

## HALF-CENTERED OPERATORS

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*Abstract.* An operator  $T$  on a Hilbert space is called half-centered if the sequence

$$T^*T, (T^*)^2T^2, \dots$$

consists of mutually commuting operators. The class of such operators contains the well-studied class of centered operators. In this paper we give a criterion for when a half-centered operator is centered and prove a structure theorem for half-centered operators that satisfy some additional conditions.

### 1. Introduction

A bounded operator  $T$  on a Hilbert space  $\mathcal{H}$  is called *centered* if the operators in the sequence

$$\{\dots T^3T^{*3}, T^2T^{*2}, TT^*, T^*T, T^{*2}T^2, T^{*3}T^3 \dots\} \quad (1)$$

are mutually commuting. Examples include weighted shifts and obviously isometries and self-adjoint operators. The structure of these operators is well understood; it has been shown in [4] that, a bit simplified, a general centered operator is a direct sum of weighted shifts (unilateral, bilateral or truncated). Another interesting article on the subject is [6], here some particular situations are investigated in relation to more general problems in operator theory.

The purpose of this paper is to investigate operators  $T$  satisfying the more general condition that the sequence

$$\{T^*T, T^{*2}T^2, T^{*3}T^3 \dots\} \quad (2)$$

consists of mutually commuting operators. As (2) is half of (1), we call such operators *half-centered*.

We will mainly consider half-centered operators satisfying  $\dim(T\mathcal{H})^\perp = 1$  and a certain technical density condition, which is however not very restrictive. It turns out, that under these assumptions, either the structure of  $T$  is very simple and can be explicitly described, or the operators in the sequence  $\{T^{*k}T^k, k \in \mathbb{N}\}$  are not linearly

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independent. More specifically, there exists  $a, b, c, d \in \mathbb{R}$ , not all zero, and strictly positive integers  $n, m \in \mathbb{Z}^>$  such that the equation

$$aI + bT^{*n}T^n + cT^{*m}T^m + dT^{*(m+n)}T^{m+n} = 0 \quad (3)$$

holds. This is the main result of the paper, and most of this text is concerned with proving it.

In section 2, we first prove a result that gives necessary and sufficient conditions for when a half-centered operator is centered; for example, any half-centered operator with dense range is centered. We will then give several examples of classes of half-centered operators that are not necessarily centered, some of which have been extensively studied in the literature. It will also be shown that some very natural operators are half-centered. For instance, any operator  $T \in \mathcal{B}(L^2(X, \mu))$ , that acts by  $f(x) \mapsto a(x)f(\phi(x))$  where  $a \in L^\infty(X, \mu)$  and  $\phi : X \rightarrow X$  is a measurable function, is half-centered by Proposition 2.5. In Section 2 will here also state the main theorem and discuss the conditions under which it holds.

This paper is written in a decreasing level of generality. In section 3 we will develop a theory for general injective operators that is needed in the latter sections and which provides a useful framework to analyze the half-centered operators. Here we will also prove some more general results about half-centered operators which do not necessarily fall under the hypothesis of the main theorem.

Section 4 concerns injective half-centered operators  $T$  with  $\dim(T\mathcal{H})^\perp = 1$ . It will be shown that in this case, the spectrum of  $T^{*k}T^k$  restricted to certain subspaces can be quite effectively analyzed.

In the last sections, 5 and 6, we will include then density condition as an assumption and prove the main result.

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## 2. Half-centered operators: examples and notations

Clearly every centered operator is half-centered. As a first basic result, we give a characterization of the half-centered operators that are actually centered. Here we use the notation  $\mathcal{E} := \ker T^* = (T\mathcal{H})^\perp$ . For a closed subspace  $V \subseteq \mathcal{H}$ , denote by  $P_V$  the orthogonal projection onto  $V$ .

**PROPOSITION 2.1.** *Let  $T \in \mathcal{B}(\mathcal{H})$  be a half-centered operator. The following are equivalent:*

1.  $T$  is centered.
2.  $T^{k*}T^k\mathcal{E} \subseteq \mathcal{E}$  for all  $k \in \mathbb{N}$ .

*Proof.* (1  $\Rightarrow$  2). Since  $(TT^*)(T^{k*}T^k) = (T^{k*}T^k)(TT^*)$  for all  $k \in \mathbb{N}$ , it is easy to see that the space  $\mathcal{E} = \ker T^* = \ker TT^*$  is invariant under the operators  $T^{*k}T^k$ .

(2  $\Rightarrow$  1). First, we notice that by 2, the projection  $P_{\mathcal{E}}$  commutes with the operators  $T^{*k}T^k$ , this is then also true for the projection  $P_{\overline{T\mathcal{H}}} = I - P_{\mathcal{E}}$ .

Now since

$$\begin{aligned} T^*((T^{*j}T^j)(TT^*))T &= (T^{*(j+1)}T^{j+1})(T^*T) \\ &= (T^*T)(T^{*(j+1)}T^{j+1}) = T^*((TT^*)(T^{*j}T^j))T, \end{aligned}$$

we have  $(P_{\overline{T\mathcal{H}}}(T^{*j}T^j)P_{\overline{T\mathcal{H}}})(TT^*) = (TT^*)(P_{\overline{T\mathcal{H}}}(T^{*j}T^j)P_{\overline{T\mathcal{H}}})$  for all  $j \in \mathbb{N}$ . This gives

$$[T^{*j}T^j, TT^*] = P_{\mathcal{E}}(T^{*j}T^j)TT^* - TT^*(T^{*j}T^j)P_{\mathcal{E}} = 0 \tag{4}$$

as  $TT^*(T^{*j}T^j)P_{\mathcal{E}} = TT^*P_{\overline{T\mathcal{H}}}(T^{*j}T^j)P_{\mathcal{E}} = 0$  by 2, so  $T^{*k}T^k$  commutes with  $TT^*$ .

Hence for any  $k \in \mathbb{N}$ , we have

$$\begin{aligned} T^{*k}((T^{*j}T^j)(T^{k+1}T^{*(k+1)}))T^k &= (T^{*(j+k)}T^{j+k})(TT^*)(T^{*k}T^k) \\ &= (T^{*k}T^k)(TT^*)(T^{*(j+k)}T^{j+k}) = T^{*k}((T^{k+1}T^{*(k+1)})(T^{*j}T^j))T^k, \end{aligned} \tag{5}$$

where the second equality follows from (4). As

$$P_{\overline{T^k\mathcal{H}}}(T^{k+1}T^{*(k+1)})P_{\overline{T^k\mathcal{H}}} = T^{k+1}T^{*(k+1)},$$

we get from (5)

$$(P_{\overline{T^k\mathcal{H}}}(T^{*j}T^j)P_{\overline{T^k\mathcal{H}}})(T^{k+1}T^{*(k+1)}) = (T^{k+1}T^{*(k+1)})(P_{\overline{T^k\mathcal{H}}}(T^{*j}T^j)P_{\overline{T^k\mathcal{H}}}) \tag{6}$$

for all  $k, j \in \mathbb{N}$ . We claim that (6) actually implies

$$(T^kT^{*k})(T^{*j}T^j) = (T^{*j}T^j)(T^kT^{*k}) \text{ for all } j, k \in \mathbb{N}. \tag{7}$$

The proof is by induction on  $k$ . We already know that it holds for  $k = 1$ , so assume it is true for  $k - 1 \geq 1$ . Now

$$(T^{k-1}T^{*(k-1)})(T^{*j}T^j) = (T^{*j}T^j)(T^{k-1}T^{*(k-1)})$$

gives  $(T^{*j}T^j)P_{\overline{T^{k-1}\mathcal{H}}} = P_{\overline{T^{k-1}\mathcal{H}}}(T^{*j}T^j)$  since  $P_{\overline{T^{k-1}\mathcal{H}}} = I - P_{\ker(T^{k-1}T^{*(k-1)})}$ . As

$$P_{\overline{T^{k-1}\mathcal{H}}}(T^kT^{*k}) = (T^kT^{*k})$$

we see that

$$\begin{aligned} T^{*j}T^jT^kT^{*k} &= (P_{\overline{T^{k-1}\mathcal{H}}}T^{*j}T^jP_{\overline{T^{k-1}\mathcal{H}}})T^kT^{*k} \\ T^kT^{*k}T^{*j}T^j &= T^kT^{*k}(P_{\overline{T^{k-1}\mathcal{H}}}T^{*j}T^jP_{\overline{T^{k-1}\mathcal{H}}}) \end{aligned}$$

and by (6), the right hand sides are equal. Hence (6) is true also for  $k$ .

There is only the equality  $(T^kT^{*k})(T^mT^{*m}) = (T^mT^{*m})(T^kT^{*k})$  left to prove. But this follows from what has already been proven, since if, say  $m \geq k$ , then

$$\begin{aligned} (T^kT^{*k})(T^mT^{*m}) &= T^k((T^{*k}T^k)(T^{m-k}T^{*(m-k)}))T^{*k} \\ &= T^k((T^{m-k}T^{*(m-k)})(T^{*k}T^k))T^{*k} = (T^mT^{*m})(T^kT^{*k}). \end{aligned}$$

The proof is now complete.  $\square$

COROLLARY 2.2. *If  $T \in \mathcal{B}(\mathcal{H})$  is half-centered and  $\overline{T\mathcal{H}} = \mathcal{H}$ , then  $T$  is centered.*

EXAMPLE 2.3. (2-isometries) An operator  $T$  satisfying the equation

$$I - 2T^*T + T^{2*}T^2 = 0 \tag{8}$$

is called a 2-isometry. Equation (8) implies that  $T^{*k}T^k$  is a linear combination of  $I$  and  $T^*T$  for every  $k \in \mathbb{N}$  and this gives that  $T$  is half-centered. 2-isometries have been studied a lot due to their connection with the Dirichlet shift (see [1], [2], [3]). From their theory one can deduce that a centered 2-isometry must be of the form  $T = U \oplus S$ , with  $U$  an isometry and  $S$  a weighted shift. In general, 2-isometries have a quite complicated structure, so in this case the centered 2-isometries forms a strict (and quite boring) subclass.

More generally, any operator  $T$  satisfying

$$a_0I + a_1T^*T + a_2T^{*2}T^2 = 0$$

for constants  $a_0, a_1, a_2 \in \mathbb{R}$  (where at least one  $a_i \neq 0$ ) will be half-centered, since then again every  $T^{*k}T^k$  will be a linear combination of  $I$  and  $T^*T$ .

EXAMPLE 2.4. Let  $P, Q$  be two orthogonal projections and consider

$$T = PQ.$$

Then  $T$  is half-centered since

$$T^{*k}T^k = \prod_{j=1}^k QP \prod_{j=1}^k PQ = Q \prod_{j=1}^{2k-1} PQ = QT^{2k-1}$$

and so

$$\begin{aligned} (T^{*j}T^j) (T^{*k}T^k) &= QT^{2j-1}QT^{2k-1} = QT^{2k+2j-2} \\ &= QT^{2k-1}QT^{2j-1} = (T^{*k}T^k) (T^{*j}T^j). \end{aligned}$$

Now,  $TT^* = PQP$  and from this we calculate

$$(TT^*) (T^*T) = PQPQPQ = T^3$$

$$(T^*T) (TT^*) = QPQPQP = T^{*3}.$$

So if  $T^{*3} \neq T^3$  then  $T$  is half-centered but not centered. The latter holds if we take, for example

$$P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

then  $T = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{bmatrix}$  and so

$$T^3 = \begin{bmatrix} \frac{1}{2^3} & 0 \\ -\frac{1}{2^3} & 0 \end{bmatrix} \neq \begin{bmatrix} \frac{1}{2^3} & -\frac{1}{2^3} \\ 0 & 0 \end{bmatrix} = T^{*3}.$$

A large class of half-centered operators are given by the following proposition:

**PROPOSITION 2.5.** *Let  $(X, \mu)$  be a measure space with  $\sigma$ -finite measure  $\mu$  and let  $\psi : X \rightarrow X$  be a measurable function such that the linear map*

$$f(x) \mapsto f(\psi(x))$$

*induces a bounded linear operator on  $L^2(X, \mu)$  and let  $\xi \in L^\infty(X, \mu)$ . Then the operator*

$$T : f \in L^2(X, \mu) \mapsto \xi(x)f(\psi(x)) \tag{9}$$

*is half-centered.*

*Proof.* Since  $T$  is the composition of two bounded linear operators, we have  $T \in \mathcal{B}(L^2(X, \mu))$ . Take any  $h \in L^\infty(X, \mu)$  and let  $M_h$  denote multiplication by  $h$ . For all  $f, g \in L^2(X, \mu)$

$$\begin{aligned} \langle g, T^*TM_hf \rangle &= \langle Tg, TM_hf \rangle = \int_X \overline{T(g)(x)}T(hf)(x)d\mu(x) \\ &= \int_X \overline{\xi(x)g(\psi(x))}\xi(x)h(\psi(x))f(\psi(x))d\mu(x) \\ &= \int_X \overline{\xi(x)\overline{h(\psi(x))}g(\psi(x))}\xi(x)f(\psi(x))d\mu(x) \\ &= \int_X \overline{T(\overline{hg})(x)}T(f)(x)d\mu(x) \\ &= \langle TM_{\overline{h}}g, Tf \rangle = \langle TM_h^*g, Tf \rangle = \langle g, M_hT^*Tf \rangle. \end{aligned}$$

This gives  $M_hT^*T = T^*TM_h$  and since  $h$  was arbitrary  $T^*T$  commutes with all of  $L^\infty(X, \mu)$ . The von Neumann algebra  $L^\infty(X, \mu) \subseteq \mathcal{B}(L^2(X, \mu))$  is maximal abelian (see [5]) and so  $T^*T \in L^\infty(X, \mu)$ . The same argument gives  $T^{*k}T^k \in L^\infty(X, \mu)$  for every  $k \in \mathbb{N}$  and therefore the operators in the sequence (2) commute with each other.  $\square$

Notice that if, for example, the set  $\{\xi(x)f(\psi(x)); f \in L^2(X, \mu)\}$  is dense in  $L^2(X, \mu)$  then Proposition 2.1 gives that  $T$  is actually centered. However, in general, the operators defined in Proposition 2.5 will not be centered.

Before we proceed any further, let's first fix some notations.

The operators  $T^{*k}T^k$  are referred to a lot, so in order to make things appear more concise, we write them as  $T_k$ . We again denote  $\ker T^* = (T\mathcal{H})^\perp$  by  $\mathcal{E}$  and this notation will be used for the rest of the text. We also remark that in this paper we consider  $0 \in \mathbb{N}$ , and we let  $T_0 = I$ .

Next, we define a subspace that will be of utmost importance here:

Let  $\mathcal{M}_{\mathcal{E}}$  be the smallest closed subspace containing  $\mathcal{E}$  that is invariant with respect to all the operators  $T_k$ . Proposition 2.1 indicates that  $\mathcal{M}_{\mathcal{E}}$  is a natural starting point when investigating the strictly half-centered operators, since this result is saying that half-centered  $T$  is centered iff  $\mathcal{M}_{\mathcal{E}} = \mathcal{E}$ .

If we have an operator  $R \in \mathcal{B}(\mathcal{H})$  and a closed subspace  $V \subseteq \mathcal{H}$  such that  $RV \subseteq V$  and  $R^*V \subseteq V$ , then  $V$  is said to be *reducing* for  $R$ . In the case when  $R$  has no reducing subspaces,  $R$  is called *irreducible*. If  $T$  is centered, then  $\mathcal{E}$  is a reducing subspace for both  $T_k$  and  $T^k T^{*k}$ . Assuming  $\mathcal{E} \neq 0$ , then if  $T$  is centered and irreducible, we must have  $\dim \ker T^* = 1$ . This is generally not true for half-centered operators.

In this paper we will prove a structure theorem for half-centered operators  $T$  satisfying the following assumptions:

- I.  $T$  is injective and  $\mathcal{E}$  has dimension 1.
- II.  $\bigvee_{k=0}^{\infty} T^k \mathcal{M}_{\mathcal{E}} = \mathcal{H}$ .

Theorem 3.23 below shows that  $\bigvee_{k=0}^{\infty} T^k \mathcal{M}_{\mathcal{E}}$  can alternately be defined as the smallest closed subspace containing  $\mathcal{E}$  that is invariant under  $T$  and the operators  $T_k$ . However, without any further conditions this subspace will in general not be reducing for  $T$ . Notice also that these conditions imply that the Hilbert space  $\mathcal{H}$  is separable.

Spread throughout the rest of this section are some examples of half-centered operators that satisfy conditions I and II.

Let us recall the notion of *wandering subspace property* for an injective operator  $R$  on a Hilbert space  $\mathcal{H}$ . Given  $R \in \mathcal{B}(\mathcal{H})$ , let, as before,  $\mathcal{E} := \ker R^*$ , then  $R$  is said to satisfy the *wandering subspace property* if

$$\bigvee_{k=0}^{\infty} R^k \mathcal{E} = \mathcal{H}. \tag{10}$$

This condition resembles II. The subspace  $\mathcal{E}$  is often called the wandering subspace for  $R$ .

Closely related to (10) is the condition

$$\bigcap_{k=0}^{\infty} R^k \mathcal{H} = \{0\}. \tag{11}$$

If for an injective operator  $R$  with closed range we let  $R' = R(R^*R)^{-1}$ , then by results in [7] (10) holds for  $R$  iff (11) holds for  $R$ . Observe that  $\ker R^* = \ker R'^*$  and  $(R')' = R$ . The operator  $R'$  is called the Cauchy dual of  $R$ .

An important fact about injective operators satisfying (10) and having closed range is that they are unitarily equivalent to the multiplication operator  $f(z) \mapsto zf(z)$  on a Hilbert space  $\mathcal{L}(\mathcal{E})$  of  $\mathcal{E}$ -valued analytic functions (with  $\mathcal{E} = \ker R^*$ ).

The condition II is actually weaker than (10) for both  $T$  and  $T'$  since the subspace  $\bigvee_{j=0}^k T^j \mathcal{M}_{\mathcal{E}}$  contains both  $T^k \mathcal{E}$  and  $T'^k \mathcal{E}$  for every  $k \in \mathbb{N}$ . Indeed, this is trivial for  $T$ . To prove it for  $T'$ , notice that  $T^*$  is a left inverse for  $T'$ , so that  $T^{*k+1} T'^k \mathcal{E} = T^* \mathcal{E} = 0$ . This gives  $T'^k \mathcal{E} \subseteq \ker T^{*k+1}$ . It is not hard to see that  $\ker T^{*k+1}$  is spanned by the

subspaces  $T^j(T^{*j}T^j)^{-1}\mathcal{E}$  for  $0 \leq j \leq k$  and these are all subspaces of  $\bigvee_{j=0}^k T^j\mathcal{M}_{\mathcal{E}}$ . It now follows:

PROPOSITION 2.6. *II holds for  $R$  if (10) or (11) holds for  $R$  or  $R'$  (if the latter operator exists).*

For instance, this implies that if  $S$  is the shift operator on  $\ell^2$ , then as any operator of the form  $ASA^{-1}$  with  $A \in \mathcal{B}(\ell^2)$  satisfies (11), it has property II.

As two of the most distinguished cases of half-centered operators satisfying I and II are the weighted shifts and the 2-isometries (in the irreducible non-isometry case) and both of these classes of operators satisfy (10) and (11) (this claim is trivial for weighted shifts, and for 2-isometries, see [7]). It is natural to ask if (10) and (11) are true in general for a half-centered operator satisfying I and II. However, as our next example shows, this is not the case.

EXAMPLE 2.7. Let  $S$  be the isometric shift on the Hardy space  $\mathbb{H}^2$ , i.e

$$f(z) \in \mathbb{H}^2 \mapsto zf(z).$$

Now consider

$$T = aS + (I - SS^*),$$

with  $a \in \mathbb{C}$  such that  $0 < |a| < 1$ . An easy way to see that both  $T$  and  $T' = T(T^*T)^{-1}$  are half-centered is to write them down as matrices in the standard basis  $\{z^k; k \in \mathbb{N}\}$ :

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ a & 0 & 0 & 0 & \dots \\ 0 & a & 0 & 0 & \dots \\ 0 & 0 & a & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{12}$$

$$T^*T = \begin{bmatrix} 1 + |a|^2 & 0 & 0 & 0 & \dots \\ 0 & |a|^2 & 0 & 0 & \dots \\ 0 & 0 & |a|^2 & 0 & \dots \\ 0 & 0 & 0 & |a|^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{13}$$

$$T' = \begin{bmatrix} \frac{1}{1+|a|^2} & 0 & 0 & 0 & \dots \\ \frac{a}{1+|a|^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{a}{|a|^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{a}{|a|^2} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{14}$$

It is not hard to see now that for all  $k \in \mathbb{N}$ , both matrices  $T^{*k}T^k$  and  $T'^{*k}T'^k$  are diagonal. From (12), we see that  $\ker T^*$  is spanned by  $\bar{a} - z$  and from (2.7) that  $T^*T -$

$|a|^2 I$  is the operator  $f(z) \mapsto (1 + |a|^2)f(0)$ . Thus  $1, z \in \mathcal{M}_{\mathcal{E}}$  and since  $T^k z = a^k z^k$ , this gives

$$\bigvee_{k=0}^{\infty} T^k \mathcal{M}_{\mathcal{E}} = \mathbb{H}^2.$$

Hence both I and II are fulfilled by  $T$ . However, as

$$\sum_{k=0}^{\infty} a^k z^k = \frac{1}{a - z} \in \mathbb{H}^2$$

is an eigenvector for  $T$  and thus in the range of  $T^k$  for all  $k \in \mathbb{N}$ ,  $T$  does not satisfy (11) and hence the Cauchy dual  $T'$  does not possess the wandering subspace property.

EXAMPLE 2.8. The operator in Example 2.7 is a special case of a more general type of half-centered operator. Let  $\mathcal{H}$  be a separable Hilbert space with an orthonormal basis  $\{x_k : k \in \mathbb{N}\}$  and inner product  $\langle \cdot, \cdot \rangle$ . Let  $J$  be an injective weighted shift operator with respect to this basis, so that

$$Jx_k = a_k x_{k+1}$$

for some nonzero constants  $a_k \in \mathbb{C}$ . If  $x_0 \otimes x_n^*$  denotes the operator  $x \mapsto \langle x, x_n \rangle x_0$ , then for any  $n \in \mathbb{N}$  and  $a \in \mathbb{C}$ , the operator

$$T = J + a(x_0 \otimes x_n^*) \tag{15}$$

is half-centered.

In fact, the operator (15) can be seen to be of type (9) if we view  $\mathcal{H}$  as  $L^2(\mathbb{N}, \mu)$ , where  $\mu$  is the counting measure. Define  $\psi_n : \mathbb{N} \rightarrow \mathbb{N}$  by  $\psi_n(k) = k - 1$  if  $k \geq 1$  and  $\psi_n(0) = n$  and let  $\xi(k) = a_{k-1}$  if  $k \geq 1$  and  $\xi(0) = a$ . It is not hard to see that the operator

$$f(x) \mapsto \xi(x)f(\psi_n(x))$$

coincides with the operator (15). Hence, by Proposition 2.5, the latter is half-centered.

### 2.1. The main theorem

The main purpose of this paper is to prove the following result.

THEOREM 2.9. (Main) *Let  $T$  be an injective half-centered operator on  $\mathcal{H}$  such that  $\bigvee_{k=0}^{\infty} T^k \mathcal{M}_{\mathcal{E}} = \mathcal{H}$  and  $\dim \mathcal{E} = 1$ .*

*Then there are two possibilities (though not mutually exclusive).*

1. *There is an orthonormal basis  $\{x_k : k \in \mathbb{N}\}$  of common eigenvectors for the operators  $\{T_k\}_{k \in \mathbb{N}}$  such that with respect to this basis,  $T$  is either a weighted shift or there is a weighted shift  $J$  such that*

$$T = J + a(x_0 \otimes x_n^*) \tag{16}$$

*for  $n \in \mathbb{N}$  and  $a \in \mathbb{C}$ .*



2. There are constants  $a, b, c, d \in \mathbb{R}$ , not all zero and  $k, n \in \mathbb{N}^+$  such that

$$aI + bT^{*k}T^k + cT^{*n}T^n + dT^{*k+n}T^{k+n} = 0. \tag{17}$$

Moreover, if  $\dim \mathcal{M}_{\mathcal{E}} \geq 3$  then (17) holds with  $a \neq 0$  and the range of  $T$  is closed.

REMARK 2.10. Notice that if  $\dim \mathcal{M}_{\mathcal{E}} = 1$  then  $\mathcal{M}_{\mathcal{E}} = \mathcal{E}$  and hence  $T^{*k}T^k \mathcal{E} \subseteq \mathcal{E}$  for all  $k \in \mathbb{N}$ . By Proposition 2.1,  $T$  is centered and the condition  $\bigvee_{k=0}^{\infty} T^k \mathcal{E} = \mathcal{H}$  gives that  $T$  is a weighted shift.

So far, we have not given any concrete example of a half-centered operator where  $\dim \mathcal{M}_{\mathcal{E}} \geq 3$ . In order to show that this class is not just void, we construct below a half-centered operator having the property that  $\mathcal{M}_{\mathcal{E}}$  is the whole space.

EXAMPLE 2.11. Let  $\mathcal{H} = \ell^2$  with standard basis  $\{e_k : k \in \mathbb{N}\}$  and let  $S$  be the shift operator. For  $0 < q < 1$ , let  $A_q$  be the operator that in the standard basis can be written as the infinite matrix

$$A_q = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & q & 0 & \dots \\ 0 & q & 0 & q^2 & \dots \\ 0 & 0 & q^2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{18}$$

Since  $0 < q < 1$ , it is straightforward to deduce that  $A_q$  is a compact self-adjoint operator. Moreover, it is easy to see that

$$S^*A_qS = qA_q \tag{19}$$

and  $\ker A_q = \{0\}$ . Thus  $\ell^2$  has an orthonormal basis  $\{x_k : k \in \mathbb{N}\}$  consisting of eigenvectors for  $A_q$  and we can easily deduce that  $\langle x_k, e_0 \rangle \neq 0$  for all  $k$  which implies that every eigenspace of  $A_q$  must be one-dimensional.

Since  $A_q$  is self-adjoint, there is  $r > 0$  such that  $A_q + rI$  is invertible and positive. Now let

$$T = (A_q + rI)^{\frac{1}{2}}S(A_q + rI)^{-\frac{1}{2}}. \tag{20}$$

Then

$$T^n = (A_q + rI)^{\frac{1}{2}}S^n(A_q + rI)^{-\frac{1}{2}} \tag{21}$$

and so by (19), we see that

$$T^{*n}T^n = (A_q + rI)^{-\frac{1}{2}}(q^n A_q + rI)(A_q + rI)^{-\frac{1}{2}} \tag{22}$$

from which it follows that  $(T^{*n}T^n)(T^{*m}T^m) = (T^{*m}T^m)(T^{*n}T^n)$  for  $m, n \geq 0$  and hence  $T$  is half-centered. Furthermore, if  $\lambda_k$  is the eigenvalue of the eigenvector  $x_k$  for  $A_q$ , then  $x_k$  is clearly an eigenvector for  $T^{*n}T^n$ , with eigenvalue  $\frac{q^n \lambda_k + r}{\lambda_k + r}$ . Since the

function  $\frac{q^n x+r}{x+r}$  is one to one on  $(-r, \infty)$ , we get that  $T^{*n}T^n$  has only one-dimensional eigenspaces. From the formula (21), we have

$$\mathcal{E} = \ker T^* = (A_q + rI)^{-\frac{1}{2}} e_0$$

giving  $\langle \mathcal{E}, x_k \rangle \neq 0$  for all  $k$ . If  $V$  were a nontrivial closed subspace, invariant under the  $T_k$ 's and orthogonal to  $\mathcal{E}$ , then  $V$  would have to contain a nonzero eigenvector  $x_m$  of  $A_q$ , giving  $\langle \mathcal{E}, x_m \rangle = 0$ , a contradiction. Since the operators  $T_k$  are all self adjoint, also  $\mathcal{M}_{\mathcal{E}}^\perp$  is invariant with respect to them and so by the last sentence, we must have  $\mathcal{M}_{\mathcal{E}}^\perp = \{0\}$  giving  $\mathcal{M}_{\mathcal{E}} = \ell^2$ .

It can be seen from (21) that the operator defined in Example 2.11 satisfies the equation

$$I - (1 + q^{-1})T^*T + q^{-1}T^{*2}T^2 = 0. \tag{23}$$

This is similar to the one that defines the 2-isometries. Indeed, the 2-isometries are a natural occurring example where often  $\dim \mathcal{M}_{\mathcal{E}} \geq 3$ , although the way they usually are constructed makes this a bit cumbersome to check.

### 3. Theory for general injective operators

Before we can tackle the main theorem we must first build up some machinery.

While the theory presented in this section was developed specifically to deal with the half-centered operators, it turned out that it could, with minor extra work, be generalized to a more general setting. Hence it is presented in this fashion.

Let us fix some more notation:

Let  $\mathcal{H}$  be a separable complex Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and  $\mathcal{B}(\mathcal{H})$  be the  $C^*$ -algebra of bounded operators on  $\mathcal{H}$ . Let  $R \in \mathcal{B}(\mathcal{H})$  be a fixed bounded injective linear operator. As in the introduction, we denote  $\ker R^*$  by  $\mathcal{E}$ , and the smallest subspace containing  $\mathcal{E}$  that is also invariant with respect to the set of operators  $\{R^{*k}R^k; k \in \mathbb{N}\}$  is denoted by  $\mathcal{M}_{\mathcal{E}}$ . Throughout the rest of this paper the letter  $T$  will be reserved for injective half-centered operators. Given a closed subspace  $V$  of the Hilbert space  $\mathcal{H}$  we write  $P_V$  for the orthogonal projection onto  $V$ . Also, for an operator  $B$  and a subspace  $V$  of  $\mathcal{H}$  we write the restriction of  $B$  to  $V$  as  $B|_V$  (or sometimes, to avoid multiple index, we write  $B|V$  instead). Notice that if  $V$  is an invariant subspace for  $B$  then

$$(B|_V)^k = B^k|_V$$

for all  $k \in \mathbb{N}$ . When we have an algebra of operators  $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$  and a subspace  $V$  which is invariant under all operators in  $\mathcal{A}$ , then  $\mathcal{A}|_V \subseteq \mathcal{B}(V)$  is the algebra of all operators that are elements in  $\mathcal{A}$  restricted to  $V$ .

The main idea of this section is to decompose the subspace  $\bigvee_{m=0}^\infty R^m \mathcal{M}_{\mathcal{E}}$  into a direct sum  $\bigoplus_{m=0}^\infty V_m$  of orthogonal subspaces  $V_m$  with  $V_0 = \mathcal{M}_{\mathcal{E}}$ , such that  $R$  acts on each  $V_m$  in a “reasonable” predictable way. Moreover, each  $V_m$  will be an invariant subspace for all the operators  $R^{*k}R^k$ . We will furthermore show that there is a strong relation between the restrictions of  $R^{*k}R^k$  to different  $V_m$ 's in the sense that there is a

natural surjective homomorphism from a sub-algebra of the von Neumann algebra generated by the operators  $\{R^{*k}R^k|V_m; k \in \mathbb{N}\}$  onto the von Neumann algebra generated by  $\{R^{*k}R^k|V_n; k \in \mathbb{N}\}$  when  $n \geq m$ . This construction makes up the technical core of this text, and it will take some time to complete.

**3.1. The  $C^*$ -algebras  $\mathbb{M}_{R,n}$  and  $\mathbb{M}_R^n$**

The purpose of this subsection is to introduce two sequences of  $C^*$ -algebras  $\mathbb{M}_{R,n}$  and  $\mathbb{M}_R^n$ , both indexed over  $\mathbb{N}$ . We refer the reader to [5] for the background on  $C^*$ -algebras.

We also remind the reader of the notation

$$R_k = R^{*k}R^k$$

that will be used for the remainder of the text. Note that if  $V$  is an invariant subspace for  $R$ , then

$$(R|_V)_k = (R|_V)^{*k}(R|_V)^k = P_V R^{*k}R^k P_V|_V = R_k|_V. \tag{24}$$

We will for technical reasons often not differentiate between the restriction of an operator  $A$  to a subspace  $V$  and  $P_V A P_V$ , so for example, we write the equality (24) as  $(R|_V)_k = P_V R_k P_V$ . This is hopefully never a source of confusion. To further simplify notation, we write

$$\mathcal{H}_n = \overline{R^n \mathcal{H}}.$$

Notice that although we may have  $\mathcal{H}_1 \neq \mathcal{H}$ , this does not in general imply  $\mathcal{H}_{n+1} \neq \mathcal{H}_n$  for all  $n \in \mathbb{N}$ .

Next, we are going to define some of the main objects studied in this section:

Let  $\mathbb{M}_R$  be the von Neumann algebra generated by the operators  $R_k$  for all  $k \in \mathbb{N}$ .

If  $\theta_R$  is the isometric part of the polar decomposition of  $R$  i.e  $R = \theta_R R_1^{\frac{1}{2}}$ , let  $\mathbb{M}_R^1$  be the von Neumann algebra generated by the operators

$$\theta_R^* R_j \theta_R \text{ for all } j \in \mathbb{N}.$$

If  $R$  has a closed range, then  $R_1$  is invertible, so  $\theta_R = R R_1^{-\frac{1}{2}}$  and thus in this case we have

$$\theta_R^* R_j \theta_R = R_1^{-\frac{1}{2}} R_{j+1} R_1^{-\frac{1}{2}} \in \mathbb{M}_R.$$

So for closed range  $R$  it is easy to see that  $\mathbb{M}_R^1$  is a sub-algebra of  $\mathbb{M}_R$ . This is also true in general:

**PROPOSITION 3.1.** *The von Neumann algebra  $\mathbb{M}_R^1$  is a sub-algebra of  $\mathbb{M}_R$ . Moreover,  $\mathbb{M}_R^1$  is isomorphic to  $\mathbb{M}_{R|_{\mathcal{H}_1}}$ .*

*Proof.* Since  $\mathbb{M}_R$  is von Neumann algebra, we have by the double commutant theorem

$$(\mathbb{M}_R^1)' = \mathbb{M}_R$$

where  $\mathcal{A}'$  denotes the commutant of the algebra  $\mathcal{A}$ . Let  $m$  be an element in  $\mathbb{M}'_R$ . Since  $\theta_R R^{\frac{1}{2}} = R$ , we have

$$R^{\frac{1}{2}} \theta_R^* R_j \theta_R R^{\frac{1}{2}} = R^* R_j R = R_{j+1}.$$

Thus

$$R^{\frac{1}{2}} \theta_R^* R_j \theta_R m R^{\frac{1}{2}} = R^{\frac{1}{2}} \theta_R^* R_j \theta_R R^{\frac{1}{2}} m = m R^{\frac{1}{2}} \theta_R^* R_j \theta_R R^{\frac{1}{2}} = R^{\frac{1}{2}} m \theta_R^* R_j \theta_R R^{\frac{1}{2}}.$$

If  $R$  is injective then the range of  $R^{\frac{1}{2}}$  is dense in  $\mathcal{H}$ , this gives

$$\theta_R^* R_j \theta_R m = m \theta_R^* R_j \theta_R$$

for all  $m \in \mathbb{M}'_R$  so that

$$\theta_R^* R_j \theta_R \in \mathbb{M}''_R = \mathbb{M}_R.$$

For the second claim, note that the map

$$B \in \mathcal{B}(\mathcal{H}) \mapsto \theta_R B \theta_R^*$$

is an isomorphism  $\mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}_1)$  such that

$$\theta_R^* R_j \theta_R \mapsto \theta_R \theta_R^* R_j \theta_R \theta_R^* = P_{\mathcal{H}_1} R_j P_{\mathcal{H}_1} = (R|_{\mathcal{H}_1})_j$$

for all  $j \in \mathbb{N}$ . Since  $\mathbb{M}_{R|_{\mathcal{H}_1}}$  is generated by these operators and the map is weakly continuous, the range must be equal to  $\mathbb{M}_{R|_{\mathcal{H}_1}}$ .  $\square$

By Propositions 3.1, there is an injective homomorphism

$$\mathbb{M}_R \leftarrow \mathbb{M}_{R|_{\mathcal{H}_1}}$$

$$(R|_{\mathcal{H}_1})_j = P_{\mathcal{H}_1} R_j P_{\mathcal{H}_1} \mapsto \theta_R^* R_j \theta_R.$$

If we now consider  $R|_{\mathcal{H}_1}$  instead, we get by the same reasoning that there is an injective homomorphism

$$\mathbb{M}_{R|_{\mathcal{H}_1}} \leftarrow \mathbb{M}_{R|_{\mathcal{H}_2}}.$$

So by induction, there is a sequence of injective homomorphisms

$$\mathbb{M}_R \leftarrow \mathbb{M}_{R|_{\mathcal{H}_1}} \leftarrow \mathbb{M}_{R|_{\mathcal{H}_2}} \leftarrow \mathbb{M}_{R|_{\mathcal{H}_3}} \leftarrow \mathbb{M}_{R|_{\mathcal{H}_4}} \leftarrow \dots \tag{25}$$

where the  $n$ 'th arrow is induced by  $\theta_{R|_{\mathcal{H}_{n-1}}} : \mathcal{H}_{n-1} \rightarrow \mathcal{H}_n$ . Since the maps in (25) are all injective, we can deduce

PROPOSITION 3.2. *If  $T$  is half-centered, then  $T|_{\mathcal{H}_n}$  is also half-centered.*

We see that the composition  $\mathbb{M}_R \leftarrow \mathbb{M}_{R|\mathcal{H}_n}$  is induced by an isometry  $\theta_{R,n} : \mathcal{H} \rightarrow \mathcal{H}_n$  given by the product

$$\theta_{R,n} = \theta_{R|\mathcal{H}_n} \cdot \theta_{R|\mathcal{H}_{n-1}} \cdots \theta_{R|\mathcal{H}_0}. \tag{26}$$

We set  $\theta_{R,1} = \theta_R$  and  $\theta_{R,0} = I$ . We will identify  $\theta_{R,n}$  with the map on  $\mathcal{H}$  given by

$$x \in \mathcal{H} \mapsto (0, \theta_{R,n}x) \in \mathcal{H}_n^\perp \oplus \mathcal{H}_n = \mathcal{H},$$

so that

$$\theta_{R,n}\theta_{R,n}^* = P_{\mathcal{H}_n}.$$

More generally,  $\theta_{R|\mathcal{H}_n}$  is interpreted as a partial isometry (that fails to be left-invertible if  $\mathcal{H}_n \neq \mathcal{H}$ ) that is zero on  $\mathcal{H}_n^\perp$  and maps  $\mathcal{H}_n \rightarrow \mathcal{H}_{n+1}$ .

For a half-centered operator  $T$  the isometries (26) can be described as follows.

PROPOSITION 3.3. *If  $T$  is injective and half-centered, and  $T^n = \theta_{T^n}T_n^{\frac{1}{2}}$  is the polar decomposition of  $T^n$ , then*

$$\theta_{T,n} = \theta_{T^n}.$$

Therefore, if  $T$  has closed range, then

$$\theta_{T,n} = T^n T_n^{-\frac{1}{2}}.$$

The proof will be given after we prove Lemma 3.11.

REMARK 3.4. An important result in the theory of centered operators is that  $\theta_T^n = \theta_{T^n}$ ; the above proposition can be seen as a generalization of this.

Next we define a class of sub-algebras of  $\mathbb{M}_R$ .

DEFINITION 3.5. For every  $n \in \mathbb{N}$ , we define the von Neumann algebra  $\mathbb{M}_R^n$  to be the weakly closed sub-algebra of  $\mathbb{M}_R$  generated by the operators  $\theta_{R,n}^* R_j \theta_{R,n}$ .

REMARK 3.6. By Lemma 3.8 below, this algebra can alternatively be defined as the image of  $\mathbb{M}_{R|\mathcal{H}_n}$  inside  $\mathbb{M}_R$  under the composition of homomorphisms in (25).

We write down some direct consequences the preceding definitions:

LEMMA 3.7. *For all  $k, n \in \mathbb{N}$*

$$\theta_{R|\mathcal{H}_{k,n}} \theta_{R,k} = \theta_{R,n+k} \tag{27}$$

LEMMA 3.8. *The image of  $(R|\mathcal{H}_n)_k \in \mathbb{M}_{R|\mathcal{H}_n}$  in  $\mathbb{M}_R$  is  $\theta_{R,n}^* R_k \theta_{R,n}$ , and hence  $\mathbb{M}_R^n$  is isomorphic to  $\mathbb{M}_{R|\mathcal{H}_n}$  via compositions of the homomorphisms in (25).*

*Proof.* We have  $(R|_{\mathcal{H}_n})_k = P_{\mathcal{H}_n} R_k P_{\mathcal{H}_n}$  since  $\mathcal{H}_n$  is an invariant subspace for  $R$  and also  $P_{\mathcal{H}_n} \theta_{R,n} = \theta_{R,n}$ . Therefore the image of  $(R|_{\mathcal{H}_n})_k$  in  $\mathbb{M}_R$  is given by  $\theta_{R,n}^* R_k \theta_{R,n}$ . The second part is obvious as the operators  $(R|_{\mathcal{H}_n})_k$  generate  $\mathbb{M}_{R|_{\mathcal{H}_n}}$  and the homomorphisms in (25) are weakly continuous.  $\square$

**COROLLARY 3.9.** *If  $T$  is half-centered and has closed range, then  $\mathbb{M}_R^n$  is generated by the operators  $T_{k+n} T_n^{-1}$  as a von Neumann algebra.*

*Proof.* By Proposition 3.3

$$\theta_{T,n} = T T_n^{-\frac{1}{2}}.$$

From this we get

$$\theta_{T,n}^* T_k \theta_{T,n} = T_{k+n} T_n^{-1}. \quad \square$$

Next, we introduce another class of  $C^*$ -algebras  $\mathbb{M}_{R,n}$ , where  $n \in \mathbb{N}$ , which are also associated to  $R$ . These algebras will in general be non-unital weakly closed sub-algebras of  $\mathcal{B}(\mathcal{H})$  that have  $\mathcal{H}_n$  as an invariant subspace and  $\mathbb{M}_{R,n} \mathcal{H}_n^\perp = 0$ . Moreover,  $\mathbb{M}_{R,n}|_{\mathcal{H}_n}$  is a von Neumann algebra such that  $\mathbb{M}_{R,n}|_{\mathcal{H}_n} \cong \mathbb{M}_R$  by Proposition 3.12 below.

For every  $n \in \mathbb{N}$ , take the set of operators

$$R^n \mathbb{M}_R R^{*n} = \{R^{*n} a R^n : a \in \mathbb{M}_R\}$$

and let  $\mathbb{M}_{R,n}$  to be the weak closure of this set. We let  $\mathbb{M}_R = \mathbb{M}_{R,0}$ .

**LEMMA 3.10.**  *$\mathbb{M}_{R,n}$  is a  $C^*$ -algebra.*

*Proof.* Additive and adjoint closeness are obvious. If  $a, b \in \mathbb{M}_R$ , then

$$R^n a R^{*n} R^n b R^n = R^n c R^{*n}$$

with  $c = a R^{*n} R^n b \in \mathbb{M}_R$ . The rest follows now from continuity.  $\square$

Next, we will see that  $\theta_{R,n}$  induces an isomorphism between  $\mathbb{M}_R$  and  $\mathbb{M}_{R,n}$  given by the mapping

$$m \mapsto \theta_{R,n} m \theta_{R,n}^*.$$

To prove this, we first need a technical lemma.

**LEMMA 3.11.** *For every  $n \in \mathbb{N}$ , there is an operator  $r_n \in \mathbb{M}_R$  such that*

$$\theta_{R,n} r_n = R^n. \tag{28}$$

and

$$r_n^* r_n = R_n. \tag{29}$$

Moreover,  $r_n$  has dense range and is given by the formula

$$r_n = (\theta_{R,n-1}^* R_1 \theta_{R,n-1})^{\frac{1}{2}} \cdot (\theta_{R,n-2}^* R_1 \theta_{R,n-2})^{\frac{1}{2}} \cdot \dots \cdot (\theta_{R,0}^* R_1 \theta_{R,0})^{\frac{1}{2}}. \tag{30}$$

*Proof.* We use induction. For  $n = 1$ , then  $\theta_{R,1} = \theta_R$  and so  $r_n = R_1^{\frac{1}{2}}$ . Now assume (30) is true for  $n \geq 1$ , then

$$R^{n+1} = RP_{\mathcal{H}_n}R^n = RP_{\mathcal{H}_n}\theta_{R,n}r_n.$$

We have

$$RP_{\mathcal{H}_n} = \theta_{R|\mathcal{H}_n}(P_{\mathcal{H}_n}R_1P_{\mathcal{H}_n})^{\frac{1}{2}}.$$

Since  $P_{\mathcal{H}_n} = \theta_{R,n}\theta_{R,n}^*$  and  $m \mapsto \theta_{R,n}m\theta_{R,n}^*$  is a homomorphism of  $C^*$ -algebras (recall that  $\theta_{R,n}$  is an isometry), we have

$$\begin{aligned} \theta_{R|\mathcal{H}_n}(P_{\mathcal{H}_n}RP_{\mathcal{H}_n})^{\frac{1}{2}} &= \theta_{R|\mathcal{H}_n}\theta_{R,n}(\theta_{R,n}^*R_1\theta_{R,n})^{\frac{1}{2}}\theta_{R,n}^* \\ &= \theta_{R,n+1}(\theta_{R,n}^*R_1\theta_{R,n})^{\frac{1}{2}}\theta_{R,n}^*. \end{aligned}$$

Putting this together, we get

$$R^{n+1} = \theta_{R,n+1}(\theta_{R,n}^*R_1\theta_{R,n})^{\frac{1}{2}}r_n.$$

From this (28), (29) and (30) follow for  $n + 1$ . Since every operator  $(\theta_{R,k}^*R_1\theta_{R,k})^{\frac{1}{2}}$  has dense range, the same is true for their product  $r_n$ .  $\square$

We can now prove Proposition 3.3. Let  $t_n \in \mathbb{M}_T$  be the operator from Lemma 3.11 such that  $\theta_{T,n}t_n = T^n$ . As

$$t_n = \prod_{k=0}^{n-1} (\theta_{T,k}^*T_1\theta_{T,k})^{\frac{1}{2}}$$

and every  $(\theta_{T,k}^*T_1\theta_{T,k})^{\frac{1}{2}} \in \mathbb{M}_T$ ,  $t_n$  is a product of positive operators that commute with each other, hence it is also positive. Now, since

$$(t_n\theta_{T,n}^*)(\theta_{T,n}t_n) = t_n^2 = T_n,$$

we must have  $t_n = T_n^{\frac{1}{2}}$ , by the uniqueness of the square root of a positive operator. So  $\theta_{T,n}T_n^{\frac{1}{2}} = \theta_{T,n}t_n = T^n$  and as  $T_n^{\frac{1}{2}}$  has dense range, we have  $\theta_{T,n} = \theta_{T^n}$ .

**PROPOSITION 3.12.** *For every  $n \in \mathbb{N}$ , the homeomorphism  $m \mapsto \theta_{R,n}m\theta_{R,n}^*$  is an isomorphism*

$$\mathbb{M}_R \rightarrow \mathbb{M}_{R,n}.$$

*Proof.* For any  $c \in \mathbb{M}_R$ , we have  $r_n c r_n^* \in \mathbb{M}_R$  and this operator is mapped to  $R^n c R^{*n}$  by Lemma 3.11. The homomorphism preserves weak closure (since it is induced by an isometry) so

$$\mathbb{M}_{R,n} \subseteq \theta_{R,n}\mathbb{M}_R\theta_{R,n}^*.$$

To prove the reverse inclusion, take any  $m \in \mathbb{M}_R$ . Since  $r_n r_n^*$  has dense range, there is a sequence of self-adjoint  $y_k \in \mathbb{M}_R$  such that

$$y_k r_n r_n^*, r_n r_n^* y_k \rightarrow I$$

strongly in  $\mathcal{H}$  as  $k \rightarrow \infty$  (this follows from a basic application of the general spectral theorem). Now take the product

$$r_n^* y_k m y_k r_n \in \mathbb{M}_R$$

for every  $k \in \mathbb{N}$ . Then we have

$$R^n (r_n^* y_k m y_k r_n) R^{*n} \in \mathbb{M}_{R,n}$$

for all  $k \in \mathbb{N}$ . But since  $R^{*n} = r_k^* \theta_{R,n}^*$ , we get

$$R^n (r_n^* y_k m y_k r_n) R^{*n} = \theta_{R,n} (r_n r_n^* y_k) m (y_k r_n r_n^*) \theta_{R,n}^* \rightarrow \theta_{R,n} m \theta_{R,n}^*$$

strongly. So  $\theta_{R,n} m \theta_{R,n}^* \in \mathbb{M}_{R,n}$  and thus

$$\theta_{R,n} \mathbb{M}_R \theta_{R,n}^* = \mathbb{M}_{R,n}. \quad \square$$

A consequence can be directly drawn from Proposition 3.12.

**COROLLARY 3.13.** *For every  $n \in \mathbb{N}$ , the  $C^*$ -algebra  $\mathbb{M}_{R|\mathcal{H}_n}$  is a sub-algebra of  $\mathbb{M}_{R,n}$ .*

**REMARK 3.14.** Similar to what was mentioned in the introduction to this subsection, we mostly view  $\mathbb{M}_{R,n}$  and  $\mathbb{M}_{R|\mathcal{H}_n}$  as non-unital weakly closed  $C^*$ -algebras in  $\mathcal{B}(\mathcal{H})$  rather than unital  $C^*$ -algebras in  $\mathcal{B}(\mathcal{H}_n)$  that perhaps would seem more natural. This is because in the upcoming sections, the main job of these algebras are to act on  $\mathcal{H}$  and therefore it would be cumbersome if we always first have to project down to  $\mathcal{H}_n$  before they can be applied.

### 3.2. A subspace decomposition

Here we will first decompose the Hilbert space  $\mathcal{H}$  into  $\mathcal{H}_\mathcal{E} \oplus \mathcal{H}_\mathcal{E}^\perp$ , where  $\mathcal{H}_\mathcal{E}$  is the smallest closed subspace containing  $\mathcal{E}$  that is invariant with respect to both  $R$  and  $\mathbb{M}_R$ . We then show that there is a further decomposition of  $\mathcal{H}_\mathcal{E}$  into orthogonal subspaces

$$\mathcal{H}_\mathcal{E} = \bigoplus_{k=0}^\infty V_k$$

with  $V_0 = \mathcal{M}_\mathcal{E}$  such that all the  $V_k$ 's are invariant subspaces for the algebra  $\mathbb{M}_R$ . The important point of this construction emerges in the next subsection where we show that  $\mathbb{M}_R|V_k$  and  $\mathbb{M}_R|V_0$  are related in a certain way.

From now on  $R$  will, as well as being injective, also be subject to the condition  $\mathcal{E} \neq 0$  (recall  $\mathcal{E} = \mathcal{H}_1^\perp = (R\mathcal{H})^\perp = \ker R^*$ ).



Also recall from the introduction that  $\mathcal{M}_\mathcal{E}$  was defined as the linear closure of

$$my, \quad m \in \mathbb{M}_R, \quad y \in \mathcal{E}.$$

For notational purposes, we sometimes abbreviate this as  $\mathbb{M}_R \mathcal{E}$  and this notation will be used from now on in general, when we have a  $C^*$ -algebra or a set of operators acting on some subspace. All subspaces here will be considered as norm-closed, unless explicitly stated otherwise. So, for example, given subspaces  $V, X \subseteq \mathcal{H}$  the subspace  $RX + V$  will denote the norm-closure of

$$\{Rx + v; x \in X, v \in V\}.$$

We remark that, as  $\mathcal{E} \subseteq \mathcal{M}_\mathcal{E}$ , the subspace  $\mathcal{M}_\mathcal{E}$  is invariant for  $P_{\mathcal{H}_1}$  and hence also an invariant subspace for the operators

$$(R|_{\mathcal{H}_1})_k = P_{\mathcal{H}_1} R_k P_{\mathcal{H}_1}.$$

LEMMA 3.15. *For all  $m \in \mathbb{N}$  and  $n \leq m$*

$$\mathbb{M}_{R,m} R^m \mathcal{E} = R^n \mathbb{M}_{R,m-n} R^{m-n} \mathcal{E} = R^m \mathcal{M}_\mathcal{E}.$$

*Proof.* We will prove the equality

$$\mathbb{M}_{R,m} R^m \mathcal{E} = R^m \mathcal{M}_\mathcal{E}. \tag{31}$$

The rest of the Lemma then follows from

$$R^n \mathbb{M}_{R,m-n} R^{m-n} \mathcal{E} = R^n \mathbb{M}_{R,m-n} R^{m-n} \mathcal{E} = R^n (R^{m-n} \mathcal{M}_\mathcal{E}) = R^m \mathcal{M}_\mathcal{E}.$$

Since  $\mathbb{M}_{R,m} \mathcal{E}$  and  $R^m \mathcal{M}_\mathcal{E}$  are both closed subspaces of  $\mathcal{H}_m$  and  $P_{\mathcal{H}_m} = \theta_{R,m} \theta_{R,m}^*$ , we can prove (31) by proving that

$$\theta_{R,m}^* \mathbb{M}_{R,m} R^m \mathcal{E} = \mathcal{M}_\mathcal{E} = \theta_{R,m}^* R^m \mathcal{M}_\mathcal{E}. \tag{32}$$

But since by Proposition 3.12  $\mathbb{M}_{R,m} = \theta_{R,m}^* \mathbb{M}_R \theta_{R,m}^*$  and  $\theta_{R,m}^* R^m = r_m$  where  $r_m \in \mathbb{M}_R$  is as in Lemma 3.11, we have  $\theta_{R,m}^* R^m \mathcal{M}_\mathcal{E} = r_m \mathcal{M}_\mathcal{E}$  and

$$\theta_{R,m}^* \mathbb{M}_{R,m} R^m \mathcal{E} = \mathbb{M}_R r_m \mathcal{E}.$$

From this it is now obvious that they are both subspaces of  $\mathcal{M}_\mathcal{E}$ . Since the range of  $r_m$  is dense in  $\mathcal{H}$  and  $\mathcal{M}_\mathcal{E}$  is an invariant subspace for both  $r_m$  and  $r_m^*$ , we must have  $r_m \mathcal{M}_\mathcal{E} = \mathcal{M}_\mathcal{E}$ . So the second equality in (32) is proven. To prove the first, recall  $r_m^* r_m = R_m$  and take a sequence  $\{a_k\} \subseteq \mathbb{M}_R$  such that  $a_k R_m \rightarrow I$  strongly. Then the sequence  $\{a_k r_m^*\} \subseteq \mathbb{M}_R$  is such that  $(a_k r_m^*) r_m \rightarrow I$  strongly. From this we see that  $\mathcal{E} \subseteq \mathbb{M}_R r_m \mathcal{E}$  and since the space in question is also invariant under  $\mathbb{M}_R$ , it must be equal to  $\mathcal{M}_\mathcal{E}$ .  $\square$

LEMMA 3.16. *Let  $\mathcal{E}_n = \mathcal{H}_n \ominus \mathcal{H}_{n+1}$  be the kernel of  $R^*$  restricted to  $\mathcal{H}_n$ . For all  $n \in \mathbb{N}$ , we have  $\mathcal{E}_n \subseteq R^n \mathcal{M}_\mathcal{E}$ .*

*Proof.* Since  $\mathcal{E}_n \perp \mathcal{H}_{n+1}$ , we have for any  $x \in \mathcal{H}$  and  $e \in \mathcal{E}_n$  that

$$0 = \langle e, R^{n+1}x \rangle = \langle R^{*n}e, Rx \rangle.$$

From this, we see  $R^{*n}\mathcal{E}_n \subseteq (R\mathcal{H})^\perp = \mathcal{E}$  so that  $R^n\mathbb{M}_R R^{*n}\mathcal{E}_n \subseteq R^n\mathcal{M}_\mathcal{E}$ . Since  $\mathbb{M}_{R,n}$  is the weak closure of the set  $R^n\mathbb{M}_R R^{*n}$ , we get  $\mathbb{M}_{R,n}\mathcal{E}_n \subseteq R^n\mathcal{M}_\mathcal{E}$ . Now, we have  $P_{\mathcal{H}_n} \in \mathbb{M}_{R,n}$  and so  $P_{\mathcal{H}_n}\mathcal{E}_n = \mathcal{E}_n \subseteq R^n\mathcal{M}_\mathcal{E}$ .  $\square$

DEFINITION 3.17. For  $n \in \mathbb{N}$ , let  $X_n = \bigvee_{j=0}^n R^j \mathcal{M}_\mathcal{E}$ . When  $n \geq 1$ , define

$$V_n = X_n \ominus X_{n-1}$$

and when  $n = 0$ , let  $V_0 = \mathcal{M}_\mathcal{E}$ .

LEMMA 3.18.  $V_n$  and  $X_n$  are invariant subspaces for  $\mathbb{M}_R$ .

*Proof.* We use induction on  $n$ . The lemma is true by construction for  $V_0 = \mathcal{M}_\mathcal{E}$ . Since the  $R_j$ 's are self-adjoint, we only have to show that  $X_n$  is  $R_j$ -invariant for all  $j \in \mathbb{N}$ . Assume now that the lemma is true for  $0 \leq m \leq n - 1$ . The vectors of the form

$$x_n = m_n + x_{n-1},$$

with  $m_n \in R^n\mathcal{M}_\mathcal{E}$  and  $x_{n-1} \in X_{n-1}$ , are dense in  $X_n$  and for these:

$$R_j x_n = R_j m_n + R_j x_{n-1} = P_{\mathcal{H}_n} R_j P_{\mathcal{H}_n} m_n + (I - P_{\mathcal{H}_n}) R_j m_n + R_j x_{n-1}. \tag{33}$$

We have  $P_{\mathcal{H}_n} R_j P_{\mathcal{H}_n} m_n \in R^n\mathcal{M}_\mathcal{E}$  by Lemma 3.15, since  $P_{\mathcal{H}_n} R_j P_{\mathcal{H}_n} \in \mathbb{M}_{R,n}$ .

As  $\mathcal{E}_j = \mathcal{H}_j \ominus \mathcal{H}_{j+1}$  we have

$$\bigvee_{j=0}^{n-1} \mathcal{E}_j = \mathcal{H}_n^\perp$$

and therefore  $I - P_{\mathcal{H}_n} = P_{\mathcal{H}_n^\perp}$  projects down to the space generated by  $\mathcal{E}_j$  for  $0 \leq j \leq n - 1$  and as  $\mathcal{E}_j \subset \mathcal{M}_\mathcal{E}$  by Lemma 3.16, we have  $(I - P_{\mathcal{H}_n}) R_j m_n \in X_{n-1}$ .

By induction, we have  $R_j x_{n-1} \in X_{n-1}$  and hence all the vectors on the right hand side of (33) are in  $X_n$ .  $\square$

COROLLARY 3.19. For  $x \in X_{n-1}^\perp$  we have  $R_j x = (R|_{\mathcal{H}_n})_j x$  and  $X_{n-1}$  is an invariant subspace for  $\mathbb{M}_{R|_{\mathcal{H}_n}}$ .

*Proof.* As  $\ker R^{*n} = \mathcal{H}_n^\perp = \bigvee_{k=0}^{n-1} \mathcal{E}_k$ , we have by Lemma 3.16 that

$$\ker R^{*n} \subseteq \bigvee_{j=0}^{n-1} R^j \mathcal{M}_\mathcal{E} = X_{n-1}$$

and hence  $X_{n-1}^\perp \subseteq \mathcal{H}_n$ . By Lemma 3.18,  $X_{n-1}^\perp$  is invariant with respect to  $R_j$ . Hence for all  $x \in X_{n-1}^\perp$ , we have  $P_{\mathcal{H}_n}x = x$  and so

$$R_jx = P_{\mathcal{H}_n}R_jP_{\mathcal{H}_n}x = (R|_{\mathcal{H}_n})_jx.$$

To see the second claim, we observe that since  $X_{n-1}$  is an invariant subspace for  $P_{\mathcal{H}_n}$ , it is an invariant subspace for all

$$P_{\mathcal{H}_n}R_kP_{\mathcal{H}_n} = (R|_{\mathcal{H}_n})_k$$

and hence for  $\mathbb{M}_{R|_{\mathcal{H}_n}}$ .  $\square$

PROPOSITION 3.20. *For all  $m \geq n$  we have  $P_{V_m}R^{m-n}V_n = V_m$ .*

*Proof.* Since  $V_m = X_m \ominus X_{m-1}$  and  $X_m = X_{m-1} + R^m\mathcal{M}_\mathcal{E}$  we have  $V_m = P_{V_m}R^m\mathcal{M}_\mathcal{E}$ . So for every  $v_m \in V_m$  and  $\varepsilon > 0$  there is a  $v_0 \in \mathcal{M}_\mathcal{E}$  such that  $\|v_m - P_{V_m}R^mv_0\| < \varepsilon$ . Then  $P_{V_m}R^mv_0 \neq 0$  will imply  $P_{V_n}R^n v_0 \neq 0$  since  $P_{V_n}R^n v_0 = 0$  would imply  $R^n v_0 \in X_{n-1}$  and so  $R^m v_0 \in X_{m-1} \perp V_m$ . Since  $P_{V_k}R^k v_0 = R^k v_0 - x_{k-1}$  with  $x_{k-1} \in X_{k-1}$  we then have

$$P_{V_m}R^{m-n}P_{V_n}R^n v_0 = P_{V_m}R^{m-n}(R^n v_0 - x_{n-1}) = P_{V_m}R^m v_0$$

and hence

$$\|v_m - P_{V_m}R^{m-n}P_{V_n}R^n v_0\| < \varepsilon. \quad \square$$

Proposition 3.20 directly implies:

COROLLARY 3.21. *For all  $m \geq n$ , we have  $\dim V_m \leq \dim V_n$ . Especially, if  $V_N = \{0\}$  for some  $N \in \mathbb{N}$  then  $V_n = \{0\}$  for all  $n \geq N$ .*

So if  $\dim \mathcal{M}_\mathcal{E} = V_0$  is finite, then  $\dim V_n$  must be finite for all  $n \in \mathbb{N}$ . The injectivity of  $R$  now gives:

PROPOSITION 3.22. *If  $V_0 = \mathcal{M}_\mathcal{E}$  is finite dimensional, then  $V_k \neq \{0\}$  for all  $k \in \mathbb{N}$ .*

*Proof.* If there were a  $K \in \mathbb{N}$  such that  $V_K = \{0\}$  then Corollary 3.21 would give  $V_k = \{0\}$  for all  $k \geq K$ . By Theorem 3.23,  $\mathcal{H}_\mathcal{E}$  must be finite dimensional and mapped by  $R$  again into  $\mathcal{H}_\mathcal{E}$  with a nontrivial cokernel, but then there must be a nonzero vector  $v \in \mathcal{H}_1$  such that  $Rv = 0$ , a contradiction.  $\square$

THEOREM 3.23. *For an injective operator  $R$  on a Hilbert space  $\mathcal{H}$ , the subspace  $\mathcal{H}_\mathcal{E} = \bigvee_{j=0}^\infty R^j\mathcal{M}_\mathcal{E}$  is  $R$  and  $\mathbb{M}_R$  invariant. Moreover*

$$\mathcal{H}_\mathcal{E} = \bigoplus_{k=0}^\infty V_k$$

and

$$RV_k \subseteq V_{k+1} \oplus (X_k \ominus RX_{k-1}).$$

*Proof.* The decomposition  $\mathcal{H}_\mathcal{E} = \bigoplus_{k=0}^\infty V_k$  follows by definition. Since

$$\mathcal{H}_\mathcal{E} = \bigoplus_{k=0}^\infty V_k = \bigvee_{k=0}^\infty R^k \mathcal{M}_\mathcal{E}$$

and  $\bigvee_{k=0}^\infty R^k \mathcal{M}_\mathcal{E}$  are invariant under  $R$ , the first claim is proved.

To prove the last claim, notice that, since  $RX_k = \bigvee_{j=0}^k R^{j+1} \mathcal{M}_\mathcal{E}$ , we have

$$RV_k \perp V_{k+2+m},$$

for all  $m \geq 0$ . Also, as  $V_k = X_k \ominus X_{k-1}$  and  $X_{k-1}$  is invariant under  $R_1$  by Lemma 3.18, we have

$$\langle RV_k, RX_{k-1} \rangle = \langle V_k, R_1 X_{k-1} \rangle = 0.$$

Thus  $RV_k \perp R^j \mathcal{M}_\mathcal{E}$  for  $1 \leq j \leq k$  and  $RV_k \perp \bigoplus_{m=k+2}^\infty V_m$ . For  $v_k \in V_k$ , we have  $Rv_k = P_{X_k^\perp} Rv_k + P_{X_k} Rv_k$ . Now  $P_{X_k^\perp} Rv_k \in V_{k+1}$  and we have

$$\langle P_{X_k} Rv_k, R^j V_0 \rangle = \langle Rv_k, R^j V_0 \rangle = 0$$

for  $1 \leq j \leq k$  so that

$$P_{X_k} Rv_k \in X_k \ominus RX_{k-1}. \quad \square$$

**COROLLARY 3.24.** *If  $T$  is injective and half-centered, then  $T|_{\mathcal{H}_\mathcal{E}}$  is also injective and half-centered.*

By Theorem 3.23, if  $v \in V_m$  and  $Rv \perp X_m \ominus RX_{m-1}$ , then we get  $Rv \in V_{m+1}$ . In the context here, this is not a particular useful characterization of those  $v \in V_m$  that end up in  $V_{m+1}$  when applying  $R$ . As we will see in the next proposition, it turns out that while we may not have  $X_{m+1} \ominus RX_m \subseteq \mathcal{M}_\mathcal{E}$ , none of the vectors in  $RV_k \ominus V_{k+1}$  can be orthogonal to  $\mathcal{M}_\mathcal{E}$ .

**PROPOSITION 3.25.** *If  $v_m \in V_m$  and  $Rv_m \perp \mathcal{M}_\mathcal{E}$ , then  $Rv_m \in V_{m+1}$ .*

*Proof.* From the proof of Theorem 3.23 we know that  $Rv_m \perp \bigvee_{j=1}^m R^j \mathcal{M}_\mathcal{E}$ , and so  $Rv_m \perp \mathcal{M}_\mathcal{E}$  would imply that  $Rv_m \in X_{m+1}$  and  $Rv_m \perp \bigvee_{j=0}^m R^j \mathcal{M}_\mathcal{E} = X_m$ , i.e  $Rv_m \in V_{m+1}$ .  $\square$

**COROLLARY 3.26.** *If  $v_m \in V_m$  and  $\theta_R v_m \perp \mathcal{M}_\mathcal{E}$ , then  $\theta_R v_m \in V_{m+1}$ .*

*Proof.* Since  $R_1$  has dense range, there is a sequence  $R_1 x_k \in V_m$  such that  $R_1 x_k \rightarrow v_m$ . Then  $\theta_R R_1 x_k = Rx_k \rightarrow \theta_R v_m \perp V_0$  and the same arguments as in Proposition 3.25 show that  $\theta_R v_m \in V_{m+1}$ .  $\square$

**LEMMA 3.27.** *For  $m \geq n$  the projection  $P_{V_m}$  commutes with the operators in  $\mathbb{M}_{R|\mathcal{H}_n}$ .*

*Proof.* Since every  $V_m$  is invariant under  $\mathbb{M}_R$ , we have that  $P_{V_m}$  must commute with all  $R_j$ . By Corollary 3.19 we have

$$(R|_{\mathcal{H}_n})_j P_{V_m} = R_j P_{V_m} = P_{V_m} R_j = P_{V_m} (R|_{\mathcal{H}_n})_j. \quad \square$$

A consequence of Corollary 3.19 is the following:

PROPOSITION 3.28. *We have*

$$R_j|V_m = (R|_{\mathcal{H}_n})_j|V_m$$

when  $m \geq n$ , so that

$$\mathbb{M}_{R|_{\mathcal{H}_n}}|V_m = \mathbb{M}_R|V_m.$$

Moreover, for  $m \geq n$  and all  $j \in \mathbb{N}$  we have

$$\theta_{R|_{\mathcal{H}_n},j}^* (R|_{\mathcal{H}_n})_i \theta_{R|_{\mathcal{H}_n},j}|V_m = \theta_{R,j}^* R_i \theta_{R,j}|V_m \tag{34}$$

so that

$$\mathbb{M}_{R|_{\mathcal{H}_n}}^j|V_m = \mathbb{M}_R^j|V_m. \tag{35}$$

Notice that the slightly complicated expression

$$\theta_{R|_{\mathcal{H}_n},j}^* (R|_{\mathcal{H}_n})_i \theta_{R|_{\mathcal{H}_n},j}$$

is simply the image of  $(R|_{\mathcal{H}_{n+j}})_i$  under the homomorphism  $\mathbb{M}_{R|_{\mathcal{H}_{n+j}}} \rightarrow \mathbb{M}_{R|_{\mathcal{H}_n}}$  coming from (25).

*Proof.* We will prove (34) by induction on  $j$ . The other claim (35) then follows from the fact that the operators in (34) generate  $\mathbb{M}_{R|_{\mathcal{H}_n}}^j|V_m$ . From Lemma 3.18 and Corollary 3.19 it follows that

$$R_i|V_m = (R|_{\mathcal{H}_n})_i|V_m$$

so the claim is true for  $j = 0$ .

Now, assume it is true for  $j - 1 \geq k \geq 0$ . By Lemma 3.11 there is a  $r_{n,j} \in \mathbb{M}_{R|_{\mathcal{H}_n}}$  such that  $\theta_{R|_{\mathcal{H}_n},j} r_{n,j} = R^j|_{\mathcal{H}_n}$  and hence

$$\begin{aligned} r_{n,j}^* \left( \theta_{R|_{\mathcal{H}_n},j}^* (R|_{\mathcal{H}_n})_i \theta_{R|_{\mathcal{H}_n},j} \right) r_{n,j} &= \left( r_{n,j}^* \theta_{R|_{\mathcal{H}_n},j}^* \right) (R|_{\mathcal{H}_n})_i \left( \theta_{R|_{\mathcal{H}_n},j} r_{n,j} \right) \\ &= (R|_{\mathcal{H}_n})_{i+j}. \end{aligned}$$

If we also take  $r_j \in \mathbb{M}_R$  such that  $\theta_{R,j} r_j = R^j$  then

$$r_j^* \left( \theta_{R,j}^* R_i \theta_{R,j} \right) r_j = R_{i+j}.$$

We want to prove that

$$r_{n,j}|V_m = r_j|V_m.$$

But this follows from the induction hypothesis, as the formula for  $r_{n,j}$  is given by

$$r_{n,j} = \left( \theta_{R|\mathcal{H}_{n,j-1}}^* (R|_{\mathcal{H}_{n,j-1}})_1 \theta_{R|\mathcal{H}_{n,j-1}} \right)^{\frac{1}{2}} \cdot \dots \cdot \left( \theta_{R|\mathcal{H}_{n,0}}^* (R|_{\mathcal{H}_{n,0}})_1 \theta_{R|\mathcal{H}_{n,0}} \right)^{\frac{1}{2}}$$

and since we assumed that (34) was true for  $j - 1 \geq k \geq 0$ , we get  $r_{n,j}|V_m = r_j|V_m$ .

We can now calculate

$$\begin{aligned} r_j^* (\theta_{R,j}^* R; \theta_{R,j}) r_j |V_m &= R_{i+j} |V_m = (R|_{\mathcal{H}_n})_{i+j} |V_m \\ &= r_{n,j}^* \left( \theta_{R|\mathcal{H}_{n,j}}^* (R|_{\mathcal{H}_n})_i \theta_{R|\mathcal{H}_{n,j}} \right) r_{n,j} |V_m \\ &= r_j^* \left( \theta_{R|\mathcal{H}_{n,j}}^* (R|_{\mathcal{H}_n})_i \theta_{R|\mathcal{H}_{n,j}} \right) r_j |V_m \end{aligned}$$

and since  $r_j$  has dense range, we must have

$$\theta_{R,j}^* R; \theta_{R,j} |V_m = \theta_{R|\mathcal{H}_{n,j}}^* (R|_{\mathcal{H}_n})_i \theta_{R|\mathcal{H}_{n,j}} |V_m.$$

Hence (34) is also true for  $j$ .  $\square$

### 3.3. A connection between $\mathbb{M}_R|V_n$ and $\mathbb{M}_R|V_m$

In the previous subsection we found a decomposition of  $\mathcal{H}_{\mathcal{E}} = \bigvee_{k=0}^{\infty} R^k \mathcal{M}_{\mathcal{E}}$  into subspaces  $V_n$  which are invariant with respect to the algebra  $\mathbb{M}_R$ . Here we show that there is a natural way to connect the different restrictions  $\mathbb{M}_R|V_n$  and  $\mathbb{M}_R|V_m$ . This will be essential in the proof of the main theorem. To explain what this connection is, we need some results that are proven below. Theorem 3.32 shows that for all  $n, m \in \mathbb{N}$  such that  $m \geq n$ , there is a surjective homomorphism

$$\Gamma_{m,n} : \mathbb{M}_R^{m-n} |V_n \rightarrow \mathbb{M}_R |V_m$$

which, in particular, maps

$$\theta_{R,m-n}^* R_k \theta_{R,m-n} |V_n \mapsto R_k |V_m \tag{36}$$

and more generally

$$\theta_{R,m-n+j}^* R_k \theta_{R,m-n+j} |V_n \mapsto \theta_{R,j} R_k \theta_{R,j} |V_m \tag{37}$$

for all  $j \geq 0$ . There is also the inclusion homomorphism  $\mathbb{M}_R^{m-n} |V_n \hookrightarrow \mathbb{M}_R |V_n$ , so we have the following diagram:

$$\begin{array}{ccc} & \mathbb{M}_R |V_m & \tag{38} \\ & \uparrow \Gamma_{m,n} & \\ \mathbb{M}_R^{m-n} |V_n & \hookrightarrow & \mathbb{M}_R |V_n \end{array}$$

From (37), the homomorphisms  $\Gamma_{m,n}$  also “preserve” the sub-algebras  $\mathbb{M}_R^j$  in the sense that the restrictions of  $\Gamma_{m,n}$  to  $\mathbb{M}_R^{m-n+j}|V_n$  are surjective homomorphisms

$$\mathbb{M}_R^{m-n+j}|V_n \rightarrow \mathbb{M}_R^j|V_m.$$

Another property of the homomorphisms  $\Gamma_{m,n}$  is that they factor through  $m \geq i \geq n$  so that the following diagram commutes

$$\begin{array}{ccc} & & \mathbb{M}_R^j|V_m \\ & \nearrow \Gamma_{m,n} & \uparrow \Gamma_{m,i} \\ \mathbb{M}_R^{m-n}|V_n & \xrightarrow{\Gamma_{i,n}} & \mathbb{M}_R^{m-i}|V_i \end{array} \tag{39}$$

We start with a particular example.

EXAMPLE 3.29. Let  $T$  be a left invertible weighted shift on  $\ell^2$  (thus  $T$  is centered) and let  $\{x_k : k \in \mathbb{N}\}$  denote the standard basis of  $\ell^2$ , so that  $Tx_k = a_k x_{k+1}$ , with  $a_k \in \mathbb{C}$  and  $a_k \neq 0$ . Then the kernel of  $T^*$  is  $\langle x_0 \rangle$ , the subspace generated by  $x_0$ . Since there is  $\lambda_k \in \mathbb{R}$  such that

$$T_k x_0 = \lambda_k x_0$$

for all  $k \in \mathbb{N}$ , we have  $\mathcal{M}_\mathcal{E} = \langle x_0 \rangle$ . From this we can deduce

$$V_k = \langle x_k \rangle.$$

Moreover, it is also easy to see that  $\theta_T = TT_1^{-\frac{1}{2}}$  is an isometric shift on the basis  $\{x_k : k \in \mathbb{N}\}$  and

$$\theta_{T,k} = \theta_{T^k} = T^k T_k^{-\frac{1}{2}}$$

(for a proof of this, use Proposition 3.3). Then (38) and (39) can be seen as a generalization of the fact that for any  $m, n, j \in \mathbb{N}$  with  $m \geq n$ , we have

$$T_j x_m = \frac{\lambda_{m+j}}{\lambda_m} x_m$$

and

$$\theta_{T^{m-n}}^* T_j \theta_{T^{m-n}} x_n = T_{m-n}^{-1} T_{j+m-n} x_n = \frac{\lambda_{n+(m-n)+j}}{\lambda_n} \frac{\lambda_n}{\lambda_m} x_n = \frac{\lambda_{m+j}}{\lambda_m} x_n.$$

It is good to keep Example 3.29 in mind, since all the components defined in this section ( $\mathbb{M}_R, V_k, \Gamma_{m,n}$  etc) becomes very simple in this case.

LEMMA 3.30. For  $m \geq n$ , the operator  $\theta_{R|\mathcal{H}_{n,m-n}}$  is a bijective isometry

$$R^n \mathcal{M}_\mathcal{E} \rightarrow R^m \mathcal{M}_\mathcal{E}$$

that induces an isomorphism

$$\Theta_{m,n} : \mathbb{M}_{R,n} |R^n \mathcal{M}_\mathcal{E} \rightarrow \mathbb{M}_{R,m} |R^m \mathcal{M}_\mathcal{E}$$

given by

$$\Theta_{m,n} : b \mapsto \theta_{R|\mathcal{H}_n, m-n} b \theta_{R|\mathcal{H}_n, m-n}^* \tag{40}$$

for  $b \in \mathbb{M}_{R,n}$ . Moreover, if  $m \geq i \geq n$ , then

$$\Theta_{m,i} \Theta_{i,n} = \Theta_{m,n}. \tag{41}$$

*Proof.* We have

$$R^m = \theta_{R|\mathcal{H}_n, k} r_{n, m-n} R^n$$

where  $r_{n, m-n}$  is the same as in Proposition 3.28. Since  $r_{n, m-n} \in \mathbb{M}_{R,n}$  has dense range in  $\mathcal{H}_n$ , we get

$$R^m \mathcal{M}_\mathcal{E} = \theta_{R|\mathcal{H}_n, m-n} r_{n, m-n} R^n \mathcal{M}_\mathcal{E} = \theta_{R|\mathcal{H}_n, m-n} R^n \mathcal{M}_\mathcal{E}.$$

Now, since

$$\theta_{R|\mathcal{H}_n, m-n} \mathbb{M}_{R,n} \theta_{R|\mathcal{H}_n, m-n}^* = \mathbb{M}_{R,m},$$

as  $\theta_{R|\mathcal{H}_n, m-n} \theta_{R,n} = \theta_{R,m}$ , it is not hard to see that (40) defines an isomorphism

$$\Theta_{m,n} : \mathbb{M}_{R,n} |R^n \mathcal{M}_\mathcal{E} \rightarrow \mathbb{M}_{R,m} |R^m \mathcal{M}_\mathcal{E}.$$

The property (41) follows from

$$\theta_{R|\mathcal{H}_n, m-n} = \theta_{R|\mathcal{H}_i, m-i} \theta_{R|\mathcal{H}_n, i-n}$$

so that

$$\begin{aligned} \theta_{R|\mathcal{H}_n, m-n} \mathbb{M}_{R,n} \theta_{R|\mathcal{H}_n, m-n}^* &= \theta_{R|\mathcal{H}_i, m-i} \theta_{R|\mathcal{H}_n, i-n} \mathbb{M}_{R,n} \theta_{R|\mathcal{H}_n, i-n}^* \theta_{R|\mathcal{H}_i, m-i}^* \\ &= \theta_{R|\mathcal{H}_n, m-i} \mathbb{M}_{R,i} \theta_{R|\mathcal{H}_n, m-i}^* = \mathbb{M}_{R,m}. \quad \square \end{aligned}$$

LEMMA 3.31. For every  $n \in \mathbb{N}$  there is a surjective homomorphism

$$\Phi_n : \mathbb{M}_{R|\mathcal{H}_n} |R^n \mathcal{M}_\mathcal{E} \rightarrow \mathbb{M}_R |V_n$$

given by

$$(R|\mathcal{H}_n)_j |R^n \mathcal{M}_\mathcal{E} \mapsto R_j |V_n.$$

Furthermore,  $\Phi_n$  restricts to a surjective homomorphism

$$\mathbb{M}_{R|\mathcal{H}_n}^k |R^n \mathcal{M}_\mathcal{E} \rightarrow \mathbb{M}_R^k |V_n$$

that maps

$$\theta_{R|\mathcal{H}_n, k} (R|\mathcal{H}_n)_j \theta_{R|\mathcal{H}_n, k} |R^n \mathcal{M}_\mathcal{E} \mapsto \theta_{R,k}^* R_j \theta_{R,k} |V_n$$

for all  $k \geq 0$ .



*Proof.* Since  $\mathbb{M}_{R|\mathcal{H}_n}$  is a sub-algebra of  $\mathbb{M}_{R,n}$  and by Lemma 3.15  $\mathbb{M}_{R,n}R^n\mathcal{M}_\mathcal{E} \subseteq R^n\mathcal{M}_\mathcal{E}$ , the restriction map

$$\begin{aligned} \eta_n : \mathbb{M}_{R|\mathcal{H}_n} &\rightarrow \mathbb{M}_{R|\mathcal{H}_n}|R^n\mathcal{M}_\mathcal{E} \\ b &\mapsto b|R^n\mathcal{M}_\mathcal{E} \end{aligned}$$

is a homomorphism. By Proposition 3.28, the map

$$\xi_n : \mathbb{M}_{R|\mathcal{H}_n} \rightarrow \mathbb{M}_{R|\mathcal{H}_n}|V_n = \mathbb{M}_R|V_n$$

that sends  $(R|\mathcal{H}_n)_j$  to  $R_j|V_n$  is a homomorphism. Now  $P_{V_n}R^n\mathcal{M}_\mathcal{E} = V_n$ , so if we take any  $m_n \in \ker \eta_n$ , then by Lemma 3.27,

$$m_n P_{V_n} R^n \mathcal{M}_\mathcal{E} = P_{V_n} m_n R^n \mathcal{M}_\mathcal{E} = 0.$$

Hence the map  $\Phi_n : \xi_n(b) \mapsto \eta_n(b), b \in \mathbb{M}_{R|\mathcal{H}_n}$  is a well-defined surjective homomorphism from  $\mathbb{M}_{R|\mathcal{H}_n}|R^n\mathcal{M}_\mathcal{E}$  to  $\mathbb{M}_R|V_n$ . The second claim follows from Proposition 3.28.  $\square$

**THEOREM 3.32.** *There are surjective homomorphisms*

$$\Gamma_{m,n} : \mathbb{M}_R^{m-n}|V_n \rightarrow \mathbb{M}_R|V_m$$

that map

$$\theta_{R,m-n+j}^* R_k \theta_{R,m-n+j} |V_n \mapsto \theta_{R,j}^* R_k \theta_{R,j} |V_m \tag{42}$$

for all  $j \geq 0$ . Furthermore, for every  $n \leq i \leq m$ ,  $\Gamma_{i,n}$  restricts to a homomorphism

$$\Gamma_{i,n} : \mathbb{M}_R^{m-n}|V_n \rightarrow \mathbb{M}_R^{m-i}|V_i$$

such that

$$\Gamma_{m,i} \Gamma_{i,n} = \Gamma_{m,n}.$$

*Proof.* Combining Lemma 3.31 and Lemma 3.30, we get a diagram

$$\begin{array}{ccc} \mathbb{M}_{R|\mathcal{H}_n}^{m-n}|R^n\mathcal{M}_\mathcal{E} & \xrightarrow{\Theta_{m,n}} & \mathbb{M}_{R|\mathcal{H}_m}|R^m\mathcal{M}_\mathcal{E} \\ \downarrow \Phi_n & & \downarrow \Phi_m \\ \mathbb{M}_R^{m-n}|V_n & & \mathbb{M}_R|V_m \end{array} \tag{43}$$

and we want to prove that there is a unique

$$\Gamma_{m,n} : \mathbb{M}_R^{m-n}|V_n \rightarrow \mathbb{M}_R|V_m$$

that makes this diagram commutative. Making the composition

$$\Phi_m \Theta_{m,n} : \mathbb{M}_{R|\mathcal{H}_n}^{m-n}|R^n\mathcal{M}_\mathcal{E} \rightarrow \mathbb{M}_R|V_m$$

we need to prove that  $\ker \Phi_n \subseteq \ker \Phi_m \Theta_{m,n}$ , because then we can define  $\Gamma_{m,n}$  as the map sending  $\Phi_n(b)$  to  $\Phi_m \Theta_{m,n}(b)$  for  $b \in \mathbb{M}_{R|\mathcal{H}_n}^{m-n} | R^n \mathcal{M}_\mathcal{E}$ .

If we take  $b \in \mathbb{M}_{R|\mathcal{H}_n}^{m-n} | R^n \mathcal{M}_\mathcal{E}$  such that  $\Phi_n(b) = 0$ , then as  $P_{V_n} R^n \mathcal{M}_\mathcal{E} = V_n$  and  $b$  commutes with  $P_{V_n}$  by Lemma 3.27, we obtain

$$\Phi_n(b) V_n = b P_{V_n} R^n \mathcal{M}_\mathcal{E} = P_{V_n} b R^n \mathcal{M}_\mathcal{E} = 0,$$

so

$$b R^n \mathcal{M}_\mathcal{E} \subseteq X_{n-1}. \tag{44}$$

We have also that  $\Phi_m \Theta_{m,n}(b) = 0$  implies

$$\theta_{R|\mathcal{H}_n, m-n} b R^n \mathcal{M}_\mathcal{E} \subseteq X_{m-1}. \tag{45}$$

We want to prove that (44) implies (45). To show this, we prove the more general statement that

$$\theta_{R|\mathcal{H}_n, m-n} X_{n-1} \subseteq X_{m-1}.$$

The partial isometry  $\theta_{R|\mathcal{H}_n, m-n}$  has a kernel equal to  $\mathcal{H}_n^\perp$ , so

$$\theta_{R|\mathcal{H}_n, m-n} X_{n-1} = \theta_{R|\mathcal{H}_n, m-n} (X_{n-1} \ominus \mathcal{H}_n^\perp).$$

We know that there is a  $r_{n,m-n} \in \mathbb{M}_{R|\mathcal{H}_n}$  with dense range in  $\mathcal{H}_n$  such that  $\theta_{R|\mathcal{H}_n, m-n} r_{n,m-n} = R^{m-n} |_{\mathcal{H}_n}$  and by Corollary 3.19

$$r_{n,m-n} (X_{n-1} \ominus \mathcal{H}_n^\perp) = X_{n-1} \ominus \mathcal{H}_n^\perp.$$

From this we can deduce

$$\theta_{R|\mathcal{H}_n, m-n} (X_{n-1} \ominus \mathcal{H}_n^\perp) = R^{m-n} (X_{n-1} \ominus \mathcal{H}_n^\perp) \subseteq X_{m-1}.$$

This gives the existence of  $\Gamma_{m,n}$ . The surjectivity follows from  $\Phi_m \Theta_{m,n} = \Gamma_{m,n} \Phi_n$  and the surjectivity of  $\Phi_m \Theta_{m,n}$ . The uniqueness follows from the surjectivity of  $\Phi_n$ .

Property (37) follows by applying the commutative diagram to

$$\theta_{R|\mathcal{H}_n, m-n+j} (R |_{\mathcal{H}_n})_i \theta_{R|\mathcal{H}_n, m-n+j} \in \mathbb{M}_{R|\mathcal{H}_n}^{m-n} | R^n \mathcal{M}_\mathcal{E}$$

for  $j \geq 0$ . Property (39) follows, as remarked, from (37).  $\square$

With the help of Theorem 3.32 we can now express the spectrum of  $\theta_{T,k}^* T_j \theta_{T,k} | V_n$  via the spectrum of  $\mathbb{M}_T | \mathcal{M}_\mathcal{E}$ .

**PROPOSITION 3.33.** *If  $T$  is half-centered and if  $\gamma$  is a point of the spectrum of  $\mathbb{M}_T | V_n$ , then there is a point  $\lambda$  in the spectrum of  $\mathbb{M}_T | \mathcal{M}_\mathcal{E}$  such that*

$$\gamma (\theta_{T,k}^* T_j \theta_{T,k}) = \lambda (\theta_{T,k+n}^* T_j \theta_{T,k+n})$$

for all  $j, k \in \mathbb{N}$ .

Note that for every point  $\gamma$  in the spectrum of  $\mathbb{M}_T$  and all  $j, k \in \mathbb{N}$ , we have

$$\gamma(\theta_{T,k}^* T_j \theta_{T,k}) \gamma(T_k) = \gamma\left(T_k^{\frac{1}{2}} \theta_{T,k}^* T_j \theta_{T,k} T_k^{\frac{1}{2}}\right) = \gamma(T_{k+j}).$$

So, if  $\gamma(T_k) \neq 0$ , then

$$\gamma(\theta_{T,k}^* T_j \theta_{T,k}) = \frac{\gamma(T_{k+j})}{\gamma(T_k)}.$$

#### 4. Fundamentals for half-centered operators

Here we present some initial results that hold for all injective half-centered operators with  $\dim \mathcal{E} = 1$ . Much of the work in this section will aim towards showing that the operator  $T_k$  has a simple form when restricted to  $\mathcal{M}_{\mathcal{E}}$ . We will see that there are real parameters  $\tau_k, \beta_k$  and a self adjoint operator  $A \in \mathcal{B}(\mathcal{M}_{\mathcal{E}})$  which is independent of  $k$ , such that  $T_k|_{\mathcal{M}_{\mathcal{E}}}$  is given by the formula

$$T_k|_{\mathcal{M}_{\mathcal{E}}} = \tau_k I + \beta_k A,$$

where  $I$  is the identity on  $\mathcal{M}_{\mathcal{E}}$ . This implies that there are  $a, b, c \in \mathbb{R}$ , not all zero, and  $k, m \in \mathbb{N}^+$  such that

$$aI + bT_k + cT_m|_{\mathcal{M}_{\mathcal{E}}} = 0, \tag{46}$$

which can be seen as a weaker form of the main theorem. Indeed if  $\mathcal{M}_{\mathcal{E}} = \mathcal{H}$ , then (46) directly implies it. However, we cannot conclude from (46) that the same identity must hold for the whole space (and in general it will not). The step from the linear dependence in  $\mathcal{M}_{\mathcal{E}}$  to the linear dependence in  $\mathcal{H}$  is the main obstacle here and much of the theory in section 2 was introduced as a way to deal with this.

Since the subspace  $\mathcal{E}$  is now one dimensional, we take  $\mathcal{E}$  to mean a unit vector that spans the subspace. To keep the notations simpler, we also write  $P$  instead of  $P_{\mathcal{H}_1}$ .

We recall the earlier result (Proposition 3.2):

$$\text{If } T \text{ is half-centered then so is } T|_{\mathcal{H}_1}.$$

This implies that  $PT_kPT_jP = PT_jPT_kP$  for all  $j, k \in \mathbb{N}$ . As  $P_{\mathcal{E}} = I - P$ , we can deduce

$$\begin{aligned} PT_kP_{\mathcal{E}}T_jP &= PT_kT_jP - PT_kPT_jP \\ &= PT_jT_kP - PT_jPT_kP = PT_jP_{\mathcal{E}}T_kP \end{aligned}$$

so that

$$PT_kP_{\mathcal{E}}T_jP = PT_jP_{\mathcal{E}}T_kP. \tag{47}$$

This equation leads to the following.

PROPOSITION 4.1. *For every  $x \in \mathcal{H}_1$  and  $u \in \mathcal{M}_{\mathcal{E}}$*

$$\langle x, T_m \mathcal{E} \rangle (T_k - \langle T_k \mathcal{E}, \mathcal{E} \rangle I) u = \langle x, T_k \mathcal{E} \rangle (T_m - \langle T_m \mathcal{E}, \mathcal{E} \rangle I) u. \tag{48}$$

*Proof.* First we prove

$$\langle Ty, T_m \mathcal{E} \rangle (T_k - \langle T_k \mathcal{E}, \mathcal{E} \rangle I) \mathcal{E} = \langle Ty, T_k \mathcal{E} \rangle (T_m - \langle T_m \mathcal{E}, \mathcal{E} \rangle I) \mathcal{E} \tag{49}$$

for each  $y \in \mathcal{H}$ . Since

$$PT_m \mathcal{E} = (T_m - \langle T_m \mathcal{E}, \mathcal{E} \rangle I) \mathcal{E},$$

we have

$$PT_m P_{\mathcal{E}} T_k PT = (T_m - \langle T_m \mathcal{E}, \mathcal{E} \rangle I) P_{\mathcal{E}} T_k T.$$

By (47), this is the same as

$$PT_k P_{\mathcal{E}} T_m PT = (T_k - \langle T_k \mathcal{E}, \mathcal{E} \rangle I) P_{\mathcal{E}} T_m T.$$

So we have

$$(T_m - \langle T_m \mathcal{E}, \mathcal{E} \rangle I) P_{\mathcal{E}} T_k Ty = (T_k - \langle T_k \mathcal{E}, \mathcal{E} \rangle I) P_{\mathcal{E}} T_m Ty$$

for all  $y \in \mathcal{H}$ . Equation (49) now follows from

$$P_{\mathcal{E}} T_m Ty = \langle Ty, T_m \mathcal{E} \rangle \mathcal{E}.$$

As  $\mathbb{M}_T$  is commutative, we have for any  $a \in \mathbb{M}_T$  that

$$\begin{aligned} \langle Ty, T_m \mathcal{E} \rangle (T_k - \langle T_k \mathcal{E}, \mathcal{E} \rangle I) a \mathcal{E} &= a \langle Ty, T_m \mathcal{E} \rangle (T_k - \langle T_k \mathcal{E}, \mathcal{E} \rangle I) \mathcal{E} \\ &= a \langle Ty, T_k \mathcal{E} \rangle (T_m - \langle T_m \mathcal{E}, \mathcal{E} \rangle I) \mathcal{E} \\ &= \langle Ty, T_k \mathcal{E} \rangle (T_m - \langle T_m \mathcal{E}, \mathcal{E} \rangle I) a \mathcal{E} \end{aligned}$$

for every  $T_n$ . The statement now follows by continuity arguments.  $\square$

The following statement must be known, but since we could not find an exact reference for it, we include the proof for the sake of completeness.

LEMMA 4.2. *Let  $\mathcal{A}$  be a commutative  $C^*$ -algebra of operators on a Hilbert space  $\mathcal{H}$  with a cyclic vector  $x \in \mathcal{H}$ . Then given  $a_1, a_2 \in \mathcal{A}$  and a point  $\lambda$  in the spectrum of  $\mathcal{A}$  there is a sequence of vectors  $x_l \in \mathcal{H}$  such that*

$$a_i x_l - \lambda(a_i) x_l \rightarrow 0$$

as  $l \rightarrow \infty$  for  $i = 1, 2$  and

$$\frac{\langle a_i x_l, x \rangle}{\langle x_l, x \rangle} \rightarrow \lambda(a_i)$$

as  $l \rightarrow \infty$ .

*Proof.* For simplicity, we write  $\hat{a}$  for the Gelfand transform of  $a \in \mathcal{A}$ . As  $x$  is a cyclic vector for  $\mathcal{A}$ , there is an isometric representation  $u : \mathcal{H} \rightarrow L^2(X, \mu_x)$ , where  $X$  is the Gelfand spectrum of  $\mathcal{A}$  and  $\mu_x$  is the Borel measure on  $X$  induced by the positive linear functional on  $C(X)$  given by

$$\hat{a} \mapsto \langle ax, x \rangle.$$

Let  $B_\varepsilon[\hat{a}_i(\lambda)]$  denote the open ball in  $\mathbb{C}$  centered on  $\hat{a}_i(\lambda)$  and with radius  $\varepsilon$ . Now define

$$W_\varepsilon = \hat{a}_1^{-1}(B_\varepsilon[\hat{a}_1(\lambda)]) \cap \hat{a}_2^{-1}(B_\varepsilon[\hat{a}_2(\lambda)])$$

i.e the subset of  $X$  such that both  $\hat{a}_1$  and  $\hat{a}_2$  have distance less than  $\varepsilon$  from their value at  $\lambda$ . Since  $\hat{a}_1$  and  $\hat{a}_2$  are both continuous,  $W_\varepsilon$  is an open set and thus there is a non-constant positive continuous function  $g_\varepsilon$ , that is zero on  $W_\varepsilon^c$ . Since  $\mu_x$  is finite and has  $X$  as its support (due to the fact that  $x$  is cyclic), we can further assume that  $\int_X |g_\varepsilon(z)|^2 d\mu_x(z) = 1$  and as  $g_\varepsilon$  is positive, we have  $0 < \int_X g_\varepsilon(z) d\mu_x(z) < \infty$ .

Now we see that

$$\begin{aligned} & \int_X |\hat{a}_i(\lambda)g_\varepsilon(z) - \hat{a}_i(z)g_\varepsilon(z)|^2 d\mu_x(z) \\ &= \int_{W_\varepsilon} |(\hat{a}_i(\lambda) - \hat{a}_i(z))|^2 |g_\varepsilon(z)|^2 d\mu_x(z) < \varepsilon^2 \end{aligned}$$

for  $1 \leq i \leq 2$  and thus  $\hat{a}_i g_\varepsilon - \hat{a}_i(\lambda)g_\varepsilon \rightarrow 0$  in  $L^2(X, \mu_x)$  as  $\varepsilon \rightarrow 0$ . Moreover

$$\begin{aligned} \left| \frac{\int_X \hat{a}_i(z)g_\varepsilon(z) d\mu_x(z)}{\int_X g_\varepsilon(z) d\mu_x(z)} - \hat{a}_i(\lambda) \right| &= \left| \frac{\int_X \hat{a}_i(z)g_\varepsilon(z) - \hat{a}_i(\lambda)g_\varepsilon(z) d\mu_x(z)}{\int_X g_\varepsilon(z) d\mu_x(z)} \right| \\ &\leq \frac{\int_{W_\varepsilon} |\hat{a}_i(z) - \hat{a}_i(\lambda)| g_\varepsilon(z) d\mu_x(z)}{\int_{W_\varepsilon} g_\varepsilon(z) d\mu_x(z)} \\ &< \varepsilon \cdot \frac{\int_{W_\varepsilon} g_\varepsilon(z) d\mu_x(z)}{\int_{W_\varepsilon} g_\varepsilon(z) d\mu_x(z)} = \varepsilon \end{aligned}$$

for  $1 \leq i \leq 2$ . Taking  $x_l = u^{-1}g_{\frac{1}{l}}$ , we obtain the statement.  $\square$

**COROLLARY 4.3.** *Given two points  $\lambda, \mu$  of the spectrum of  $\mathbb{M}_T$  restricted to  $\mathcal{M}_\mathcal{E}$  and  $m_1, m_2 \in \mathbb{N}$ , there are two sequences of unit vectors  $x_l, y_l \in \mathcal{M}_\mathcal{E}$  such that*

$$\frac{\langle T_{m_i} x_l, \mathcal{E} \rangle}{\langle x_l, \mathcal{E} \rangle} \rightarrow \lambda(T_{m_i})$$

and

$$\frac{\langle T_{m_i} y_l, \mathcal{E} \rangle}{\langle y_l, \mathcal{E} \rangle} \rightarrow \mu(T_{m_i})$$

as  $l \rightarrow \infty$  for  $i = 1, 2$ .

Now, let  $(\lambda, \mu)$ ,  $m_1, m_2 \in \mathbb{N}$  and  $x_l, y_l \in \mathcal{M}_\mathcal{E}$  be as in Corollary 4.3. Consider the new sequence

$$v_l = \frac{x_l}{\langle x_l, \mathcal{E} \rangle} - \frac{y_l}{\langle y_l, \mathcal{E} \rangle}.$$

Then  $v_l \perp \mathcal{E}$  for all  $l \in \mathbb{N}$  so that  $v_l \in \mathcal{H}_1$ . Moreover, for  $i = 1, 2$

$$\langle v_l, T_{m_i} \mathcal{E} \rangle \rightarrow \lambda(T_{m_i}) - \mu(T_{m_i})$$

as  $l \rightarrow \infty$ .

If we apply Proposition 4.1 with the sequence  $v_l$  in the place of  $x$  and  $k = m_1, m = m_2$  and let  $l \rightarrow \infty$ , then we get for every  $u \in \mathcal{M}_\mathcal{E}$

$$(\lambda(T_m) - \mu(T_m))(T_k - \langle T_k \mathcal{E}, \mathcal{E} \rangle I)u = (\lambda(T_k) - \mu(T_k))(T_m - \langle T_m \mathcal{E}, \mathcal{E} \rangle I)u. \tag{50}$$

We can draw some conclusions from this formula. Write  $\sigma(\mathcal{A})$  for the spectrum of a  $C^*$ -algebra  $\mathcal{A}$ .

PROPOSITION 4.4. *Let  $\lambda, \mu \in \sigma(\mathbb{M}_T | \mathcal{M}_\mathcal{E})$  and  $\lambda \neq \mu$ . Then  $\lambda(T_m) = \mu(T_m)$  for some  $m \in \mathbb{N}$  if and only if*

$$T_m \mathcal{E} = \langle T_m \mathcal{E}, \mathcal{E} \rangle \mathcal{E}$$

*i.e.  $\mathcal{E}$  is an eigenvector for  $T_m$ .*

*Proof.* If  $k$  is such that  $\lambda(T_k) \neq \mu(T_k)$  and  $m$  is such that  $\lambda(T_m) = \mu(T_m)$ , then the left-hand side of (50) is zero and therefore so is the right-hand side, but since  $\lambda(T_k) \neq \mu(T_k)$ , we obtain

$$(T_m - \langle T_m \mathcal{E}, \mathcal{E} \rangle I) \mathcal{E} = 0.$$

The other direction is trivial.  $\square$

If  $\dim \mathcal{M}_\mathcal{E} \geq 2$  then there must be at least two different point in the spectrum of  $\mathbb{M}_T$  restricted to  $\mathcal{M}_\mathcal{E}$ , this makes it possible to do the following definition.

DEFINITION 4.5. Let  $\dim \mathcal{M}_\mathcal{E} \geq 2$  and let  $(\lambda, \mu)$  be two different points in  $\sigma(\mathbb{M}_T | \mathcal{M}_\mathcal{E})$ . For every  $k \in \mathbb{N}$ , let

$$\beta_k := \lambda(T_k) - \mu(T_k). \tag{51}$$

REMARK 4.6. Clearly  $\beta_0 = 0$ . We note also that if  $(\lambda', \mu')$  is another couple of points in  $\sigma(\mathbb{M}_T | \mathcal{M}_\mathcal{E})$  then by Lemma 4.8 below we have  $\lambda(T_k) - \mu(T_k) = c(\lambda'(T_k) - \mu'(T_k))$  for a nonzero constant  $c \in \mathbb{R}$  and every  $k \in \mathbb{N}$ , so the sequence  $\{\beta_k\}$  is defined up to a multiplicative constant by a couple of different points in the spectrum  $\sigma(\mathbb{M}_T | \mathcal{M}_\mathcal{E})$ .

DEFINITION 4.7. We let

$$\tau_k := \langle T_k \mathcal{E}, \mathcal{E} \rangle \tag{52}$$

for all  $k \in \mathbb{N}$ .

LEMMA 4.8. *If  $\lambda$  is in the spectrum of  $\mathbb{M}_T | \mathcal{M}_\mathcal{E}$  then*

$$\lambda(T_k) = \tau_k + A_\lambda \beta_k$$

*for some constant  $A_\lambda \in \mathbb{R}$  only depending on  $\lambda$ .*

*Proof.* With our new notations (50) can be now rewritten as

$$\beta_k (T_m - \tau_m) u = \beta_m (T_k - \tau_k) u. \tag{53}$$

By Lemma 4.2 we can find a sequence  $\{x_j\} \in \mathcal{M}_\mathcal{E}$  such  $\frac{\langle T_k x_j, \mathcal{E} \rangle}{\langle x_j, \mathcal{E} \rangle} \rightarrow \lambda(T_k)$  and  $\frac{\langle T_m x_j, \mathcal{E} \rangle}{\langle x_j, \mathcal{E} \rangle} \rightarrow \lambda(T_m)$ . Substituting  $v$  with  $\frac{x_j}{\langle x_j, \mathcal{E} \rangle}$  in (53), then taking the scalar product with  $\mathcal{E}$  on both sides and letting  $j \rightarrow \infty$ , we get

$$\beta_k (\lambda(T_m) - \tau_m) = \beta_m (\lambda(T_k) - \tau_k). \tag{54}$$

If  $\dim \mathcal{M}_\mathcal{E} \geq 2$  then there must be at least one  $m \in \mathbb{N}$  such that  $\beta_m \neq 0$  and if we take

$$A_\lambda = \frac{(\lambda(T_m) - \tau_m)}{\beta_m}$$

then we see from (54) that  $A_\lambda$  is independent of the choice of  $k \in \mathbb{N}$  as long as  $\beta_k \neq 0$ . So we have

$$\lambda(T_k) = \tau_k + \frac{\lambda(T_k) - \tau_k}{\beta_k} \beta_k = \tau_k + A_\lambda \beta_k$$

when  $\beta_k \neq 0$  and when  $\beta_j = 0$  we have from Proposition 4.4 that

$$\lambda(T_j) = \tau_j = \tau_j + A_\lambda \beta_j$$

so that the formula is valid in this case also.  $\square$

The results of this subsection can be summarized as follows:

**THEOREM 4.9.** *If  $T \in \mathcal{B}(\mathcal{H})$  is half-centered and injective with  $\dim(T\mathcal{H})^\perp = 1$ , then there are self adjoint operators  $A, C \in \mathcal{B}(\mathcal{M}_\mathcal{E})$ , such that for every  $k \in \mathbb{N}$*

$$T_k|_{\mathcal{M}_\mathcal{E}} = \tau_k I + \beta_k A. \tag{55}$$

$$PT_k P|_{\mathcal{M}_\mathcal{E}} = \tau_k P + \beta_k C. \tag{56}$$

where  $C = PAP$ .

While  $T$  is assumed to be injective, we cannot rule out the possibility that  $0 \notin \sigma(\mathbb{M}_T)$ , in fact we can not even rule out  $0 \notin \sigma(\mathbb{M}_T|_{\mathcal{M}_\mathcal{E}})$ . In the end of Section 5, we will see that if  $\bigvee_{k=0}^\infty R^k \mathcal{M}_\mathcal{E} = \mathcal{H}$ , then actually  $0 \notin \sigma(\mathbb{M}_T|_{\mathcal{M}_\mathcal{E}})$ , but in general this may not be the case. However, the property  $0 \in \sigma(\mathbb{M}_T|_{\mathcal{M}_\mathcal{E}})$  does give quite strong implications regarding the structure of  $T$  and we must take these into account in the next section when we add the condition  $\bigvee_{k=0}^\infty R^k \mathcal{M}_\mathcal{E} = \mathcal{H}$ , even though we end up showing the non-existence of such points.

**LEMMA 4.10.** *If  $\gamma(T_k) = 0$  for some  $\gamma \in \sigma(\mathbb{M}_T)$  and  $k \in \mathbb{N}$ , then  $\gamma(T_{k+j}) = 0$  for all  $j \in \mathbb{N}$ .*

*Proof.* We have

$$0 = \gamma(T_k) \gamma(\theta_{T,k}^* T_j \theta_{T,k}) = \gamma\left( (T_k^{\frac{1}{2}} \theta_{T,k}^*) T_j (\theta_{T,k} T_k^{\frac{1}{2}}) \right) = \gamma(T_{k+j}). \quad \square$$

PROPOSITION 4.11. *If  $0 \in \sigma(T_k |_{\mathcal{M}_{\mathcal{E}}})$  for some  $k \in \mathbb{N}$ , then*

$$\beta_{k+j} = \frac{\tau_{j+k}}{\tau_k} \beta_k$$

and

$$\theta_{T,k}^* T_j \theta_{T,k} |_{\mathcal{M}_{\mathcal{E}}} = \frac{\tau_{j+k}}{\tau_k} I |_{\mathcal{M}_{\mathcal{E}}}$$

for all  $j \in \mathbb{N}$ .

*Proof.* It follows from Theorem 4.9 that if  $0 \in \sigma(T_k |_{\mathcal{M}_{\mathcal{E}}})$ , then there is  $\lambda \in \sigma(\mathbb{M}_T |_{\mathcal{M}_{\mathcal{E}}})$  such that  $0 = \lambda(T_k) = \tau_k + \beta_k A_\lambda$  for some  $A_\lambda \in \mathbb{R}$ . By Lemma 4.10 we have  $\tau_{k+j} + \beta_{k+j} A_\lambda = 0$ . Since  $\tau_j \neq 0$  for all  $j \in \mathbb{N}$  we must have  $\tau_{k+j} = -\beta_{k+j} A_\lambda \neq 0$  for all  $j \in \mathbb{N}$ . Hence

$$\tau_{k+j} + \beta_{k+j} A_\lambda = \tau_{k+j} + \frac{\tau_{k+j}}{\tau_k} \beta_k A_\lambda \tag{57}$$

giving  $\beta_{k+j} = \frac{\tau_{k+j}}{\tau_k} \beta_k$ . Furthermore, the formula (57) shows that

$$\left( \frac{\tau_{k+j}}{\tau_k} I \right) T_k |_{\mathcal{M}_{\mathcal{E}}} = T_{k+j} |_{\mathcal{M}_{\mathcal{E}}}.$$

As also  $(\theta_{T,k}^* T_j \theta_{T,k}) T_k |_{\mathcal{M}_{\mathcal{E}}} = T_{k+j} |_{\mathcal{M}_{\mathcal{E}}}$  and the range of  $T_k$  is dense in  $\mathcal{M}_{\mathcal{E}}$ , we must have

$$\theta_{T,k}^* T_j \theta_{T,k} |_{\mathcal{M}_{\mathcal{E}}} = \frac{\tau_{j+k}}{\tau_k} I |_{\mathcal{M}_{\mathcal{E}}}. \quad \square$$

### 5. Structure properties of injective half-centered operators

The aim of this section is to establish structure results for injective half-centered operators that satisfy the main assumptions:  $\dim \mathcal{E} = 1$  and  $\mathcal{H}_{\mathcal{E}} = \mathcal{H}$ .

As it was mentioned after the statement of the main theorem, if  $\dim \mathcal{M}_{\mathcal{E}} = 1$  then  $T$  is centered and moreover if  $\bigvee_{k=0}^{\infty} T^k \mathcal{E} = \mathcal{H}$ , then  $T$  is a weighted shift. Hence in what follows, we assume that  $\dim \mathcal{M}_{\mathcal{E}} \geq 2$ .

First we discuss the spectrum of  $\mathbb{M}_{T |_{\mathcal{H}_1}} |_{\mathcal{M}_{\mathcal{E}}} \ominus \mathcal{E}$ .

PROPOSITION 5.1. *If  $\dim \mathcal{M}_{\mathcal{E}} \geq 3$  then the spectrum of  $\mathbb{M}_{T |_{\mathcal{H}_1}} |_{\mathcal{M}_{\mathcal{E}}} \ominus \mathcal{E}$  contains at least two points.*

*Proof.* As before, we denote by  $P$  the orthogonal projection onto  $\mathcal{H}_1 = \overline{T \mathcal{H}}$ . To prove the statement it is enough to see that if  $\dim \mathcal{M}_{\mathcal{E}} \geq 2$  and  $PT_k \mathcal{E} \neq 0$  (such  $k$  exists, otherwise  $\dim \mathcal{M}_{\mathcal{E}} = 1$ ), then

$$\mathbb{M}_{T |_{\mathcal{H}_1}} PT_k \mathcal{E} = \mathcal{M}_{\mathcal{E}} \ominus \mathcal{E}. \tag{58}$$



Since if  $PT_k\mathcal{E} \in \mathcal{M}_\mathcal{E} \ominus \mathcal{E}$  is a cyclic vector for  $\mathbb{M}_T|_{\mathcal{H}_1}|_{\mathcal{M}_\mathcal{E} \ominus \mathcal{E}}$ , then the number of points in  $\sigma(\mathbb{M}_T|_{\mathcal{H}_1}|_{\mathcal{M}_\mathcal{E} \ominus \mathcal{E}})$  is equal to  $\dim \mathcal{M}_\mathcal{E} \ominus \mathcal{E}$  and by assumption, it is larger than two.

Let  $A$  be the operator from Theorem 4.9, then  $PT_k\mathcal{E} = \beta_k PA\mathcal{E}$  so for any  $j, k \in \mathbb{N}$  the two vectors  $PT_k\mathcal{E}$  and  $PT_j\mathcal{E}$  differ only by a constant multiple. Hence  $PT_j\mathcal{E} \in \mathbb{M}_T|_{\mathcal{H}_1}PT_k\mathcal{E}$  for any  $j \in \mathbb{N}$ .

The space  $\mathbb{M}_T|_{\mathcal{H}_1}PT_k\mathcal{E}$  is of course a subspace of  $\mathcal{M}_\mathcal{E}$ , so if we can prove that  $(\mathbb{M}_T|_{\mathcal{H}_1}PT_k\mathcal{E}) \oplus \mathcal{E}$  is invariant for every  $T_j$ , then, since  $\mathcal{M}_\mathcal{E}$  is the smallest closed subspace containing  $\mathcal{E}$  that is invariant under  $\mathbb{M}_T$ , this would imply  $(\mathbb{M}_T|_{\mathcal{H}_1}PT_k\mathcal{E}) \oplus \mathcal{E} = \mathcal{M}_\mathcal{E}$  and therefore  $\mathbb{M}_T|_{\mathcal{H}_1}PT_k\mathcal{E} = \mathcal{M}_\mathcal{E} \ominus \mathcal{E}$ .

So take any  $x + c\mathcal{E} \in (\mathbb{M}_T|_{\mathcal{H}_1}PT_k\mathcal{E}) \oplus \mathcal{E}$  with  $c \in \mathbb{C}$  and  $x \in \mathbb{M}_T|_{\mathcal{H}_1}PT_k\mathcal{E}$ . Then since  $P + P_\mathcal{E} = I$  we have

$$T_jx + cT_j\mathcal{E} = (P + P_\mathcal{E})(T_jx + cT_j\mathcal{E}) = PT_jPx + P_\mathcal{E}T_jx + cPT_j\mathcal{E} + cP_\mathcal{E}T_j\mathcal{E}.$$

As  $PT_jP \in \mathbb{M}_T|_{\mathcal{H}_1}$  and  $P_\mathcal{E}T_j\mathcal{E} = \langle T_j\mathcal{E}, \mathcal{E} \rangle \mathcal{E} = \tau_j\mathcal{E}$ , we obtain

$$(PT_jPx + cPT_j\mathcal{E}) + (\langle T_jx, \mathcal{E} \rangle + c\tau_j)\mathcal{E} \in \mathbb{M}_T|_{\mathcal{H}_1}PT_k\mathcal{E} \oplus \mathcal{E}. \quad \square$$

**5.1. Relating  $\mathbb{M}_T|_{\mathcal{M}_\mathcal{E}}$  to  $\mathbb{M}_T|_{\mathcal{H}_1}|_{\mathcal{M}_\mathcal{E} \ominus \mathcal{E}}$ ; the discrete case**

The purpose of the next two subsections is to show that when  $\dim \mathcal{M}_\mathcal{E} \geq 2$ , then there is a relation between the spectrum of  $\mathbb{M}_T|_{\mathcal{M}_\mathcal{E}}$  and that of  $\mathbb{M}_T|_{\mathcal{H}_1}|_{\mathcal{M}_\mathcal{E} \ominus \mathcal{E}}$ .

To see where this relation comes from, assume for a moment that  $\mathbb{M}_T|_{\mathcal{M}_\mathcal{E}}$  has an orthonormal basis of eigenvectors  $x_i \in \mathcal{M}_\mathcal{E}$ . Since  $\bigvee_{k=0}^\infty T^k \mathcal{M}_\mathcal{E} = \mathcal{H}$  and  $\dim \mathcal{E} = 1$ , we can find an eigenvector  $x_k$  and a smallest integer  $m \geq 1$  such that  $T^m x_k$  is not orthogonal to  $\mathcal{M}_\mathcal{E}$  but  $T^j x_k \perp \mathcal{M}_\mathcal{E}$  for  $1 \leq j \leq m - 1$  (if this set of  $j$ 's is non-empty!). Such  $x_k$  and  $m$  must exist; in fact the converse would imply  $T\mathcal{H} \perp \mathcal{M}_\mathcal{E}$  and hence  $\mathcal{M}_\mathcal{E} \subseteq \mathcal{E}$ , contradicting  $\dim \mathcal{M}_\mathcal{E} \geq 2$ .

Now fix such  $x_k$  and  $m$ . From Proposition 3.25 we get  $T^j x_k \in V_j$  for  $1 \leq j \leq m - 1$ . Moreover,  $T^j x_k$  is an eigenvector for  $\mathbb{M}_T|_{V_j}$ . This is due to the following calculation: given  $l \in \mathbb{N}$

$$\begin{aligned} T_l T^j x_k &= P_{\mathcal{H}_j} T_l P_{\mathcal{H}_j} T^j x_k = (\theta_{T,j} \theta_{T,j}^*) T_l (\theta_{T,j} \theta_{T,j}^*) T^j x_k \\ &= \theta_{T,j} (\theta_{T,j}^* T_l \theta_{T,j}) \theta_{T,j}^* T^j x_k = \theta_{T,j} (\theta_{T,j}^* T_l \theta_{T,j}) T_j^{\frac{1}{2}} x_k \\ &= \theta_{T,j} T_j^{\frac{1}{2}} (\theta_{T,j}^* T_l \theta_{T,j}) x_k = \lambda (\theta_{T,j}^* T_l \theta_{T,j}) T^j x_k \end{aligned}$$

where  $\lambda \in \sigma(\mathbb{M}_T)$  is the eigenvalue corresponding to  $x_k$ .

Next we observe that  $T^m x_k$  can not be an eigenvector for  $\mathbb{M}_T$ . In fact, assume contrary to our claim, that  $bT^m x_k = \gamma(b)T^m x_k$  for all  $b \in \mathbb{M}_T$ , where  $\gamma \in \sigma(\mathbb{M}_T)$ . Then as  $T^m x_k \perp \mathcal{E}$ , we obtain

$$\langle T^m x_k, b\mathcal{E} \rangle = \langle bT^m x_k, \mathcal{E} \rangle = \gamma(b) \langle T^m x_k, \mathcal{E} \rangle = 0$$

as  $T^m x_k \perp \mathcal{E}$ . Hence  $T^m x_k \perp \mathcal{M}_\mathcal{E}$ , a contradiction.

However,  $T^m x_k$  must be an eigenvector for  $\mathbb{M}_{T|\mathcal{H}_1}$  since

$$\begin{aligned} PT_l PT^m x_k &= (\theta_T \theta_T^*) T_l (\theta_T \theta_T^*) T^m x_k = \theta_T (\theta_T^* T_l \theta_T) T_l^{\frac{1}{2}} T^{m-1} x_k \\ &= \gamma_{m-1} (\theta_T^* T_l \theta_T) T^m x_k = \lambda (\theta_{T,m}^* T_l \theta_{T,m}) T^m x_k \end{aligned}$$

where  $\gamma_{m-1} \in \sigma(\mathbb{M}_T|V_{m-1})$  is the eigenvalue corresponding to  $T^{m-1} x_k \in V_{k-1}$  (the last equality follows from Proposition 3.33). If we project  $T^m x_k$  onto  $\mathcal{M}_\mathcal{E}$ , then this will still be an eigenvector, since the projection commutes with  $\mathbb{M}_{T|\mathcal{H}_1}$ .

From this we see that for one of the points  $\gamma$  in the spectrum of  $\mathbb{M}_{T|\mathcal{H}_1}|\mathcal{M}_\mathcal{E} \ominus \mathcal{E}$  there is  $\lambda$  in the spectrum of  $\mathbb{M}_T|\mathcal{M}_\mathcal{E}$  is such that

$$\gamma(PT_l P) = \lambda (\theta_{T,m}^* T_l \theta_{T,m}) \tag{59}$$

for all  $l \in \mathbb{N}$ . If we multiply both sides of (59) with  $\lambda(T_m)$  and use

$$\lambda(T_m) \lambda (\theta_{T,m}^* T_l \theta_{T,m}) = \lambda(T_{m+l}),$$

$$\lambda(T_m) \gamma(PT_l P) = \lambda(T_{m+l}) \tag{60}$$

which is valid for all  $l \in \mathbb{N}$ . This shows how it is possible to express some points in the spectrum of  $\mathbb{M}_{T|\mathcal{H}_1}|\mathcal{M}_\mathcal{E} \ominus \mathcal{E}$  in terms of the spectrum of  $\mathbb{M}_T|\mathcal{M}_\mathcal{E}$ .

**5.2. Relating  $\mathbb{M}_T|\mathcal{M}_\mathcal{E}$  to  $\mathbb{M}_{T|\mathcal{H}_1}|\mathcal{M}_\mathcal{E} \ominus \mathcal{E}$ ; the general case**

A similar reasoning as one used to derive (59) can be generalized to work even in the general case, but due to the possible lack of eigenvectors, the proof of Proposition 5.3 uses the above arguments in a “reversed” way. However, this approach has a disadvantage of making less clear what the central idea is. This is why we included the discrete case as motivation.

First we need an easy result.

LEMMA 5.2. *There is an isomorphism*

$$\Psi : \mathbb{M}_{T|\mathcal{H}_1}|\mathcal{M}_\mathcal{E} \ominus \mathcal{E} \rightarrow \mathbb{M}_T|\theta_T^* \mathcal{M}_\mathcal{E}$$

induced by

$$b \in \mathbb{M}_{T|\mathcal{H}_1}|\mathcal{M}_\mathcal{E} \mapsto \theta_T^* b \theta_T|\theta_T^* \mathcal{M}_\mathcal{E}.$$

We can now proceed to prove the generalization of the result in the last subsection to the case when we may not have any non-trivial eigenvectors of  $\mathbb{M}_T$ .

PROPOSITION 5.3. *If  $\bigvee_{k=0}^\infty T^k \mathcal{M}_\mathcal{E} = \mathcal{H}$ , then there is a dense subset  $M$  of the spectrum of  $\mathbb{M}_{T|\mathcal{H}_1}|\mathcal{M}_\mathcal{E} \ominus \mathcal{E}$  such that for every  $\gamma \in M$  there is a point  $\lambda$  in the spectrum of  $\mathbb{M}_T|\mathcal{M}_\mathcal{E}$  and an integer  $m \in \mathbb{N}$  such that*

$$\gamma(PT_k P) = \lambda (\theta_{T,m}^* T_k \theta_{T,m})$$

for all  $k \in \mathbb{N}$ .

*Proof.* Consider the subspace  $\theta_T^* \mathcal{M}_\mathcal{E}$ . Since  $\dim \mathcal{M}_\mathcal{E} \geq 2$  and  $\dim \ker \theta_T^* = 1$ , this subspace is nonzero. By assumption,  $\mathcal{H}_\mathcal{E} = \mathcal{H}$ , and hence  $\sum_{k=0}^\infty P_{V_k} = I$ , so there exists  $m \in \mathbb{N}$  such that  $P_{V_m} \theta_T^* \mathcal{M}_\mathcal{E} \neq 0$ . Since the projection  $P_{V_k}$  commutes with  $\mathbb{M}_T$ , we have a homomorphism

$$s_k : \mathbb{M}_T|_{\mathcal{H}_1} | \mathcal{M}_\mathcal{E} \ominus \mathcal{E} \rightarrow \mathbb{M}_T^1|_{P_{V_k} \theta_T^* \mathcal{M}_\mathcal{E}}$$

which is defined as the composition

$$\mathbb{M}_T|_{\mathcal{H}_1} | \mathcal{M}_\mathcal{E} \ominus \mathcal{E} \xrightarrow{\Psi} \mathbb{M}_T^1|_{\theta_T^* \mathcal{M}_\mathcal{E}} \rightarrow \mathbb{M}_T^1|_{P_{V_k} \theta_T^* \mathcal{M}_\mathcal{E}}$$

where  $\Psi$  is the isomorphism from Lemma 5.2 and the second arrow is the restriction. The homomorphism  $s_k$  induces an injective continuous map

$$s_k^* : \sigma(\mathbb{M}_T^1|_{P_{V_k} \theta_T^* \mathcal{M}_\mathcal{E}}) \rightarrow \sigma(\mathbb{M}_T|_{\mathcal{H}_1} | \mathcal{M}_\mathcal{E} \ominus \mathcal{E}).$$

As  $\sum P_{V_k} = I$  and  $\Psi$  is an isomorphism, given  $a \in \mathbb{M}_T|_{\mathcal{H}_1} | \mathcal{M}_\mathcal{E} \ominus \mathcal{E}$ , we have  $a = 0$  iff  $s_k(a) = 0$  for all  $k \in \mathbb{N}$ . So the union of the ranges of all  $s_k^*$  must be dense in  $\sigma(\mathbb{M}_T|_{\mathcal{H}_1} | \mathcal{M}_\mathcal{E} \ominus \mathcal{E})$ .

If  $\mu \in \sigma(\mathbb{M}_T^1|_{P_{V_k} \theta_T^* \mathcal{M}_\mathcal{E}})$  then there is  $\mu_k \in \sigma(\mathbb{M}_T|_{V_k})$  such that

$$\mu(\theta_T^* T_j \theta_T) = \mu_k(\theta_T^* T_j \theta_T)$$

and so by Proposition 3.33 there is  $\lambda \in \mathbb{M}_T | \mathcal{M}_\mathcal{E}$  such that

$$\mu(\theta_T^* T_j \theta_T) = \mu_k(\theta_T^* T_j \theta_T) = \lambda(\theta_{T,k+1} T_j \theta_{T,k+1}).$$

Taking  $\gamma = s_k^*(\mu)$ , we have  $\gamma(PT_jP) = \mu(\theta_T^* T_j \theta_T)$  and so

$$\gamma(PT_jP) = \mu(\theta_T^* T_j \theta_T) = \lambda(\theta_{T,k+1} T_j \theta_{T,k+1})$$

for all  $j \in \mathbb{N}$ . This implies the statement with  $m = k + 1$ .  $\square$

Proposition 5.3 motivates the following definition:

DEFINITION 5.4. Let  $\mathcal{F}$  be the set of all triples  $(\lambda, \gamma, m)$  consisting of

$$\lambda \in \sigma(\mathbb{M}_T | \mathcal{M}_\mathcal{E})$$

$$\gamma \in \sigma(\mathbb{M}_T|_{\mathcal{H}_1} | \mathcal{M}_\mathcal{E} \ominus \mathcal{E})$$

and  $m \in \mathbb{N}^+$  such that

$$\gamma(PT_kP) = \lambda(\theta_{T,m}^* T_k \theta_{T,m})$$

for all  $k \in \mathbb{N}$ . We say that the triples  $(\lambda, \gamma, m)$  and  $(\lambda', \gamma', m')$  are not equal if either  $\lambda \neq \lambda'$  or  $\gamma \neq \gamma'$  or  $m \neq m'$ .

Recall from Theorem 4.9 that if  $\lambda \in \sigma(\mathbb{M}_T | \mathcal{M}_\mathcal{E})$  and  $\gamma \in \sigma(\mathbb{M}_T | \mathcal{H}_1 | \mathcal{M}_\mathcal{E} \ominus \mathcal{E})$  then

$$\begin{aligned} \lambda(T_k) &= \tau_k + \beta_k A_\lambda \\ \gamma(PT_kP) &= \tau_k + \beta_k C_\gamma \end{aligned}$$

for some  $A_\lambda, C_\gamma \in \mathbb{R}$ .

Next proposition shows how every triple  $(\lambda, \gamma, m) \in \mathcal{F}$  gives rise to a relation between the  $\tau_k$ 's and  $\beta_k$ 's.

PROPOSITION 5.5. *For any triple  $(\lambda, \gamma, m) \in \mathcal{F}$  and every  $k \in \mathbb{N}$  we have*

$$\lambda(T_m)\gamma(PT_kP) = \lambda(T_{m+k}). \tag{61}$$

Moreover, if  $\lambda(T_m) = \tau_m + A_\lambda \beta_m$  and  $\gamma(PT_mP) = \tau_m + C_\gamma \beta_m$  then for all  $k \in \mathbb{N}$

$$\tau_k - \frac{\tau_{m+k}}{\lambda(T_m)} = \frac{A_\lambda \beta_{m+k}}{\lambda(T_m)} - C_\gamma \beta_k \tag{62}$$

when  $\lambda(T_m) \neq 0$  and

$$\tau_k - \frac{\tau_{m+k}}{\tau_m} = -C_\gamma \beta_k \tag{63}$$

when  $\lambda(T_m) = 0$ .

*Proof.* We have  $\gamma(PT_kP) = \lambda(\theta_{T,m}^* T_k \theta_{T,m})$ , so

$$\lambda(T_m)\gamma(PT_kP) = \lambda(T_m)\lambda(\theta_{T,m}^* T_k \theta_{T,m}) = \lambda(T_{m+k})$$

proving the first part. If  $\lambda(T_m) \neq 0$ , then

$$\lambda(T_m)\tau_k + \lambda(T_m)C_\gamma\beta_k = \lambda(T_m)\gamma(PT_kP) = \lambda(T_{m+k})$$

by (61). As  $\lambda(T_{m+k}) = \tau_{m+k} + A_\lambda \beta_{m+k}$ , we obtain (62). When  $\lambda(T_m) = 0$ , we get the formula from Propositions 3.33 and 4.11.  $\square$

### 6. Main theorem: the case $|\mathcal{F}| \geq 2$

The aim of this section is to show that when  $\mathcal{F}$  has at least two elements, then  $T$  satisfies equation (17) in the main theorem.

Let  $\{\tau_k\}$  and  $\{\beta_k\}$  be the sequences of real numbers associated to  $T$  that are defined by (52) and (51). Let

$$\tau(z) = \sum_{j=0}^{\infty} \tau_j z^j$$

and

$$B(z) = \sum_{j=0}^{\infty} \beta_j z^j.$$

be formal power series associated to  $\{\tau_k\}$  and  $\{\beta_k\}$ .

Let  $S^*$  be the backwards shift operator, defined on power series as

$$S^* : \sum_{k=0}^{\infty} a_k z^k \mapsto \sum_{k=0}^{\infty} a_{k+1} z^k$$

and pick  $(\lambda, \gamma, m) \in \mathcal{F}$ . Then (62) and (63) can be rewritten as follows:

$$\left( I - \frac{S^{*m}}{\lambda(T_m)} \right) \tau(z) = - \left( C_\gamma I - \frac{A_\lambda S^{*m}}{\lambda(T_m)} \right) B(z). \tag{64}$$

when  $\lambda(T_m) \neq 0$  and

$$\left( I - \frac{S^{*m}}{\tau_m} \right) \tau(z) = -C_\gamma B(z) \tag{65}$$

otherwise.

Taking another triple  $(\mu, \omega, n) \in \mathcal{F}$  we obtain similar equalities with  $(\lambda, \gamma, m)$  replaced by  $(\mu, \omega, n)$

Letting

$$P_1(z) = \begin{cases} 1 - \frac{z^m}{\lambda(T_m)} & \lambda(T_m) \neq 0 \\ 1 - \frac{z^m}{\tau_m} & \text{otherwise} \end{cases}, \quad P_2(z) = \begin{cases} C_\gamma - \frac{A_\lambda z^m}{\lambda(T_m)} & \lambda(T_m) \neq 0 \\ C_\gamma & \text{otherwise} \end{cases},$$

$$Q_1(z) = \begin{cases} 1 - \frac{z^n}{\mu(T_n)} & \mu(T_n) \neq 0 \\ 1 - \frac{z^n}{\tau_n} & \text{otherwise} \end{cases}, \quad Q_2(z) = \begin{cases} C_\omega - \frac{A_\mu z^n}{\mu(T_n)} & \mu(T_n) \neq 0 \\ C_\omega & \text{otherwise} \end{cases}.$$

We have

$$\begin{aligned} P_1(S^*)\tau(z) &= -P_2(S^*)\beta(z) \\ Q_1(S^*)\tau(z) &= -Q_2(S^*)\beta(z). \end{aligned} \tag{66}$$

Now let  $P(z) = P_1(z)Q_2(z) - P_2(z)Q_1(z)$ .

LEMMA 6.1. *We have*

$$\begin{aligned} P(S^*)\tau(z) &= 0 \\ P(S^*)\beta(z) &= 0 \end{aligned}$$

*Proof.* It follows from (66) that

$$\begin{aligned} P_1(S^*)Q_1(S^*)\tau(z) &= -P_1(S^*)Q_2(S^*)\beta(z) \\ Q_1(S^*)P_1(S^*)\tau(z) &= -Q_1(S^*)P_2(S^*)\beta(z) \end{aligned}$$

giving the first equality  $P(S^*)\beta(z) = 0$ . A similar calculation gives  $P(S^*)\tau(z) = 0$ .  $\square$

Our next goal is to show that if  $\mathcal{F}$  contains at least two triples, then we can choose  $(\lambda, \gamma, m)$  and  $(\mu, \omega, n)$  such that  $P(z)$  is not identically zero.

Since we will always work with only two triples at the time, we can without any resulting confusion denote the polynomial corresponding to  $(\lambda, \gamma, m), (\mu, \omega, n)$  by  $P(z)$ .

LEMMA 6.2. *If  $\dim \mathcal{M}_\mathcal{E} = 2$  then for every triple  $(\lambda, \gamma, m) \in \mathcal{F}$  we have  $A_\lambda \neq C_\gamma$ .*

*Proof.* Choose  $k \in \mathbb{N}$  such that  $\beta_k \neq 0$ . Then the spectrum of  $T_k|_{\mathcal{M}_\mathcal{E}}$  consists of two points, since from (55) we find that in this case,  $I$  and  $T_k$  are generators for  $\mathbb{M}_T|_{\mathcal{M}_\mathcal{E}}$ . Now consider the function  $H(z) = \langle (T_k - z)^{-1} \mathcal{E}, \mathcal{E} \rangle$ . This is a rational function with simple poles at the eigenvalues of  $T_k$ . Since for real  $z$

$$H'(z) = \langle (T_k - z)^{-2} \mathcal{E}, \mathcal{E} \rangle = \langle (T_k - z)^{-1} \mathcal{E}, (T_k - z)^{-1} \mathcal{E} \rangle > 0$$

we see that  $H(z)$  has a zero  $\chi$  between its two poles. As  $\chi$  is not in the spectrum of  $T_k$  we must have  $(T_k - \chi)^{-1} \mathcal{E} \neq 0$  and then from  $\langle (T_k - \chi)^{-1} \mathcal{E}, \mathcal{E} \rangle = 0$  we get  $(T_k - \chi)^{-1} \mathcal{E} \perp \mathcal{E}$ . Now we can calculate

$$\begin{aligned} PT_kP(T_k - \chi)^{-1} \mathcal{E} &= PT_k(T_k - \chi)^{-1} \mathcal{E} = P(T_k - \chi + \chi)(T_k - \chi)^{-1} \mathcal{E} \\ &= \chi(T_k - \chi)^{-1} \mathcal{E}. \end{aligned}$$

Hence  $\chi$  is in the spectrum of  $PT_kP|_{\mathcal{M}_\mathcal{E} \ominus \mathcal{E}}$  so

$$\chi = \tau_k + C_\gamma \beta_k.$$

But  $\chi$  is not in the spectrum of  $T_k|_{\mathcal{M}_\mathcal{E}}$  and therefore  $\chi \neq \tau_k + \beta_k A_\lambda$ . As  $\beta_k \neq 0$ , we get  $C_\gamma \neq A_\lambda$ .  $\square$

If  $\dim \mathcal{M}_\mathcal{E} = 2$ , there is only one element in  $\sigma(\mathbb{M}_T|_{\mathcal{H}_1}|_{\mathcal{M}_\mathcal{E} \ominus \mathcal{E}})$ . Hence for two triples  $(\lambda, \gamma, m), (\mu, \omega, n) \in \mathcal{F}$  we must have  $\gamma = \omega$ .

LEMMA 6.3. *Let  $\dim \mathcal{M}_\mathcal{E} = 2$ . If there are two different triples*

$$(\lambda, \gamma, m), (\mu, \gamma, n) \in \mathcal{F},$$

*then  $P(z) \not\equiv 0$ . However, we have  $P(0) = 0$ .*

*Proof.* First, note that since  $\dim \mathcal{M}_\mathcal{E} = 2$  and  $T$  is injective we can not have  $\lambda(T_k) = 0$  for any  $\lambda \in \sigma(\mathbb{M}_T|_{\mathcal{M}_\mathcal{E}})$  and  $k \in \mathbb{N}$ , since otherwise we will have a nonzero  $u \in \mathcal{M}_\mathcal{E}$  such that  $T_k u = 0$  and hence  $0 = \langle T_k u, u \rangle = \|T^k u\|^2$ . Let  $(\lambda, \gamma, m), (\mu, \gamma, n) \in \mathcal{F}$ , then  $\lambda(T_m) \neq 0$  and  $\mu(T_n) \neq 0$ . The corresponding polynomial  $P(z)$  is then of the form

$$P(z) = \left( C_\gamma - \frac{A_\lambda z^m}{\lambda(T_m)} \right) \left( 1 - \frac{z^n}{\mu(T_n)} \right) - \left( C_\gamma - \frac{A_\mu z^n}{\mu(T_n)} \right) \left( 1 - \frac{z^m}{\lambda(T_m)} \right). \tag{67}$$

Assume on the contrary that  $P(z) \equiv 0$ . By expanding the right-hand side of (67) and use  $A_\lambda \neq C_\gamma$  and  $A_\mu \neq C_\gamma$ , we easily see that  $P(z) \equiv 0$  implies  $m = n$  and  $A_\lambda = A_\mu$  and hence the triples are equal. The second claim follows from the fact that the constant term on the right-hand side of (67) vanishes.  $\square$

PROPOSITION 6.4. *If the set  $\mathcal{F}$  has more than two elements, then there are two triples  $(\lambda, \gamma, k), (\mu, \omega, m) \in \mathcal{F}$  such that the polynomial  $P(z)$  is not the zero polynomial. Moreover, if  $\dim \mathcal{M} \geq 3$ , then there are different triples such that  $P(0) \neq 0$ .*

*Proof.* We already know that if  $\dim \mathcal{M}_\mathcal{E} = 2$  and there are two different triples, then the polynomial  $P(z)$  is not constantly zero. If  $\dim \mathcal{M}_\mathcal{E} \geq 3$  then by Lemma 5.1 the  $C^*$ -algebra generated by the  $PT_jP$ 's restricted to  $\mathcal{M}_\mathcal{E}$  must have a spectrum consisting of at least 2 different points. Proposition 5.3 now gives that there are  $\gamma, \omega \in M$  with  $\gamma \neq \omega$  and thus also with  $C_\gamma \neq C_\omega$ . An easy calculation gives that the constant term of  $P(z)$  is  $C_\gamma - C_\omega$  and hence  $P(z) \neq 0$ .  $\square$

Now we can prove the main result of this section:

THEOREM 6.5. *If  $\mathcal{F}$  has at least two elements, then there are constants  $a, b, c, d \in \mathbb{R}$ , not all zero, and integers  $n, m \in \mathbb{N}^+$ , such that*

$$aI + bT_n + cT_m + dT_{n+m} = 0. \tag{68}$$

*In particular, if  $\dim \mathcal{M}_\mathcal{E} \geq 3$  then we may assume that  $a \neq 0$ .*

*Proof.* If  $\mathcal{F}$  has at least two elements, it follows from Proposition 6.4 that there exist  $(\lambda, \gamma, k)$  and  $(\mu, \omega, m)$  in  $\mathcal{F}$  such that the corresponding polynomial  $P(z)$  is of the form  $a + bz^n + cz^m + dz^{n+m}$ , where  $a, b, c, d \in \mathbb{R}$  are not all zero and  $n, m \in \mathbb{N}^+$ . As  $P(S^*)B(z) = 0$  and  $P(S^*)\tau(z) = 0$ , we obtain that for all  $k \in \mathbb{N}$

$$a\tau_k + b\tau_{k+n} + c\tau_{k+m} + d\tau_{k+n+m} = 0 \tag{69}$$

$$a\beta_k + b\beta_{k+n} + c\beta_{k+m} + d\beta_{k+n+m} = 0. \tag{70}$$

By Theorem 4.9, these equations imply

$$aT_k + bT_{n+k} + cT_{m+k} + dT_{n+m+k}|_{\mathcal{M}_\mathcal{E}} = 0$$

for all  $k \in \mathbb{N}$ . Now fix  $k \in \mathbb{N}$  and consider

$$aI + b(\theta_{T,k}^* T_n \theta_{T,k}) + c(\theta_{T,k}^* T_m \theta_{T,k}) + d(\theta_{T,k}^* T_{n+m} \theta_{T,k}).$$

Restricted to  $\mathcal{M}_\mathcal{E}$ , we have

$$\begin{aligned} T_k^{\frac{1}{2}} (aI + b(\theta_{T,k}^* T_n \theta_{T,k}) + c(\theta_{T,k}^* T_m \theta_{T,k}) + d(\theta_{T,k}^* T_{n+m} \theta_{T,k})) T_k^{\frac{1}{2}}|_{\mathcal{M}_\mathcal{E}} \\ = aT_k + bT_{n+k} + cT_{m+k} + dT_{n+m+k}|_{\mathcal{M}_\mathcal{E}} = 0. \end{aligned}$$

Since  $T_k^{\frac{1}{2}}$  has dense range, we must have

$$aI + b(\theta_{T,k}^* T_n \theta_{T,k}) + c(\theta_{T,k}^* T_m \theta_{T,k}) + d(\theta_{T,k}^* T_{n+m} \theta_{T,k})|_{\mathcal{M}_\mathcal{E}} = 0.$$

By Theorem 3.32, this implies that

$$aI + bT_n + cT_m + dT_{n+m}|_{V_k} = 0.$$

As this is true for every  $k \in \mathbb{N}$  and the subspaces  $V_k$  span  $\mathcal{H}$ , we have

$$aI + bT_n + cT_m + dT_{n+m} = 0.$$

If  $\dim \mathcal{M}_\mathcal{E} \geq 3$  then by Proposition 6.4 there are  $(\lambda, \gamma, n), (\mu, \omega, m) \in \mathcal{F}$  such that  $P(0) = a \neq 0$ .  $\square$

**COROLLARY 6.6.** *For all  $k, j \in \mathbb{N}$ , the restriction  $T_j|_{V_k}$  is invertible. If  $\dim \mathcal{M}_\mathcal{E} \geq 3$ , then  $T_j$  is invertible for all  $j \in \mathbb{N}$ , or equivalently,  $T$  has closed range.*

*Proof.* Since  $T$  is injective, every  $T_j$  has dense range. If  $\dim \mathcal{M}_\mathcal{E} \leq 2$  then  $\dim V_k \leq 2$  for all  $k \in \mathbb{N}$ , so  $T_j|_{V_k}$  must be invertible.

When  $\dim \mathcal{M}_\mathcal{E} \geq 3$ , it follows from Theorem 6.5 that there are  $b, c, d \in \mathbb{R}$  and  $m, n \in \mathbb{N}^+$  such that

$$I + bT_n + cT_m + dT_{n+m} = 0$$

(we can divide (68) by  $a \neq 0$ ). If, say,  $n \leq m$  then consider

$$-bI - c(\theta_{T,n}^* T_{m-n} \theta_{T,n}) - d(\theta_{T,n}^* T_m \theta_{T,n}).$$

This is an inverse of  $T_n$  since

$$T_n (-bI - c(\theta_{T,n}^* T_{m-n} \theta_{T,n}) - d(\theta_{T,n}^* T_m \theta_{T,n})) = -bT_n - cT_m - dT_{n+m} = I.$$

But if  $T_n$  is invertible, then so is  $T_1$ , since  $T_n = T_1(\theta_T^* T_{n-1} \theta_T)$ .  $\square$

**6.1. Main theorem: the case  $|\mathcal{F}| = 1$**

The final case to consider is when there is only one triple in  $\mathcal{F}$ .

Take  $J$  to be a weighted shift on  $\ell^2$  with the standard basis  $\{e_k; k \in \mathbb{N}\}$ . Now for some  $n \in \mathbb{N}$  and  $a \in \mathbb{C}$  consider

$$L = J + a(e_0 \otimes e_n^*)$$

(recall that by  $e_0 \otimes e_n^*$  we denote the rank one operator  $x \mapsto \langle x, e_n \rangle e_0$ ). With respect to the standard basis, this infinite matrix will look as follows

$$L = \begin{bmatrix} a & 0 & 0 & 0 & \dots \\ a_0 & 0 & 0 & 0 & \dots \\ 0 & a_1 & 0 & 0 & \dots \\ 0 & 0 & a_2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{71}$$



when  $n = 0$  and

$$L = \begin{bmatrix} 0 & \dots & a & 0 & \dots \\ a_0 & \dots & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & a_n & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{72}$$

for a general  $n$ .

LEMMA 6.7. *The operator  $L$  is half-centered and for every  $k \in \mathbb{N}$  we have that  $L_k$  is diagonal with respect to the standard basis  $\{e_k; k \in \mathbb{N}\}$ .*

*Proof.* As it was mention after Example 2.8, this is a corollary of Proposition 2.5.  $\square$

In this section, we prove the following:

THEOREM 6.8. *If  $\mathcal{F}$  has only one triple  $(\lambda, \gamma, n)$ , then there is an orthonormal basis  $\{x_k; k \in \mathbb{N}\}$  of  $\mathcal{H}$ , a wighted shift  $J$  on this basis and  $a \in \mathbb{C}$  such that*

$$T = J + a(x_0 \otimes x_n^*).$$

There is an orthonormal basis  $v, w$  of  $\mathcal{M}_\mathcal{E}$  consisting of common eigenvectors for all the  $T'_j$ s restricted to this space. Let, say,  $w$  be an eigenvector corresponding to  $\lambda$ . As there is only one triple, we must have  $T^{k}v \in V_k$  for all  $k \in \mathbb{N}$ , otherwise the reasoning used in subsection 4.1 would yield a different triple. Furthermore:

LEMMA 6.9. *If there is only one triple  $(\lambda, \gamma, n)$  in  $\mathcal{F}$ , then  $V_{n+j} = \langle T^{n+j}v \rangle$  for all  $j \in \mathbb{N}$ .*

*Proof.* The reasoning used in the proof of Proposition 5.3 shows that the only way we could end up with only one triple  $(\lambda, \gamma, n)$  is if  $\dim \mathcal{M}_\mathcal{E} = 2$  and  $T^* \mathcal{M}_\mathcal{E}$  is a subspace of  $V_{n-1}$ . This in turn gives  $TV_k \perp \mathcal{M}_\mathcal{E}$  for  $k \neq n - 1$ . Hence  $TV_{n+j} = V_{n+j+1}$  for all  $j \in \mathbb{N}$  by Proposition 3.20 and Proposition 3.25. This shows that the subspace  $\oplus_{m=n+2}^\infty V_m$  is  $T$ -invariant and so  $\oplus_{m=0}^{n+1} V_m$  is  $T^*$ -invariant. Now  $T^*V_{n+1} = T^*TV_n = V_n$  and

$$\langle T^*V_j, V_{n+1} \rangle = \langle V_j, TV_{n+1} \rangle = \langle V_j, V_{n+2} \rangle = 0$$

for  $j \neq n + 2$ , so  $V_{n+1} \perp T^* \oplus_{m=0}^{n+1} V_m$ . But  $T^*$  restricted to  $\oplus_{m=0}^{n+1} V_m$  still has just  $\mathcal{E}$  as its kernel and since the space  $\oplus_{m=0}^{n+1} V_m$  has finite dimension, the dimension of the kernel must be equal to that of the cokernel. So  $\dim V_{n+1} = 1$  and since  $V_{n+1} = TV_n$  this must also be true for  $V_n$ . Since  $T^{n+j}v \in V_{n+j}$ , the whole space must be spanned by this vector.  $\square$

PROPOSITION 6.10. *We have  $T^n w \in \mathcal{M}_\mathcal{E}$ .*

*Proof.* If  $m \geq n + 1$ , then as  $T^n w \in X_n$ , we have  $T^n w \perp V_m$  by definition of  $V_m$ . Also  $V_n = \langle T^n v \rangle$  and so

$$\langle T^n v, T^n w \rangle = \langle T_n v, w \rangle = \lambda(T_n) \langle v, w \rangle = 0.$$

The same argument shows that  $T^n w \perp V_m$  for  $1 \leq m \leq n - 1$  since

$$\langle T^m w, T^n w \rangle = \langle T_m w, T^{n-m} w \rangle = 0$$

$$\langle T^m v, T^n w \rangle = \langle T_m e_0, T^{n-m} w \rangle = 0$$

and the vectors  $T^m w, T^m v$  span  $V_m$  for  $1 \leq m \leq n - 1$ . So  $T^n w \in (\bigoplus_{k=1}^\infty V_k)^\perp = \mathcal{M}_\mathcal{E}$ .  $\square$

**COROLLARY 6.11.** *We have  $\mathcal{M}_\mathcal{E} \ominus \mathcal{E} = \langle T^n w \rangle$  and  $T^* \mathcal{M}_\mathcal{E} = \langle T^{n-1} w \rangle$ .*

*Proof.*  $T^n w \in \mathcal{M}_\mathcal{E}$  is orthogonal to  $\mathcal{E}$  and since  $\dim \mathcal{M}_\mathcal{E} = 2$ , the subspace  $\mathcal{M}_\mathcal{E} \ominus \mathcal{E}$  must be generated by  $T^n w$ . The second claim now follows from

$$T^* \mathcal{M}_\mathcal{E} = T^* \mathcal{M}_\mathcal{E} \ominus \mathcal{E} = \langle T_1 T^{n-1} w \rangle = \langle T^{n-1} w \rangle$$

since  $T^{n-1} w$  is an eigenvector for  $\mathbb{M}_T$  by the introduction to subsection 4.1.  $\square$

With the help of these result we can now proceed to prove Theorem 6.8:

*Proof.* For  $0 \leq k \leq n - 1$  take

$$x_k = \frac{T^k w}{\|T^k w\|}$$

and when  $n \leq k$  take

$$x_k = \frac{T^{k-n} v}{\|T^{k-n} v\|}.$$

Thus  $\{x_k; k \in \mathbb{N}\}$  is an orthonormal basis for the Hilbert space  $\mathcal{H}$  and by the results above, there are constants  $a_k, a \in \mathbb{C}$  such that

$$T x_k = a_k x_{k+1}$$

when  $0 \leq k \leq n - 2$  or  $n \leq j$  and

$$T x_{n-1} = a_n x_n + a x_0$$

(since  $T^n w$  was in  $\mathcal{M}_\mathcal{E}$  generated by  $w = x_0, v = x_n$ ). If we now take  $J$  to be the shift

$$J x_k = a_k x_{k+1}$$

then

$$T = J + a(x_0 \otimes x_n^*). \quad \square$$

The main theorem can now be proven by combining the results from sections 5 and 6 :

*Proof of the main theorem.* When  $\dim \mathcal{M}_\varepsilon = 1$ , we refer to the remarks given after the statement of the main theorem in section 2. When  $\dim \mathcal{M}_\varepsilon \geq 2$ , it follows from Propositions 5.1 and 5.3 that  $|\mathcal{F}| \geq 1$  and hence we can split the arguments into the cases  $|\mathcal{F}| = 1$  and  $|\mathcal{F}| \geq 2$ . When  $|\mathcal{F}| = 1$ , we get from Theorem 6.8 that this corresponds to the second part of case 1. When  $|\mathcal{F}| \geq 2$ , we get (17) from Theorem 6.5. Finally, when  $\dim \mathcal{M}_\varepsilon \geq 3$ , the claim follows from Theorem (6.5) and Corollary 6.6.  $\square$

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