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ON THE DENSENESS OF MINIMUM ATTAINING OPERATORS

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Abstract. Let H_1, H_2 be complex Hilbert spaces and T be a densely defined closed linear operator (not necessarily bounded). It is proved that for each $\varepsilon > 0$, there exists a bounded operator S with $\|S\| \leqslant \varepsilon$ such that $T+S$ is minimum attaining. Further, if T is bounded below, that is if there exists $m > 0$ such that $\|Tx\| \geqslant m\|x\|$ for every x in the domain of T, then S can be chosen to be rank one.

1. Introduction

It is well known that the set of all norm attaining operators defined between two complex Hilbert spaces is norm dense in the space of all bounded linear operators defined between complex Hilbert spaces. This result is even true for operators defined between Banach spaces, when the domain space is reflexive, which is proved by Lindenstrauss [14]. A simple proof of this fact, in the case of Hilbert space operators is given by Enflo et al. in [6]. Moreover, the authors proved that rank one perturbation of a bounded operator can be made as norm attaining operator.

Similar to the norm attaining operators, bounded operators that attain their minimum modulus is introduced in [5]. The unbounded case is dealt in [13] and the authors established basic properties of minimum attaining closed densely defined operators.

It is very natural to ask whether the Lindenstrauss theorem is true in case of minimum attaining operators. In this article we answer this question affirmatively. We show that the set of all minimum attaining densely defined closed operators is dense in the class of densely defined closed operators with respect to the gap metric. As a consequence, we can conclude that the same is true for bounded operators with respect to the operator norm. In a special case, we also show that rank one perturbations lead to minimum attaining operators. This leads to the perturbations of minimum attaining operators.

In the second section we recall some basic definitions and results which we need for proving our main results. In the third section we prove the Lindenstrauss type theorem for minimum attaining operators.

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2. Notations and preliminaries

Throughout we consider infinite dimensional complex Hilbert spaces which will be denoted by H, H_1, H_2 etc. The inner product and the induced norm are denoted by $\langle\cdot\rangle$ and $||.||$, respectively. The closure of a subspace M of H is denoted by $\overline{M}$. We denote the unit sphere of M by $S_M = \{x \in M : \|x\| = 1\}$.

If M is a closed subspace of a Hilbert space H, then P_M denotes the orthogonal projection $P_M : H \to H$ with range M.

Let T be a linear operator with domain $D(T)$ (a subspace of H_1) and taking values in H_2. If $D(T)$ is dense in H_1, then T is called a *densely defined* operator. The graph $G(T)$ of T is defined by $G(T) := \{(x, Tx) : x \in D(T)\} \subseteq H_1 \times H_2$. If $G(T)$ is closed, then T is called a *closed operator*. Equivalently, T is closed if (x_n) is a sequence in $D(T)$ such that $x_n \to x \in H_1$ and $Tx_n \to y \in H_2$, then $x \in D(T)$ and $Tx = y$.

For a densely defined linear operator T, there exists a unique linear operator (in fact, a closed operator) $T^* : D(T^*) \to H_1$, with

$$D(T^*) := \{y \in H_2 : x \mapsto \langle Tx, y\rangle \text{ for all } x \in D(T) \text{ is continuous}\} \subseteq H_2$$

satisfying

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for all } x \in D(T) \text{ and } y \in D(T^*).$$

We say T to be *bounded* if there exists $M > 0$ such that $\|Tx\| \leqslant M\|x\|$ for all $x \in D(T)$. Note that if T is densely defined and bounded then T can be extended to whole of H_1 in a unique way.

By the closed graph Theorem, an everywhere defined closed operator is bounded. Hence the domain of an unbounded closed operator is a proper subspace of a Hilbert space.

The space of all bounded linear operators between H_1 and H_2 is denoted by $\mathscr{B}(H_1, H_2)$ and the class of all closed linear operators between H_1 and H_2 is denoted by $\mathscr{C}(H_1, H_2)$. We write $\mathscr{B}(H, H) = \mathscr{B}(H)$ and $\mathscr{C}(H, H) = \mathscr{C}(H)$.

If $T \in \mathscr{B}(H_1, H_2)$ is such that for every bounded sequence (x_n) of H_1, (Tx_n) has a convergent subsequence in H_2, then T is called a compact operator. Equivalently, T is compact if and only if for every bounded set B of H_1, $T(B)$ is pre compact in H_2.

If $T \in \mathscr{C}(H_1, H_2)$, then the null space and the range space of T are denoted by $N(T)$ and $R(T)$, respectively and the space $C(T) := D(T) \cap N(T)^{\perp}$ is called the carrier of T. In fact, $D(T) = N(T) \oplus^{\perp} C(T)$ [3, Page 340]. Here $\oplus^{\perp}$ denote the orthogonal direct sum of subspaces.

Let $S, T \in \mathscr{C}(H)$ be densely defined operators with domains $D(S)$ and $D(T)$, respectively. Then $S + T$ is an operator with domain $D(S + T) = D(S) \cap D(T)$ defined by $(S + T)(x) = Sx + Tx$ for all $x \in D(S + T)$. The operator ST has the domain $D(ST) = \{x \in D(T) : Tx \in D(S)\}$ and is defined as $(ST)(x) = S(Tx)$ for all $x \in D(ST)$.

If S and T are closed operators with the property that $D(T) \subseteq D(S)$ and $Tx = Sx$ for all $x \in D(T)$, then S is called the *restriction* of T and T is called an *extension* of S.

A densely defined operator $T \in \mathscr{C}(H)$ is said to be

1. *normal* if $T^*T = TT^*$
2. *self-adjoint* if $T = T^*$
3. *positive* if $T = T^*$ and $\langle Tx, x\rangle \geqslant 0$ for all $x \in D(T)$.

Let $V \in \mathscr{B}(H_1, H_2)$. Then V is called

1. an *isometry* if $\|Vx\| = \|x\|$ for all $x \in H_1$
2. a *partial isometry* if $V|_{N(V)^\perp}$ is an isometry. The space $N(V)^\perp$ is called the *initial space* or the *initial domain* and the space $R(V)$ is called the *final space* or the *final domain* of V.

DEFINITION 2.1. Let $T \in \mathscr{B}(H_1, H_2)$. Then T is said to be *norm attaining* if there exists $x_0 \in S_{H_1} \subseteq H_1$ such that $\|Tx_0\| = \|T\|$.

An analogous concept to the norm of an operator is the minimum modulus of an operator. This concept was introduced in [7].

DEFINITION 2.2. [7] Let $T \in \mathscr{C}(H_1, H_2)$ be densely defined. Then

$$m(T) = \inf\{\|Tx\| : x \in S_{D(T)}\}$$

is called the *minimum modulus* of T.

DEFINITION 2.3. Let $T \in \mathscr{C}(H_1, H_2)$ be densely defined. Then T is said to be *bounded below* if there exists $m > 0$ such that $\|Tx\| \geqslant m\|x\|$ for all $x \in D(T)$.

REMARK 2.4.

1. This definition of bounded below operators is analogous to the usual definition given in Functional Analysis books (for example [1, Definition 2.1, Page 69]) for the case of bounded operators.
2. It is easy to see that T is bounded below if and only if $m(T) > 0$. In this case $m(T) = \sup\{m > 0 : \|Tx\| \geqslant m\|x\|, \text{ for all } x \in D(T)\}$.
3. T is bounded below if and only $R(T)$ is closed and T is one-to-one.

If $T \in \mathscr{C}(H_1, H_2)$ is densely defined and one-to-one, then the *inverse operator* is the linear operator from H_2 into H_1 with $D(T^{-1}) = R(T)$ and $T^{-1}Tx = x$ for all $x \in D(T)$. In this case we say T is *invertible* and $R(T^{-1}) = D(T)$ (see [20, Page 4] for details). In particular, if T is one-to-one and onto, then T^{-1} is defined on whole of H_2 and is continuous by the closed graph theorem.

Note that if $T \in \mathscr{C}(H)$ is normal, then T has a bounded inverse if and only if $m(T) > 0$.

DEFINITION 2.5. [13, Definition 3.1] Let $T \in \mathscr{C}(H_1, H_2)$ be densely defined. Then we say T to be *minimum attaining* if there exists $x_0 \in S_{D(T)}$ such that $\|Tx_0\| = m(T)$.

We denote the class of minimum attaining densely defined closed operators between H_1 and H_2 by $\mathscr{M}_c(H_1, H_2)$ and $\mathscr{M}_c(H,H)$ by $\mathscr{M}_c(H)$.

NOTE 2.6. If $T \in \mathscr{C}(H_1, H_2)$ is densely defined and $N(T) \neq \{0\}$, then $m(T) = 0$ and there exists $x \in S_{N(T)}$ such that $Tx = 0$. Hence $T \in \mathscr{M}_c(H_1, H_2)$.

THEOREM 2.7. [19, Theorem 13.31, Page 369][4, Theorem 4, Page 144] *Let $T \in \mathscr{C}(H)$ be densely defined and positive. Then there exists a unique positive operator S such that $T = S^2$. The operator S is called the square root of T and is denoted by $S = T^{\frac{1}{2}}$.*

For $T \in \mathscr{C}(H_1, H_2)$ densely defined, the operator $|T| := (T^*T)^{\frac{1}{2}}$ is called the modulus of T. Moreover, $D(|T|) = D(T)$, $N(|T|) = N(T)$ and $\overline{R(|T|)} = \overline{R(T^*)}$. As $\|Tx\| = \||T|x\|$ for all $x \in D(T)$, we can conclude that $m(T) = m(|T|)$ and $T \in \mathscr{M}_c(H_1, H_2)$ if and only if $|T| \in \mathscr{M}_c(H_1)$.

THEOREM 2.8. [4, Theorem 2, Page 184] *Let $T \in \mathscr{C}(H_1, H_2)$ be densely defined. Then there exists a unique partial isometry $V : H_1 \to H_2$ with the initial space $\overline{R(T^*)}$ and the final space $\overline{R(T)}$ such that $T = V|T|$.*

DEFINITION 2.9. [19, Definition 13.26, Theorem 13.27, Page 365–366] Let $T \in \mathscr{C}(H)$ be densely defined. The resolvent of T is defined by

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda I : D(T) \to H \text{ is invertible and } (T - \lambda I)^{-1} \in \mathscr{B}(H)\}$$

and

$$\begin{aligned} \sigma(T) &:= \mathbb{C} \setminus \rho(T) \\ \sigma_p(T) &:= \{\lambda \in \mathbb{C} : T - \lambda I : D(T) \to H \text{ is not one-to-one}\}, \end{aligned}$$

are called the spectrum and the point spectrum of T, respectively.

A formula to compute the minimum modulus of a densely defined closed operator is given as follows.

PROPOSITION 2.10. [13, Proposition 3.3] *Let $T \in \mathscr{C}(H)$ be normal. Then*

$$m(T) = d(0, \sigma(T)),$$

where $d(x,A)$ denote the distance between the point x and the set A.

REMARK 2.11. If $T \in \mathscr{C}(H)$ is normal, then $\sigma(T)$ is closed (the details can be found in [20, Proposition 2.6(ii), Page 29]). Hence by Proposition (2.10), it is clear that there exists a $\lambda \in \sigma(T)$ such that $|\lambda| = m(T)$.

DEFINITION 2.12. [20, Definition 8.3 Page 178] Let $T = T^* \in \mathscr{C}(H)$. Then the *discrete spectrum* $\sigma_d(T)$ of T is defined as the set of all eigenvalues of T with finite multiplicities which are isolated points of the spectrum $\sigma(T)$ of T. The complement set $\sigma_{ess}(T) := \sigma(T) \setminus \sigma_d(T)$ is called the *essential spectrum* of T.

The essential spectrum is stable under compact perturbations.

PROPOSITION 2.13. (Weyl's theorem) [20, Corollary 8.16, Page 182] *Let* $T \in \mathscr{C}(H)$ *be self-adjoint and* C *is compact, self-adjoint. Then* $\sigma_{ess}(T+C) = \sigma_{ess}(T)$.

LEMMA 2.14. [8, 9, 16] *Let* $T \in \mathscr{C}(H_1, H_2)$ *be densely defined. Denote* $\check{T} = (I+T^*T)^{-1}$ *and* $\widehat{T} = (I+TT^*)^{-1}$. *Then*

1. $\check{T} \in \mathscr{B}(H_1)$, $\widehat{T} \in \mathscr{B}(H_2)$
2. $\widehat{T}T \subseteq T\check{T}$, $\quad ||T\check{T}|| \leqslant \frac{1}{2}$ *and* $\check{T}T^* \subseteq T^*\widehat{T}$, $\quad ||T^*\widehat{T}|| \leqslant \frac{1}{2}$
3. *if* $g : [0,1] \to \mathbb{C}$ *is a continuous function, then*
   (a) $T^*g(\widehat{T})y = g(\check{T})T^*y$ *for all* $y \in D(T^*)$
   (b) $Tg(\check{T})x = g(\widehat{T})Tx$ *for all* $x \in D(T)$.

2.1. Gap metric

The gap between closed subspaces M and N of H is defined by $\theta(M,N) := \|P_M - P_N\|$. This defines a metric on the class of closed subspaces of H, known as the gap metric. The topology induced by the gap metric is known as the gap topology. We have the following alternative formula for the gap;

$$\theta(M,N) := \max\Big\{||P_M(I-P_N)||, ||P_N(I-P_M)||\Big\}.$$

For the details we refer to [2, Page 70].

Let $A, B \in \mathscr{C}(H_1, H_2)$ be densely defined. Then $G(A)$ and $G(B)$ are closed subspaces of $H_1 \times H_2$. The gap between A and B is defined as

$$\theta(A,B) = \|P_{G(A)} - P_{G(B)}\|,$$

where $P_M : H_1 \times H_2 \to H_1 \times H_2$, is an orthogonal projection onto the closed subspace M of $H_1 \times H_2$. This defines a metric on the class of closed operators and induced topology is known as the gap topology. The gap topology restricted to the space of bounded linear operators coincides with the norm topology. Also the convergence with respect to the gap metric on the set of self-adjoint bounded operators coincide with the resolvent convergence [18, Chapter VII, Page 235]).

We have the following formula for the gap between two closed operators;

THEOREM 2.15. [12, Theorem 3.5] *Let* $S,T \in \mathscr{C}(H_1,H_2)$ *be densely defined. Then the operators* $\widehat{T}^{\frac{1}{2}}S\check{S}^{\frac{1}{2}}$, $T\check{T}^{\frac{1}{2}}\check{S}^{\frac{1}{2}}$, $S\check{S}^{\frac{1}{2}}\check{T}^{\frac{1}{2}}$ *and* $\widehat{S}^{\frac{1}{2}}T\check{T}^{\frac{1}{2}}$ *are bounded and*

$$\theta(S,T) = \max\left\{\left\|T\check{T}^{\frac{1}{2}}\check{S}^{\frac{1}{2}} - \widehat{T}^{\frac{1}{2}}S\check{S}^{\frac{1}{2}}\right\|, \left\|S\check{S}^{\frac{1}{2}}\check{T}^{\frac{1}{2}} - \widehat{S}^{\frac{1}{2}}T\check{T}^{\frac{1}{2}}\right\|\right\}. \tag{2.1}$$

3. Denseness of minimum attaining operators

In this section we discuss the denseness of minimum attaining operators. First let us consider the case of functionals:

Let H be a Hilbert space and $\phi : H \to \mathbb{C}$ be a non zero linear functional. Then ϕ is continuous (bounded) if and only if ϕ is closed. Since H is infinite dimensional and $H/N(\phi)$ is isomorphic with $\mathbb{C}$, we can clearly conclude that $N(\phi) \neq \{0\}$. Hence ϕ is minimum attaining. Thus, the class of minimum attaining bounded linear functionals coincide with the space of all bounded linear functionals. So in this case, the minimum attaining functionals are dense. If H is finite dimensional, then clearly every linear functional is minimum attaining. Hence in this case also the result holds trivially.

In the above discussion we can replace Hilbert space by a Banach space. By a theorem of James we can conclude that a normed linear space X is reflexive if and only if every non zero bounded linear functional is norm attaining (see [10, 11] for details). This is no more true if we replace the norm attaining property of functionals by minimum attaining property, as we have noted in the above paragraph.

Now we consider the case of densely defined closed operators defined between two different Hilbert spaces. We prove that the set of all minimum attaining densely defined closed operators defined between two Hilbert spaces is dense in the class of all densely defined closed operators with respect to the gap metric. To prove this we need the following result, which is proved in [17, Remark 3.7, Corollary 3.9] for regular operators defined between two Hilbert C^*-modules. For the sake of completeness, we provide the details here.

THEOREM 3.1. *Let* $S,T \in \mathscr{C}(H_1,H_2)$ *be densely defined and* $D(S) = D(T)$. *Then*

1. *the operators* $\widehat{T}^{\frac{1}{2}}(T-S)\check{S}^{\frac{1}{2}}$ *and* $\widehat{S}^{\frac{1}{2}}(S-T)\check{T}^{\frac{1}{2}}$ *are bounded and*

$$\theta(S,T) = \max\left\{\|\widehat{T}^{\frac{1}{2}}(T-S)\check{S}^{\frac{1}{2}}\|,\ \|\widehat{S}^{\frac{1}{2}}(S-T)\check{T}^{\frac{1}{2}}\|\right\}$$

2. *if* $T-S$ *is bounded, then* $\theta(S,T) \leqslant \|S-T\|$.

Proof. First we simplify the term $T\check{T}^{\frac{1}{2}}\check{S}^{\frac{1}{2}}$. For $x \in H_1$, we have that $\check{S}^{\frac{1}{2}}x \in D(S) = D(T)$, consequently, $T\check{T}^{\frac{1}{2}}\check{S}^{\frac{1}{2}}x = \widehat{T}^{\frac{1}{2}}T\check{S}^{\frac{1}{2}}x$, by (3) of Lemma 2.14. Thus,

$$T\check{T}^{\frac{1}{2}}\check{S}^{\frac{1}{2}}x - \widehat{T}^{\frac{1}{2}}S\check{S}^{\frac{1}{2}}x = \left(\widehat{T}^{\frac{1}{2}}T\check{S}^{\frac{1}{2}} - \widehat{T}^{\frac{1}{2}}S\check{S}^{\frac{1}{2}}\right)x \tag{3.1}$$

$$= \widehat{T}^{\frac{1}{2}}(T-S)\check{S}^{\frac{1}{2}}x. \tag{3.2}$$

With a similar argument we can show that

$$S\check{S}^{\frac{1}{2}}\check{T}^{\frac{1}{2}} - \widehat{S}^{\frac{1}{2}}T\check{T}^{\frac{1}{2}} = \widehat{S}^{\frac{1}{2}}(S-T)\check{T}^{\frac{1}{2}}.$$

Now the conclusion follows by Theorem 2.15.

If $T-S$ is bounded, then

$$\|\widehat{T}^{\frac{1}{2}}(T-S)\check{S}^{\frac{1}{2}}\| \leqslant \|\widehat{T}^{\frac{1}{2}}\|\,\|T-S\|\,\|\check{S}^{\frac{1}{2}}\| \leqslant \|T-S\|. \tag{3.3}$$

Similarly, we can conclude that $\|\widehat{S}^{\frac{1}{2}}(S-T)\check{T}^{\frac{1}{2}}\| \leqslant \|T-S\|$.

Hence by the above two observations the conclusion follows. $\square$

PROPOSITION 3.2. [13, Propositions 3.8, 3.9] *Let $T \in \mathscr{C}(H)$ be positive. Then*

1. *$T \in \mathscr{M}_c(H)$ if and only if $T^{\frac{1}{2}} \in \mathscr{M}_c(H)$*
2. *$T \in \mathscr{M}_c(H)$ if and only if $m(T)$ is an eigenvalue of T.*

PROPOSITION 3.3. [13, Proposition 3.5] *Let $T \in \mathscr{C}(H)$ be positive. Then*

$$m(T) = \inf\{\langle Tx, x\rangle : x \in S_{D(T)}\}.$$

THEOREM 3.4. *Let $T \in \mathscr{C}(H)$ be positive. Then for each $\varepsilon > 0$, there exists an operator $S \in \mathscr{B}(H)$ such that*

1. *$\|S\| \leqslant \varepsilon$*
2. *$T+S$ is minimum attaining*
3. *$\theta(S+T, T) \leqslant \varepsilon$.*

Moreover, if $m(T) > 0$, then S can be chosen such that $\operatorname{rank}(S) = 1$ *and $m(T+S) > 0$.*

Proof. We prove the results by considering the following three cases which exhaust all possibilities.

Case (1): $m(T) > 0$

It suffices to prove the assertion for $\varepsilon \in (0, m(T))$. Since, $T \geqslant 0$ and $m(T) = \inf\limits_{x \in S_{D(T)}} \langle Tx, x\rangle$ by Proposition 3.3, there exists $x_\varepsilon \in S_{D(T)}$, such that

$$\langle Tx_\varepsilon, x_\varepsilon\rangle < m(T) + \frac{\varepsilon}{2}. \tag{3.4}$$

Now, define

$$C_\varepsilon(x) = \varepsilon\,\langle x, x_\varepsilon\rangle x_\varepsilon, \text{ for every } x \in H. \tag{3.5}$$

Then clearly, C_ε is a rank one positive, bounded operator with $\|C_\varepsilon\| = \|C_\varepsilon(x_\varepsilon)\| = \varepsilon$.

Let $T_\varepsilon = T - C_\varepsilon$. It is easy to see that T_ε is a closed densely defined operator with $D(T_\varepsilon) = D(T)$. Clearly, T_ε is self-adjoint. In fact, we show that $T_\varepsilon \geqslant 0$. To this end, let $x \in D(T_\varepsilon) = D(T)$. Then

$$\begin{aligned}\langle T_\varepsilon x, x\rangle &= \langle Tx, x\rangle - \varepsilon|\langle x, x_\varepsilon\rangle|^2 \\ &\geqslant (m(T)-\varepsilon)\langle x, x\rangle \ \text{ (by Cauchy-Schwarz inequality).}\end{aligned}$$

From this we can conclude that $T_\varepsilon \geqslant 0$ and by Proposition 3.3, we have that $m(T_\varepsilon) \geqslant m(T) - \varepsilon > 0$.

We claim that $T_\varepsilon \in \mathscr{M}_c(H)$. By Remark 2.11, we have that $m(T_\varepsilon) \in \sigma(T_\varepsilon)$. Next, we show that $m(T_\varepsilon) \in \sigma_d(T_\varepsilon)$. Assume that $m(T_\varepsilon) \in \sigma_{ess}(T_\varepsilon)$. Then by the Weyl's theorem we have $\sigma_{ess}(T_\varepsilon) = \sigma_{ess}(T)$. Note as $m(T) \in \sigma(T)$ and $m(T)$ is the smallest spectral value, we can conclude that $m(T) \leqslant m(T_\varepsilon)$.

But we have

$$m(T_\varepsilon) \leqslant \langle T_\varepsilon x_\varepsilon, x_\varepsilon\rangle = \langle Tx_\varepsilon, x_\varepsilon\rangle - \varepsilon < m(T) - \frac{\varepsilon}{2} < m(T).$$

Thus our assumption that $m(T_\varepsilon) \in \sigma_{ess}(T_\varepsilon)$ is wrong. Consequently $m(T_\varepsilon) \in \sigma_d(T_\varepsilon)$, and hence $T_\varepsilon \in \mathscr{M}_c(H)$.

Note that $T_\varepsilon - T = -C_\varepsilon|_{D(T)}$ is a bounded operator with domain $D(T)$. By Theorem 3.1, it follows that

$$\theta(T_\varepsilon, T) \leqslant \|T_\varepsilon - T\| = \|C_\varepsilon|_{D(T)}\| \leqslant \|C_\varepsilon\| = \varepsilon.$$

Take $S = -C_\varepsilon$. Then S satisfies the stated conditions. Note that $m(T+S) = m(T_\varepsilon) > 0$.

Case (2): $m(T) = 0$ and T is not one-to-one

Clearly T is minimum attaining. In this case $S = 0$ satisfy the required properties.

Case (3): $m(T) = 0$ and T is one-to-one

We can use case (1) to get the desired operator S. Note that $T + \frac{\varepsilon}{2}I$ is positive and $m\Big(T + \frac{\varepsilon}{2}I\Big) = \frac{\varepsilon}{2}$. Hence by Case (1) above, there exists a positive rank one operator C with $\|C\| \leqslant \frac{\varepsilon}{2}$ such that $T + \frac{\varepsilon}{2}I - C$ is minimum attaining and $\theta\Big(T, T + \frac{\varepsilon}{2}I - C\Big) \leqslant \Big\|\frac{\varepsilon}{2}I - C\Big\| \leqslant \varepsilon$. Let $S = \frac{\varepsilon}{2}I - C$. Then S satisfies all the required conditions. $\square$

Now we prove the above result for the general case.

THEOREM 3.5. *Let $T \in \mathscr{C}(H_1, H_2)$ be densely defined. Then for each $\varepsilon > 0$ there exists an operator $S \in \mathscr{B}(H_1, H_2)$ with $\|S\| \leqslant \varepsilon$ such that $T + S$ is minimum attaining and $\theta(T+S, T) \leqslant \varepsilon$. Moreover, if $m(T) > 0$ then S can be chosen such that* $\operatorname{rank}(S) = 1$ *and $m(T+S) > 0$.*

Proof. By Note 2.6, it suffices to consider the case for operators that are injective. Let $T = V|T|$ be the polar decomposition of T. Applying Theorem 3.4 to $|T|$, there exists $A \in \mathscr{B}(H_1)$ with $\|A\| \leqslant \varepsilon$ and $|T| + A$ is minimum attaining.

Define $S = VA$. Then $S \in \mathscr{B}(H_1, H_2)$ with $\|S\| \leqslant \varepsilon$. Next, we claim that $T + S = V(|T| + A)$ is minimum attaining. Since T is one-to-one, V is an isometry and $T + S$ is minimum attaining as $|T| + A$ minimum attaining. Note that $m(T + S) = m(|T| + A)$.

In case if $m(T) > 0$, then by Theorem 3.4, A can be chosen so that $\operatorname{rank}(A) = 1$ and $m(|T| + A) > 0$. Hence $\operatorname{rank}(S) = \operatorname{rank}(A) = 1$ and $m(T + S) = m(|T| + A) > 0$.

Finally, by Theorem 3.1, we have $\theta(T + S, T) \leqslant \|S\| \leqslant \varepsilon$. $\square$

The following Corollary is an immediate consequence of Theorem 3.5.

COROLLARY 3.6. *The set $\mathscr{M}_c(H_1, H_2)$ is dense in the set of all densely defined operators in $\mathscr{C}(H_1, H_2)$ with respect to the gap topology.*

COROLLARY 3.7. *The set of all minimum attaining bounded operators is dense in $\mathscr{B}(H_1, H_2)$ with respect to the norm topology of $\mathscr{B}(H_1, H_2)$.*

Proof. The gap topology restricted to $\mathscr{B}(H_1, H_2)$ coincide with the norm topology of $\mathscr{B}(H_1, H_2)$ by [15, Theorem 2.5]. Hence the conclusion follows by Corollary 3.6. $\square$

THEOREM 3.8. *Let*

$$\mathscr{G} := \{T \in \mathscr{M}_c(H_1, H_2) : T \textit{ is bounded below }\}$$

and $C_b(H_1, H_2) = \{T \in \mathscr{C}(H_1, H_2) : T$ is densely defined and bounded below$\}$. If $T \in C_b(H_1, H_2)$ and $\varepsilon > 0$, there exists $\tilde{T} \in \mathscr{G}$ such that $\theta(T, \tilde{T}) \leqslant \varepsilon$. In other words, $\mathscr{G}$ is dense in $C_b(H_1, H_2)$ with respect to the gap metric.

Proof. Let $T \in C_b(H_1, H_2)$. Since T is bounded below $m(T) > 0$. Hence by Theorem 3.5, there exists $S \in \mathscr{B}(H_1, H_2)$ such that $T + S$ is minimum attaining, $\theta(T + S, T) \leqslant \varepsilon$, $\operatorname{rank}(S) = 1$ and $m(T + S) > 0$, that is $T + S$ is bounded below. Take $\tilde{T} = T + S$. Then $\tilde{T}$ satisfy all the requirements. $\square$

Recall that $A \in \mathscr{B}(H_1, H_2)$ is bounded below if there exists $k > 0$ such that $\|Ax\| \geqslant k\|x\|$ for all $x \in H_1$. In this case, the minimum modulus

$$m(A) := \inf\{\|Ax\| : x \in S_{H_1}\}$$

is given by $m(A) = \sup\{k > 0 : \|Ax\| \geqslant k\|x\|, \text{ for all } x \in H_1\}$.

COROLLARY 3.9. *The set*

$$\mathscr{M}_b(H_1, H_2) := \{A \in \mathscr{B}(H_1, H_2) : A \textit{ is minimum attaining and bounded below}\}$$

is norm dense in $\mathscr{B}_b(H_1, H_2) = \{A \in \mathscr{B}(H_1, H_2) : A$ is bounded below$\}$.

Proof. The gap metric and the metric induced by the operator norm are equivalent on $\mathscr{B}(H_1, H_2)$ by [15, Theorem 2.5]. Hence the conclusion follows by Theorem 3.8. □

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