j} \otimes K^u_{\zeta_j})(K^u_{\zeta_k} \otimes K^u_{\zeta_k}) = \delta_{jk} |u'(\zeta_j)| K^u_{\zeta_j} \otimes K^u_{\zeta_j},$$

we obtain, as in the above example, that A generates the following semigroup of TTOs:

$$T_t = I + \sum_{j=1}^{n} \left[\frac{e^{ta_j |u'(\zeta_j)|} - 1}{|u'(\zeta_j)|} (K^u_{\zeta_j} \otimes K^u_{\zeta_j}) \right].$$

Using these examples we can now describe all the uniformly continuous semigroups of finite Toeplitz matrices.

THEOREM 5.4. *Let $u(z) = z^n$ and $\omega = e^{\frac{2\pi i}{n}}$. The uniformly continuous semigroups of TTOs on $\mathscr{K}_u$ are:*

$$T_t = I + \frac{1}{nr^{n-1} e^{i(n-1)\theta}} \sum_{j=0}^{n-1} \omega^j (e^{tc_j} - 1) \widetilde{K^u_{re^{i\theta}\omega^j}} \otimes K^u_{re^{i\theta}\omega^j},$$

$$T_t = I + \frac{1}{nr^{n-1} e^{i(1-n)\theta}} \sum_{j=0}^{n-1} \overline{\omega}^j (e^{tc_j} - 1) K^u_{re^{i\theta}\omega^j} \otimes \widetilde{K^u_{re^{i\theta}\omega^j}},$$

$$T_t = I + \frac{1}{n}\sum_{j=0}^{n-1}(e^{tc_j}-1)(K^u_{e^{i\theta}\omega^j}\otimes K^u_{e^{i\theta}\omega^j}),$$

$$T_t = A^u_{e^{t\varphi}} \text{ and } T_t = A^u_{e^{t\overline{\varphi}}},$$

where $0<r<1$, $\theta\in\mathbb{R}$, $(c_j)_{j=0}^{n-1}\in\mathbb{C}^n$ *and* $\varphi\in\mathscr{K}_u$.

Proof. Let $(T_t)_{t\geqslant 0}$ be a uniformly continuous semigroup of TTOs on $\mathscr{K}_u$ and let A be its generator. There exists $\alpha\in\widehat{\mathbb{C}}$ such that all the operators T_t and A are in $\mathscr{B}^\alpha_u$. We have one of the following cases:

(1) $0<|\alpha|<1$: We write $\alpha=(re^{i\theta})^n$, with $0<r<1$ and $\theta\in\mathbb{R}$. The solutions of the equation $u(z)=\alpha$ are $\lambda_j=re^{i\theta}\omega^{j-1}$, $j=1,\dots,n$. The family $\{\widetilde{K^u_{\lambda_j}},\ j=1,\dots,n\}$ is a basis of $\mathscr{K}_u$. Therefore $A=A^u_{\varphi+\alpha\overline{S_u\widetilde{\varphi}}}$, where φ has the form $\varphi=\sum_{j=1}^n a_j\widetilde{K^u_{\lambda_j}}$ with $(a_j)_{j=1}^n\in\mathbb{C}^n$. As in Examples 5.3 (3),

$$T_t = I+\sum_{j=1}^n\left[\frac{e^{ta_ju'(\lambda_j)}-1}{u'(\lambda_j)}(\widetilde{K^u_{\lambda_j}}\otimes K^u_{\lambda_j})\right],$$

which gives

$$T_t = I+\frac{1}{nr^{n-1}e^{i(n-1)\theta}}\sum_{j=0}^{n-1}\omega^j(e^{tc_j}-1)\widetilde{K^u_{re^{i\theta}\omega^j}}\otimes K^u_{re^{i\theta}\omega^j},$$

for some complex numbers $c_0,\dots,c_{n-1}$.

(2) $1<|\alpha|<\infty$: Applying the above case to $(T^*)_{t\geqslant 0}$, we get that T_t has the form

$$T_t = I+\frac{1}{nr^{n-1}e^{i(1-n)\theta}}\sum_{j=0}^{n-1}\overline{\omega}^j(e^{tc_j}-1)K^u_{re^{i\theta}\omega^j}\otimes\widetilde{K^u_{re^{i\theta}\omega^j}}.$$

(3) $|\alpha|=1$: Let $\alpha=e^{in\theta}$. The solutions of the equation $u(z)=\alpha$ are $\zeta_j=e^{i\theta}\omega^{j-1}$, $j=1,\dots,n$. As in Examples 5.3, (4),

$$T_t = I+\sum_{j=1}^n\left[\frac{e^{ta_j|u'(\zeta_j)|}-1}{|u'(\zeta_j)|}(K^u_{\zeta_j}\otimes K^u_{\zeta_j})\right],$$

which shows that

$$T_t = I + \frac{1}{n}\sum_{j=0}^{n-1}(e^{tc_j}-1)(K^u_{e^{i\theta}\omega^j}\otimes K^u_{e^{i\theta}\omega^j}),$$

for some complex numbers $c_0,\dots,c_{n-1}$.

(4) $\alpha=0$: There exists $\varphi\in\mathscr{K}_u$ such that $A=A^u_\varphi$. Note that φ is a bounded holomorphic function. Therefore $T_t=e^{tA^u_\varphi}=A^u_{e^{t\varphi}}$.

(5) $|\alpha|=\infty$: We apply case (4) to $(T^*)_{t\geqslant 0}$, we get then that $T_t=A^u_{e^{t\overline{\varphi}}}$.

On the other hand it is a straightforward calculation that all the families $(T_t)_{t\geqslant 0}$ cited in the theorem are uniformly continuous semigroups □

6. C_0-semigroups

Recall that the Nevanlinna class $\mathscr{N}$ is the set of holomorphic functions $\varphi = \frac{\psi}{\chi}$, where ψ, $\chi \in H^\infty$ and χ not identically zero.

The Smirnov class $\mathscr{N}^+$ consists of all members of $\mathscr{N}$ having a denominator that is an outer function. It is well known that each nonzero function φ in $\mathscr{N}^+$ has a canonical representation, that is a unique expression of the form $\varphi = \frac{b}{a}$, where a and b are in H^∞, a is an outer function, $a(0) > 0$, and $|a|^2 + |b|^2 = 1$ almost everywhere on $\mathbb{T}$.

As defined in [11], the local Smirnov class $\mathscr{N}_u^+$ consists of all $\varphi \in \mathscr{N}$ for which u, χ are relatively prime. Any $\varphi \in \mathscr{N}_u^+$ has a unique canonical representation $\varphi = \frac{b}{va}$, where $a, b \in H^\infty$, a is an outer function such that $a(0) > 0$, $|a|^2 + |b|^2 = 1$ almost everywhere on $\mathbb{T}$, v is inner and $GCD(v, b) = GCD(v, u) = 1$.

In [11], Sarason extends the definition of A_φ^u to functions φ in $\mathscr{N}_u^+$. As observed in [11], the functional calculus can be used to define these operators: For $\varphi \in \mathscr{N}_u^+$ with the unique canonical representation $\varphi = \frac{b}{va}$, $A_{\overline{\varphi}}^u = ((va)^*(S_u^*))^{-1} b^*(S_u^*)$, where, for a function ψ holomorphic in $\mathbb{D}$, $\psi^*(z) = \overline{\psi(\overline{z})}$. Then $A_{\overline{\varphi}}^u$ is a closed densely defined operator on $\mathscr{K}_u$. The operator A_φ^u is defined as the adjoint of $A_{\overline{\varphi}}^u$ ([12], Lemma 5.4) and is also the C-transform of $A_{\overline{\varphi}}^u$, that is $A_\varphi^u = CA_{\overline{\varphi}}^u C$.

In ([18]) Suárez characterises the closed densely operators on $\mathscr{K}_u$ that commute with S_u. This result was completed by Sarason in [12], giving the following theorem.

THEOREM 6.1. ([12]) *A closed operator A densely defined in $\mathscr{K}_u$ commutes with S_u if and only if $A = A_\varphi^u$ for some $\varphi \in \mathscr{N}_u^+$.*

Let $\alpha \in \mathbb{D}$. The Crofoot operator U_α is the multiplication operator by the function $(1 - |\alpha|^2)^{-1/2}(1 - \overline{\alpha}u)$. It is known that U_α is a unitary operator from $\mathscr{K}_{u_\alpha}$ onto $\mathscr{K}_u$, where $u_\alpha = \frac{u-\alpha}{1-\overline{\alpha}u}$ (see [4, 10]).

If φ is in H^∞ we have $A^u_{\frac{\varphi}{1-\alpha\overline{u}}} = U_\alpha A_\varphi^{u_\alpha} U_\alpha^{-1}$ and $A^u_{\frac{\overline{\varphi}}{1-\overline{\alpha}u}} = U_\alpha A_{\overline{\varphi}}^{u_\alpha} U_\alpha^{-1}$ (see [13]). These formulas yield the following definition.

DEFINITION 6.2. For $\alpha \in \mathbb{D}$ and $\varphi \in \mathscr{N}_{u_\alpha}^+$, we set $A^u_{\frac{\varphi}{1-\alpha\overline{u}}} = U_\alpha A_\varphi^{u_\alpha} U_\alpha^{-1}$ and $A^u_{\frac{\overline{\varphi}}{1-\overline{\alpha}u}} = U_\alpha A_{\overline{\varphi}}^{u_\alpha} U_\alpha^{-1}$.

The operator $A^u_{\frac{\varphi}{1-\alpha\overline{u}}}$ is closed and densely defined with domain $D(A^u_{\frac{\varphi}{1-\alpha\overline{u}}}) = \{f \in \mathscr{K}_u : U_\alpha^{-1} f \in D(A_\varphi^{u_\alpha})\}$. Moreover we have $(A^u_{\frac{\varphi}{1-\alpha\overline{u}}})^* = A^u_{\frac{\overline{\varphi}}{1-\overline{\alpha}u}}$ and $(A^u_{\frac{\varphi}{1-\alpha\overline{u}}})^* = CA^u_{\frac{\overline{\varphi}}{1-\overline{\alpha}u}} C$, since $CU_\alpha C_{u_\alpha} = U_\alpha$, where $C = C_u$ and C_{u_α} are respectively the conjugate operators on $\mathscr{K}_u$ and $\mathscr{K}_{u_\alpha}$. It follows from the equality $U_\alpha^{-1} S_u^\alpha U_\alpha = S_{u_\alpha}$ and Theorem 6.1, that the closed densely defined operators on $\mathscr{K}_u$ commuting with S_u^α are the operators $A^u_{\frac{\varphi}{1-\alpha\overline{u}}}$ with $\varphi \in \mathscr{N}_{u_\alpha}^+$.

Let now $\alpha \in \mathbb{T}$. Then S_u^α is unitarily equivalent to the multiplication operator $M_z(f) = zf$ on $L^2(\mu_\alpha)$. Indeed, for $f \in L^2(\mu_\alpha)$, let

$$(V_\alpha f)(z) = (1-\overline{\alpha}u(z)) \int \frac{f(\zeta)}{1-\overline{\zeta}z} d\mu_\alpha(\zeta).$$

Then V_α is a unitary operator from $L^2(\mu_\alpha)$ onto $\mathscr{K}_u$ and we have $S_u^\alpha = V_\alpha M_z V_\alpha^{-1}$.

For a measurable function $\Phi : \mathbb{T} \to \mathbb{C}$, the multiplication operator M_Φ is defined by

$$M_\Phi f = \Phi f, \quad f \in D(M_\Phi) = \{f \in L^2(\mu_\alpha) : \Phi f \in L^2(\mu_\alpha)\}.$$

M_Φ is a closed densely defined operator on $L^2(\mu_\alpha)$ (see [5], Proposition 4.10, pp. 31). We set $\Phi(S_u^\alpha) = V_\alpha M_\Phi V_\alpha^{-1}$ with domain $\{f \in \mathscr{K}_u,\ V_\alpha^{-1} f \in D(M_\Phi)\}$. The following result is probably known, but we have not found a suitable reference.

PROPOSITION 6.3. *Let $\alpha \in \mathbb{T}$. A closed densely defined operator A on $\mathscr{K}_u$ commutes with S_u^α and $(S_u^\alpha)^*$ if and only if $A = \Phi(S_u^\alpha)$ for some measurable function $\Phi : \mathbb{T} \to \mathbb{C}$.*

Proof. Let B be a densely closed operator on $L^2(\mu_\alpha)$ that commutes with M_z and $M_{\overline{z}}$. Clearly B commutes with M_P for every trigonometric polynomial P. Let $\Psi \in L^\infty(\mu_\alpha)$. There exists a uniformly bounded sequence of continuous functions on the support of μ_α that converges to Ψ, μ_α-almost everywhere. To see this one can use a Lusin theorem. By the Tietze-Urysohn extension theorem, these functions can be considered as continuous functions on $\mathbb{T}$. It follows that there exists a uniformly bounded sequence of trigonometric polynomials $(P_n)_n$ that converges to Ψ, μ_α-almost everywhere. For f in $D(B)$, we obtain by using the dominated convergence theorem

$$(M_{P_n} f, BM_{P_n} f) = (M_{P_n} f, M_{P_n} Bf) \to (M_\Psi f, M_\Psi Bf), \text{ as } n \to \infty.$$

Since B is closed we get that $M_\Psi f \in D(B)$ and that $BM_\Psi f = M_\Psi Bf$, which means that B commutes with M_Ψ. On the other side (by [3], Corollary 6.9, p. 279), the multiplication algebra $\mathscr{A} = \{M_\Psi,\ \Psi \in L^\infty(\mu_\alpha)\}$ and its commutant coincide. It follows from ([8], Theorem 5.6.4, pp. 343–344) that $B = M_\Phi$ for some measurable function $\Phi : \mathbb{T} \to \mathbb{C}$. Now we finish the proof by applying this result to $V_\alpha^{-1} A V_\alpha$ for a closed densely defined operator A on $\mathscr{K}_u$ commuting with S_u^α and $(S_u^\alpha)^*$. $\square$

The following result gives a necessary condition on the generators of TTOs semigroups.

PROPOSITION 6.4. *If A is the generator of a C_0-semigroup of TTOs on $\mathscr{K}_u$ then A has one of the following forms:*

1. *$A = \Phi(S_u^\alpha)$ for some $\alpha \in \mathbb{T}$ and a measurable function Φ.*
2. *$A = A^u_{\frac{\varphi}{1-\alpha\overline{u}}}$ for some $\alpha \in \mathbb{D}$ and some $\varphi \in \mathscr{N}_{u\alpha}^+$.*

3. $A = A^u_{\frac{\alpha\overline{\varphi}}{\alpha-u}}$ *for some* $\alpha \in \mathbb{C}\setminus\overline{\mathbb{D}}$ *and some* $\varphi \in \mathcal{N}^+_{u_{1/\overline{\alpha}}}$.

4. $A = A^u_{\overline{\varphi}}$, *for some* $\varphi \in \mathcal{N}^+_u$.

Proof. Let A be the infinitesimal generator of a C_0-semigroup $(T_t)_{t\geqslant 0}$ of TTOs on $\mathcal{K}_u$. There exists $\alpha \in \widehat{\mathbb{C}}$ such that all the operators T_t are in $\mathcal{B}^\alpha_u$. For $|\alpha| \leqslant 1$, the operators T_t commute with S^α_u. This implies that A commutes with S^α_u. If $|\alpha| = 1$, S^α_u is unitary, whence normal, and by the Putnam-Fuglede theorem T_t commutes also with $(S^\alpha_u)^*$. Therefore A commute with $(S^\alpha_u)^*$. Now cases (1) and (2) are consequences of Proposition 6.3 and the discussion that follows Definition 6.2. We deduce (3) and (4) by considering the adjoint operator A^* which is the generator of the C_0-semigroup $(T^*_t)_{t\geqslant 0}$. □

A closed linear map A having domain $D(A)$ dense in a Hilbert space is said to be dissipative on $D(A)$ if for every $x \in D(A)$, $\Re(\langle Ax, x\rangle) \leqslant 0$ for each element f in $D(A)$. In [17], Seubert examined closed, densely defined and dissipative operators on $\mathcal{K}_u$ that commute with S^*_u. He proved that these operators are of the form $A_{\overline{C}}$ for functions C holomorphic in $\mathbb{D}$ and satisfying $\Re(C) \leqslant 0$. Such an operator is the generator of the contractive semigroup of TTOs $(A^u_{e^{tC}})_{t\geqslant 0}$. In the following theorem we describe all contractive semigroups of TTOs.

THEOREM 6.5. *Let* $(T_t)_{t\geqslant 0}$ *be a family of TTOs on* $\mathcal{K}_u$. *Then* $(T_t)_{t\geqslant 0}$ *is a* C_0-*semigroup of contractions if and only if there exists* $\alpha \in \widehat{\mathbb{C}}$, *an analytic function* C *on* $\mathbb{D}$ *with a non-positive real part and a measurable function* q *on* $\mathbb{T}$ *such that one of following assertions hold:*

1. $|\alpha| < 1$ *and* $T_t = A^u_{\frac{e^{tC}}{1-\alpha\overline{u}}}$.

2. $1 < |\alpha| < +\infty$ *and* $T_t = A^u_{\frac{\alpha e^{t\overline{C}}}{\alpha-u}}$.

3. $\alpha = \infty$ *and* $T_t = A^u_{e^{t\overline{C}}}$.

4. $|\alpha| = 1$, $\operatorname*{esssup}_{\zeta\in\mathbb{T}} \Re(q(\zeta)) \leqslant 0$ *and* $T_t = e^{tq}(S^\alpha_u)$.

Proof. Let $(T_t)_{t\geqslant 0}$ be a C_0-semigroup of contractions TTOs on $\mathcal{K}_u$. Then there exist $\alpha \in \widehat{\mathbb{C}}$, such that for every $t \geqslant 0$, $T_t \in \mathcal{B}^\alpha_u$. We have three cases:

(1) $|\alpha| < 1$. Then $(U^{-1}_\alpha T_t U_\alpha)_{t\geqslant 0}$ is a C_0-semigroup on $\mathcal{K}_{u_\alpha}$ commuting with S_{u_α}. By Theorem 2 of [16], there exits an analytic function C on $\mathbb{D}$ with a non-positive real part such that

$$U^{-1}_\alpha T_t U_\alpha = A^{u_\alpha}_{e^{tC}}, \quad t \geqslant 0.$$

Therefore

$$T_t = U_\alpha A^{u_\alpha}_{e^{tC}} U^{-1}_\alpha = A^u_{\frac{e^{tC}}{1-\alpha\overline{u}}}.$$

Note that in this case the generator of $(U_\alpha^{-1}T_tU_\alpha)_{t\geqslant 0}$ is the operator A_C and that of $(T_t)_{t\geqslant 0}$ is $A^u_{\frac{C}{1-\alpha\overline{u}}}$.

(2) $1<|\alpha|\leqslant+\infty$. Then T_t^* commutes with $S_u^{1/\overline{\alpha}}$ and using the above case we get

$$T_t=(T_t^*)^*=(A^u_{\frac{e^tC}{1-\overline{(\frac{u}{\alpha})}}})^*=A^u_{\frac{\alpha e^t\overline{C}}{\alpha-u}}$$

with generator $A^u_{\frac{\alpha\overline{C}}{\alpha-u}}$.

(3) $\alpha\in\mathbb{T}$. Then for every $t\geqslant 0$, $V_\alpha^{-1}T_tV_\alpha$ is a multiplication operator on $L^2(\mu_\alpha)$, since it commutes with the multiplication operator by z (see [3], Corollary 6.9, p. 279). By Proposition 4.11 in [5] there exists a measurable function q on $\mathbb{T}$ such that $\operatorname*{esssup}_{\zeta\in\mathbb{T}}\Re(q(\zeta))\leqslant 0$, and

$$V_\alpha^{-1}T_tV_\alpha=M_{e^{tq}}.$$

It follows that

$$T_t=V_\alpha M_{e^{tq}}V_\alpha^{-1}=e^{tq}(S_u^\alpha).$$

Since $(T_t)_{t\geqslant 0}$ is a semigroup of contractions, $\operatorname*{esssup}_{\zeta\in\mathbb{T}}\Re(q(\zeta))\leqslant 0$. Moreover $q(S_u^\alpha)$ is the generator of $(T_t)_{t\geqslant 0}$. $\square$

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