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THE MODIFIED PARSEVAL EQUALITY OF STURM–LIOUVILLE PROBLEMS WITH COUPLED BOUNDARY CONDITION

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Abstract. We consider the Sturm-Liouville(S-L) problems with coupled boundary condition and transmission condition. Defining a Hilbert space related to the transmission conditions, we discuss the S-L problems in this modified Hilbert space. We prove the Parseval equality of the S-L problems with the transmission conditions in a modified Hilbert space and derive the Green's function for these problems.

1. Introduction

Sturm-Liouville (S-L) problems with transmission conditions appear in mathematics, mechanics, physics and in other applications. The S-L problems with transmission conditions are concerned in many publications [2, 4, 10, 13], however they are only for the S-L problems with the separated boundary conditions. Here we construct the Green's function of the S-L problems with coupled boundary condition and transmission condition, and establish the modified Parseval equality of the considered S-L problems.

The differential equation we considered is

$$ly := -y'' + q(x)y = \lambda y, \quad x \in J = [-1,0) \cup (0,1], \tag{1.1}$$

with the coupled boundary condition (CBC)

$$AY(-1) + Y(1) = 0, \quad Y(\pm 1) = \begin{pmatrix} y(\pm 1) \\ y'(\pm 1) \end{pmatrix}, \tag{1.2}$$

and the transmission condition (TC)

$$KY(0-) + Y(0+) = 0, \quad Y(0\pm) = \begin{pmatrix} y(0\pm) \\ y'(0\pm) \end{pmatrix}, \tag{1.3}$$

where λ is the complex eigenparameter; A, K are 2×2 matrices

$$A = e^{i\gamma} \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \quad K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}, \tag{1.4}$$

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with $-\pi \leqslant \gamma \leqslant \pi$, $\alpha_1\alpha_4 - \alpha_2\alpha_3 > 0$, $k_{11}k_{22} - k_{12}k_{21} > 0$; and the matrices $(A, -I)$, $(K, -I)$ have full rank, I is the 2×2 identity matrix; $q \in L(J, \mathbb{R})$. Note that the conditions are minimal in the sense that it is necessary and sufficient for all initial value problems of the equation (1.1) to have unique solutions on $[-1,1]$ ([6, 14]); α_j $(j = 1,2,3,4)$ and k_{mj} $(m, j = 1,2)$ are real numbers.

The organization of this paper is as follows: After the Introduction in Section 1, we give the condition for λ being the eigenvalue of the S-L problem with the CBC and TC, and the eigenvalues of the S-L problem (1.1)–(1.4) are countably infinite in Section 2. In Section 3, we construct the Green's function of the S-L problem with the CBC and TC. Finally, we derive the eigenfunction expansion for the Green's function and establish the modified Parseval equality by using the eigenfunction expansion in Section 4.

2. The eigenvalues of the S-L operators

In this section, we construct the basic solutions of the equation (1.1), which satisfy the TC, and characterize the eigenvalues of the S-L problem (1.1)–(1.4).

Let $h = \det K$, where K is the coefficient matrix in the TC (1.3), (1.4). Define a new inner product in $L^2(J)$ as follows:

$$\langle f, g\rangle = h\int_{-1}^{0} f_1\overline{g}_1 dx + \int_0^1 f_2\overline{g}_2 dx, \text{ for } f, g \in L^2(J), \tag{2.1}$$

where $f_1 = f(x)\mid_{[-1,0)}$, $f_2 = f(x)\mid_{(0,1]}$; $h = \det K > 0$, K is the coefficient matrix in the TC (1.3), (1.4). It is easy to verify that $(L^2(J), \langle\cdot,\cdot\rangle)$ is a Hilbert space. For simplicity, we denote it by H, and the norm induced by the inner product is denoted by $\|\cdot\|_H$. Now we consider the S-L problems (1.1)–(1.4) in the associated Hilbert space H.

The operator L_M related to the S-L problems (1.1)–(1.4) is defined by

$$\mathscr{D}(L_M) = \{y \in H | y_1, y_1' \in AC_{loc}[-1,0), y_2, y_2' \in AC_{loc}(0,1],\ \ ly \in H \\ \text{and } KY(0-) + Y(0+) = 0\},$$

$$L_M y = ly, \quad y \in \mathscr{D}(L_M),$$

where $AC_{loc}[-1,0)$ and $AC_{loc}(0,1]$ denote the sets of complex-valued absolutely continuous functions on whole compact subintervals of $[-1,0)$ and $(0,1]$. The S-L operator L is defined by

$$\mathscr{D}(L) = \{\, y \in \mathscr{D}(L_M) | AY(-1) + Y(1) = 0\},$$

$$Ly = ly,\ \ y \in \mathscr{D}(L).$$

THEOREM 2.1. *If the matrices A, K satisfy $AEA^* = hE$, $KEK^* = hE$, with $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then the operator L is self-adjoint.*

Proof. See [5]. □

Below, we consider the S-L problems (1.1)–(1.4) with the conditions

$$AEA^* = hE, KEK^* = hE.$$

That is, the concerned S-L operator L generated by the S-L problems (1.1)–(1.4) is self-adjoint. We shall define two fundamental solutions

$$\phi(x,\lambda) = \begin{cases} \phi_1(x,\lambda), \ x \in [-1,0), \\ \phi_2(x,\lambda), \ x \in (0,1], \end{cases} \quad \text{and} \quad \chi(x,\lambda) = \begin{cases} \chi_1(x,\lambda), \ x \in [-1,0), \\ \chi_2(x,\lambda), \ x \in (0,1], \end{cases}$$

of the differential equation (1.1), which satisfy the TC (1.3) through the following procedure.

At first we consider the initial-value problem

$$\begin{cases} -y'' + q_1(x)y = \lambda y, \quad x \in [-1,0), \\ y(-1) = 1, y'(-1) = 0. \end{cases} \tag{2.2}$$

By virtue of Theorem 1.5 in [11], the problem has a unique solution $\phi_1(x,\lambda)$ for each $\lambda \in \mathbb{C}$, which is an entire function of λ for each fixed $x \in [-1,0)$. Similarly, for the initial-value problem

$$\begin{cases} -y'' + q_1(x)y = \lambda y, \quad x \in [-1,0), \\ y(-1) = 0, y'(-1) = 1, \end{cases} \tag{2.3}$$

the problem also has a unique solution $\chi_1(x,\lambda)$ which is an entire function of λ for each fixed $x \in [-1,0)$.

The initial-value problem

$$\begin{cases} -y'' + q_2(x)y = \lambda y, \quad x \in (0,1], \\ y(0+) = k_{11}\phi_1(0-,\lambda) + k_{12}\phi_1'(0-,\lambda), \\ y'(0+) = k_{21}\phi_1(0-,\lambda) + k_{22}\phi_1'(0-,\lambda), \end{cases} \tag{2.4}$$

has a unique solution $\phi_2(x,\lambda)$ for each $\lambda \in \mathbb{C}$. Moreover $\phi_2(x,\lambda)$ is an entire function of λ for each fixed $x \in (0,1]$. Similarly, the initial-value problem

$$\begin{cases} -y'' + q_2(x)y = \lambda y, \quad x \in (0,1], \\ y(0+) = k_{11}\chi_1(0-,\lambda) + k_{12}\chi_1'(0-,\lambda), \\ y'(0+) = k_{21}\chi_1(0-,\lambda) + k_{22}\chi_1'(0-,\lambda), \end{cases} \tag{2.5}$$

also has a unique solution $\chi_2(x,\lambda)$, which is an entire function of λ for each fixed $x \in (0,1]$. Obviously, $\phi(x,\lambda)$, $\chi(x,\lambda)$ satisfy the equation (1.1) and the TC (1.3).

It is well known, from the ordinary linear differential equation theory, the Wronskian $W(\phi_j(x,\lambda),\ \chi_j(x,\lambda))$ is independent of the variable x. Let $\omega_j(\lambda) := W(\phi_j(x,\lambda),$

$\chi_j(x,\lambda))$, then we have

$$\omega_1(\lambda)=\omega_1(\lambda)|_{x=-1}=\begin{vmatrix}\phi_1(-1,\lambda) & \chi_1(-1,\lambda)\\ \phi_1'(-1,\lambda) & \chi_1'(-1,\lambda)\end{vmatrix}=1,$$

$$\begin{aligned}\omega_2(\lambda)=\omega_2(\lambda)|_{x=0+}&=\begin{vmatrix}\phi_2(0+,\lambda) & \chi_2(0+,\lambda)\\ \phi_2'(0+,\lambda) & \chi_2'(0+,\lambda)\end{vmatrix}\\ &=\begin{vmatrix}k_{11}\phi_1(0-,\lambda)+k_{12}\phi_1'(0-,\lambda) & k_{11}\chi_1(0-,\lambda)+k_{12}\chi_1'(0-,\lambda)\\ k_{21}\phi_1(0-,\lambda)+k_{22}\phi_1'(0-,\lambda) & k_{21}\chi_1(0-,\lambda)+k_{22}\chi_1'(0-,\lambda)\end{vmatrix}=h\omega_1(\lambda)=h.\end{aligned}$$

LEMMA 2.2. *Let*

$$y(x,\lambda)=\begin{cases}y_1(x,\lambda), & x\in[-1,0),\\ y_2(x,\lambda), & x\in(0,1],\end{cases}$$

be a solution of the equation (1.1), then the solution can be expressed in the following form

$$y(x,\lambda)=\begin{cases}c_1\phi_1(x,\lambda)+c_2\chi_1(x,\lambda), & x\in[-1,0),\\ d_1\phi_2(x,\lambda)+d_2\chi_2(x,\lambda), & x\in(0,1].\end{cases}\tag{2.6}$$

If $y(x,\lambda)$ satisfies the TC (1.3), then $c_1=d_1$, $c_2=d_2$.

Proof. Since $y(x,\lambda)$ satisfies the TC (1.3), namely

$$\begin{aligned}&k_{11}(c_1\phi_1(0-,\lambda)+c_2\chi_1(0-,\lambda))+k_{12}(c_1\phi_1'(0-,\lambda)+c_2\chi_1'(0-,\lambda))\\ &\qquad-(d_1\phi_2(0+,\lambda)+d_2\chi_2(0+,\lambda))=0,\\ &k_{21}(c_1\phi_1(0-,\lambda)+c_2\chi_1(0-,\lambda))+k_{22}(c_1\phi_1'(0-,\lambda)+c_2\chi_1'(0-,\lambda))\\ &\qquad-(d_1\phi_2'(0+,\lambda)+d_2\chi_2'(0+,\lambda))=0.\end{aligned}$$

From (2.4), (2.5), the last equation system becomes

$$\begin{cases}(c_1-d_1)\phi_2(0+,\lambda)+(c_2-d_2)\chi_2(0+,\lambda)=0,\\ (c_1-d_1)\phi_2'(0+,\lambda)+(c_2-d_2)\chi_2'(0+,\lambda)=0.\end{cases}$$

Since the determinant of the coefficient matrix of the equation system is

$$\begin{vmatrix}\phi_2(0+,\lambda) & \chi_2(0+,\lambda)\\ \phi_2'(0+,\lambda) & \chi_2'(0+,\lambda)\end{vmatrix}=\omega_2(\lambda)\neq 0,$$

we get $c_1=d_1, c_2=d_2$. □

Let

$$\Phi_j(x,\lambda)=\begin{pmatrix}\phi_j(x,\lambda) & \chi_j(x,\lambda)\\ \phi_j'(x,\lambda) & \chi_j'(x,\lambda)\end{pmatrix},\quad j=1,2,$$

and let

$$\Phi(x,\lambda)=\begin{cases}\Phi_1(x,\lambda), & x\in[-1,0),\\ \Phi_2(x,\lambda), & x\in(0,1].\end{cases}\tag{2.7}$$

THEOREM 2.3. *Let* $\lambda_0 \in \mathbb{C}$. λ_0 *is an eigenvalue of the S-L problems (1.1)–(1.4) if and only if* $\Delta(\lambda_0) := \det(A - \Phi(1,\lambda_0)) = 0$, *where* $A = e^{i\gamma}\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix}$.

Proof. Let λ_0 be an eigenvalue of the S-L problems (1.1)–(1.4) and $y(x,\lambda_0)$ be any corresponding eigenfunction. From Lemma 2.2, there exist c_1, c_2 such that

$$y(x,\lambda_0) = \begin{cases} c_1\phi_1(x,\lambda_0) + c_2\chi_1(x,\lambda_0), \; x \in [-1,0), \\ c_1\phi_2(x,\lambda_0) + c_2\chi_2(x,\lambda_0), \; x \in (0,1], \end{cases} \tag{2.8}$$

where at least one of the constants c_1, c_2 is not zero. Substituting (2.8) into the boundary condition (1.2) we obtain

$$e^{i\gamma}\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix}\begin{pmatrix}c_1\phi_1(-1,\lambda_0) + c_2\chi_1(-1,\lambda_0)\\ c_1\phi_1'(-1,\lambda_0) + c_2\chi_1'(-1,\lambda_0)\end{pmatrix} - \begin{pmatrix}c_1\phi_2(1,\lambda_0) + c_2\chi_2(1,\lambda_0)\\ c_1\phi_2'(1,\lambda_0) + c_2\chi_2'(1,\lambda_0)\end{pmatrix} = 0$$

that is,

$$\left[e^{i\gamma}\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix} - \begin{pmatrix}\phi_2(1,\lambda_0) & \chi_2(1,\lambda_0)\\ \phi_2'(1,\lambda_0) & \chi_2'(1,\lambda_0)\end{pmatrix}\right]\begin{pmatrix}c_1\\ c_2\end{pmatrix} = 0.$$

Since at least one of the constants $c_1,\ c_2$ is not zero, we obtain

$$\Delta(\lambda_0) = \det(A - \Phi(1,\lambda_0)) = 0, \tag{2.9}$$

where $A = e^{i\gamma}\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix}$ with $-\pi < \gamma \leqslant \pi$ and $\alpha_1\alpha_4 - \alpha_2\alpha_3 > 0$.

Conversely, if $\det(A - \Phi(1,\lambda_0)) = 0$, then the equation

$$(A - \Phi(1,\lambda_0))\begin{pmatrix}d_1\\ d_2\end{pmatrix} = 0,$$

has a nonzero solution (c_1', c_2'). Let

$$y(x,\lambda_0) = \begin{cases} c_1'\phi_1(x,\lambda_0) + c_2'\chi_1(x,\lambda_0), \; x \in [-1,0), \\ c_1'\phi_2(x,\lambda_0) + c_2'\chi_2(x,\lambda_0), \; x \in (0,1]. \end{cases} \tag{2.10}$$

Then $y(x,\lambda_0)$ is a nonzero solution of the equation (1.1) and satisfies the boundary and transmission conditions (1.2), (1.3). Hence λ_0 is an eigenvalue of the S-L problems (1.1)–(1.4), and $y(x,\lambda_0)$ is the corresponding eigenfunction. □

LEMMA 2.4. *Let L be the operator defined by the S-L problems (1.1)–(1.4). Then the eigenvalues of the operator L are countably infinite.*

Proof. From Theorem 2.3, the eigenvalues of the S-L problems (1.1)–(1.4) are zeros of the entire function $\Delta(\lambda)$. Since the S-L operator L generated by the S-L problems (1.1)–(1.4) is self-adjoint, the eigenvalues of the operator L are real. Then $\Delta(\lambda) \neq 0$ for $\lambda \in \mathbb{C}(\Im\lambda \neq 0)$, so $\Delta(\lambda)$ is not identical to zero for $\lambda \in \mathbb{C}\backslash\mathbb{R}$. By the properties of zeros of the entire function, the eigenvalues of the operator L are countably infinite. □

3. Green's function of the S-L problems

We go on to construct the Green's function of the S-L operator L generated by the S-L problems (1.1)–(1.4). Let $\lambda \in \Omega = \{\lambda \in \mathbb{C} \mid \Delta(\lambda) \neq 0.\}$ and let $f \in H$. We consider the non-homogeneous differential equation

$$ly - \lambda y = f(x), \quad x \in [-1,0) \cup (0,1], \tag{3.1}$$

together with the CBC and TC (1.2)–(1.4). We can represent the general solution of the differential equation $ly - \lambda y = f_1(x)$, $x \in [-1,0)$ in the form

$$y_1(x,\lambda) = \phi_1(x,\lambda)\int_{-1}^{x}\chi_1(\xi,\lambda)f_1(\xi)d\xi - \chi_1(x,\lambda)\int_{-1}^{x}\phi_1(\xi,\lambda)f_1(\xi)d\xi \tag{3.2}$$
$$+ d_1\phi_1(x,\lambda) + d_2\chi_1(x,\lambda),$$

where $f_1 = f(x)\mid_{[-1,0)}$ and $d_1, d_2 \in \mathbb{C}$. And the general solution of the differential equation $ly - \lambda y = f_2(x), x \in (0,1]$ can be represented in the form

$$y_2(x,\lambda) = \frac{1}{h}\phi_2(x,\lambda)\int_{0}^{x}\chi_2(\xi,\lambda)f_2(\xi)d\xi - \frac{1}{h}\chi_2(x,\lambda)\int_{0}^{x}\phi_2(\xi,\lambda)f_2(\xi)d\xi \tag{3.3}$$
$$+ e_1\phi_2(x,\lambda) + e_2\chi_2(x,\lambda).$$

where $f_2 = f(x)\mid_{(0,1]}$ and $e_1, e_2 \in \mathbb{C}$. Taking into account the TC (1.3), (1.4) and by (2.4), (2.5), we obtain

$$\begin{aligned} &k_{11}y(0-) + k_{12}y'(0-) - y(0+) \\ &\quad = \phi_2(0,\lambda)\int_{-1}^{0}\chi_1(\xi,\lambda)f_1(\xi)d\xi - \chi_2(0,\lambda)\int_{-1}^{0}\phi_1(\xi,\lambda)f_1(\xi)d\xi + d_1\phi_2(0,\lambda) \\ &\qquad + d_2\chi_2(0,\lambda) - e_1\phi_2(0,\lambda) - e_2\chi_2(0,\lambda) = 0, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &k_{21}y(0-) + k_{22}y'(0-) - y'(0+) \\ &\quad = \phi_2'(0,\lambda)\int_{-1}^{0}\chi_1(\xi,\lambda)f_1(\xi)d\xi - \chi_2'(0,\lambda)\int_{-1}^{0}\phi_1(\xi,\lambda)f_1(\xi)d\xi + d_1\phi_2'(0,\lambda) \\ &\qquad + d_2\chi_2'(0,\lambda) - e_1\phi_2'(0,\lambda) - e_2\chi_2'(0,\lambda) = 0. \end{aligned} \tag{3.5}$$

From (3.4), (3.5), we get

$$e_1 = d_1 + \int_{-1}^{0}\chi_1(\xi,\lambda)f_1(\xi)d\xi, \quad e_2 = d_2 - \int_{-1}^{0}\phi_1(\xi,\lambda)f_1(\xi)d\xi.$$

Substituting them into (3.3), we obtain

$$\begin{aligned} y_2(x,\lambda) = &\frac{1}{h}\phi_2(x,\lambda)\int_{0}^{x}\chi_2(\xi,\lambda)f_2(\xi)d\xi - \frac{1}{h}\chi_2(x,\lambda)\int_{0}^{x}\phi_2(\xi,\lambda)f_2(\xi)d\xi \\ &+ \phi_2(x,\lambda)\int_{-1}^{0}\chi_1(\xi,\lambda)f_1(\xi)d\xi + \chi_2(x,\lambda)\int_{-1}^{0}\phi_1(\xi,\lambda)f_1(\xi)d\xi \\ &+ d_1\phi_2(x,\lambda) + d_2\chi_2(x,\lambda). \end{aligned} \tag{3.6}$$

By applying the boundary condition (1.2), we have

$$\begin{aligned}
&\alpha_1 e^{i\gamma} y(-1) + \alpha_2 e^{i\gamma} y'(-1) - y(1) \\
= &\ \alpha_1 e^{i\gamma}\big(d_1\phi_1(-1,\lambda) + d_2\chi_1(-1,\lambda)\big) + \alpha_2 e^{i\gamma}\big(d_1\phi_1'(-1,\lambda) \\
&+ d_2\chi_1'(-1,\lambda)\big) - \frac{1}{h}\phi_2(1,\lambda)\int_0^1 \chi_2(\xi,\lambda) f_2(\xi)d\xi + \frac{1}{h}\chi_2(1,\lambda)\int_0^1 \phi_2(\xi,\lambda) f_2(\xi)d\xi \\
&- \phi_2(1,\lambda)\int_{-1}^0 \chi_1(\xi,\lambda) f_1(\xi)d\xi - \chi_2(1,\lambda)\int_{-1}^0 \phi_1(\xi,\lambda) f_1(\xi)d\xi \\
&- d_1\phi_2(1,\lambda) - d_2\chi_2(1,\lambda) = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\alpha_3 e^{i\gamma} y(-1) + \alpha_4 e^{i\gamma} y'(-1) - y'(1) \\
= &\ \alpha_3 e^{i\gamma}\big(d_1\phi_1(-1,\lambda) + d_2\chi_1(-1,\lambda)\big) + \alpha_4 e^{i\gamma}\big(d_1\phi_1'(-1,\lambda) \\
&+ d_2\chi_1'(-1,\lambda)\big) - \frac{1}{h}\phi_2'(1,\lambda)\int_0^1 \chi_2(\xi,\lambda) f_2(\xi)d\xi + \frac{1}{h}\chi_2'(1,\lambda)\int_0^1 \phi_2(\xi,\lambda) f_2(\xi)d\xi \\
&- \phi_2'(1,\lambda)\int_{-1}^0 \chi_1(\xi,\lambda) f_1(\xi)d\xi - \chi_2'(1,\lambda)\int_{-1}^0 \phi_1(\xi,\lambda) f_1(\xi)d\xi \\
&- d_1\phi_2'(1,\lambda) - d_2\chi_2'(1,\lambda) = 0.
\end{aligned}$$

And from the values of $\phi_1(-1,\lambda), \phi_1'(-1,\lambda), \chi_1(-1,\lambda), \chi_1'(-1,\lambda)$ in (2.2), (2.3), the following equations related to $d_1,\ d_2$ are obtained

$$\begin{cases}
d_1(\alpha_1 e^{i\gamma} - \phi_2(1,\lambda)) + d_2(\alpha_2 e^{i\gamma} - \chi_2(1,\lambda)) - \phi_2(1,\lambda)(\int_{-1}^0 \chi_1(\xi,\lambda) f_1(\xi)d\xi \\
\quad + \frac{1}{h}\int_0^1 \chi_2(\xi,\lambda) f_2(\xi)d\xi) + \chi_2(1,\lambda)(\int_{-1}^0 \phi_1(\xi,\lambda) f_1(\xi)d\xi \\
\quad + \frac{1}{h}\int_0^1 \phi_2(\xi,\lambda) f_2(\xi)d\xi) = 0, \\
d_1(\alpha_1 e^{i\gamma} - \phi_2'(1,\lambda)) + d_2(\alpha_2 e^{i\gamma} - \chi_2'(1,\lambda)) - \phi_2'(1,\lambda)(\int_{-1}^0 \chi_1(\xi,\lambda) f_1(\xi)d\xi \\
\quad + \frac{1}{h}\int_0^1 \chi_2(\xi,\lambda) f_2(\xi)d\xi) + \chi_2'(1,\lambda)(\int_{-1}^0 \phi_1(\xi,\lambda) f_1(\xi)d\xi \\
\quad + \frac{1}{h}\int_0^1 \phi_2(\xi,\lambda) f_2(\xi)d\xi) = 0.
\end{cases} \tag{3.7}$$

Since the determinant of the coefficients of the above equations is equal to $\Delta(\lambda)$, which is nonzero for $\lambda \in \Omega$, the equations have the solutions

$$\begin{cases}
d_1 = \frac{1}{\Delta(\lambda)}\phi_{24}(1)(\int_{-1}^0 \chi_1(\xi,\lambda) f_1(\xi)d\xi + \frac{1}{h}\int_0^1 \chi_2(\xi,\lambda) f_2(\xi)d\xi) \\
\qquad + \frac{1}{\Delta(\lambda)}\chi_{24}(1)(\int_{-1}^0 \phi_1(\xi,\lambda) f_1(\xi)d\xi + \frac{1}{h}\int_0^1 \phi_2(\xi,\lambda) f_2(\xi)d\xi), \\
d_2 = \frac{1}{\Delta(\lambda)}\phi_{13}(1)(\int_{-1}^0 \chi_1(\xi,\lambda) f_1(\xi)d\xi + \frac{1}{h}\int_0^1 \chi_2(\xi,\lambda) f_2(\xi)d\xi) \\
\qquad + \frac{1}{\Delta(\lambda)}\chi_{13}(1)(\int_{-1}^0 \phi_1(\xi,\lambda) f_1(\xi)d\xi + \frac{1}{h}\int_0^1 \phi_2(\xi,\lambda) f_2(\xi)d\xi),
\end{cases} \tag{3.8}$$

where

$$\phi_{13}(1) = \alpha_1 e^{i\gamma}\phi_2'(1,\lambda) - \alpha_3 e^{i\gamma}\phi_2(1,\lambda), \phi_{24}(1) = \alpha_4 e^{i\gamma}\phi_2(1,\lambda) - \alpha_2 e^{i\gamma}\phi_2'(1,\lambda) - \omega_2, \tag{3.9}$$
$$\chi_{24}(1) = \alpha_2 e^{i\gamma}\chi_2'(1,\lambda) - \alpha_4 e^{i\gamma}\chi_2(1,\lambda), \chi_{13}(1) = \alpha_3 e^{i\gamma}\chi_2(1,\lambda) - \alpha_1 e^{i\gamma}\chi_2'(1,\lambda) + \omega_2.$$

Substituting (3.9) into (3.2), (3.6), we obtain

$$y(x,\lambda) = h\int_{-1}^{0} G(x,\xi,\lambda) f_1(\xi)d\xi + \int_0^1 G(x,\xi,\lambda) f_2(\xi)d\xi, \tag{3.10}$$

where $G(x,\xi,\lambda)$ is as follows:

$$G(x,\xi,\lambda) = \tag{3.11}$$
$$\begin{cases} \frac{1}{\triangle(\lambda)h}[\chi_{13}(1)\phi_1(x,\lambda) - \phi_{13}(1)\chi_1(x,\lambda)]\chi_1(\xi,\lambda) + \frac{1}{\triangle(\lambda)h}[\phi_{24}(1)\chi_1(x,\lambda) \\ \qquad -\chi_{24}(1)\phi_1(x,\lambda)]\phi_1(\xi,\lambda), & -1<\xi\leqslant x<0, \\ \frac{1}{\triangle(\lambda)h}[\phi_{24}(1)\phi_1(x,\lambda) - \phi_{13}(1)\chi_1(x,\lambda)]\chi_1(\xi,\lambda) + \frac{1}{\triangle(\lambda)h}[\chi_{13}(1)\chi_1(x,\lambda) \\ \qquad -\chi_{24}(1)\phi_1(x,\lambda)]\phi_1(\xi,\lambda), & -1<x\leqslant\xi<0, \\ \frac{1}{\triangle(\lambda)h^2}[\phi_{24}(1)\phi_1(x,\lambda) - \phi_{13}(1)\chi_1(x,\lambda)]\chi_2(\xi,\lambda) + \frac{1}{\triangle(\lambda)h^2}[\chi_{13}(1)\chi_1(x,\lambda) \\ \qquad -\chi_{24}(1)\phi_1(x,\lambda)]\phi_2(\xi,\lambda), & -1<x<0, 0<\xi<1, \\ \frac{1}{\triangle(\lambda)}[\chi_{13}(1)\phi_2(x,\lambda) - \phi_{13}(1)\chi_2(x,\lambda)]\chi_1(\xi,\lambda) + \frac{1}{\triangle(\lambda)}[\phi_{24}(1)\chi_2(x,\lambda) \\ \qquad -\chi_{24}(1)\phi_2(x,\lambda)]\phi_1(\xi,\lambda), & -1<\xi<0, 0<x<1, \\ \frac{1}{\triangle(\lambda)h}[\chi_{13}(1)\phi_2(x,\lambda) - \phi_{13}(1)\chi_2(x,\lambda)]\chi_2(\xi,\lambda) + \frac{1}{\triangle(\lambda)h}[\phi_{24}(1)\chi_2(x,\lambda) \\ \qquad -\chi_{24}(1)\phi_2(x,\lambda)]\phi_2(\xi,\lambda), & 0<\xi\leqslant x<1, \\ \frac{1}{\triangle(\lambda)h}[\phi_{24}(1)\phi_2(x,\lambda) - \phi_{13}(1)\chi_2(x,\lambda)]\chi_2(\xi,\lambda) + \frac{1}{\triangle(\lambda)h}[\chi_{13}(1)\chi_2(x,\lambda) \\ \qquad -\chi_{24}(1)\phi_2(x,\lambda)]\phi_2(\xi,\lambda), & 0<x\leqslant\xi<1. \end{cases}$$

THEOREM 3.1. *Let $f \in H$, then the function*

$$y(x,\lambda) = h\int_{-1}^{0} G(x,\xi,\lambda) f(\xi)d\xi + \int_0^1 G(x,\xi,\lambda) f(\xi)d\xi \tag{3.12}$$

satisfies (1.1) and the CBC and TC (1.2)–(1.4).

Proof. From the above calculations in the construction of the Green's function, the theorem is obvious. □

Thus the resolvent of the S-L problems (1.1)–(1.4) is obtained, and the function $G(x,\xi,\lambda)$ is the Green's function of the S-L problems (1.1)–(1.4). From the above calculations the domain of $(L-\lambda I)^{-1}$, which is the resolvent of L at $\lambda \in \Omega$, is the space H. And the S-L operator L is self-adjoint. So by the Closed Graph Theorem, $(L-\lambda I)^{-1}$ is bounded. Then we have

THEOREM 3.2. *The operator L has only point-spectrum, i.e., $\sigma(L) = \sigma_p(L)$.*

LEMMA 3.3. *Let $\delta \in \mathbb{R} \setminus \sigma_p(L)$. And let μ be the eigenvalue of $(L-\delta I)^{-1}$ and y be the corresponding eigenfunction. Then $\frac{1}{\mu}$ is the eigenvalue of $L-\delta I$, and y is the corresponding eigenfunction, and vice versa.*

4. The modified parseval equality

In this section, we show the eigenvalues of the S-L problems (1.2)–(1.4) are simple under some conditions. And using the eigenfunction expansion of the Green's function, we prove the modified Parseval equality.

Let

$$D(\lambda) = \alpha_4 \phi_2(1,\lambda) - \alpha_2 \phi_2'(1,\lambda) - \alpha_3 \chi_2(1,\lambda) + \alpha_1 \chi_2'(1,\lambda), \tag{4.1}$$

then

$$\Delta(\lambda) = (1+e^{2i\gamma})h - D(\lambda)e^{i\gamma}, \tag{4.2}$$

where $\Delta(\lambda)$ is the same as in Theorem 2.3.

LEMMA 4.1. *Let $\lambda \in \sigma_p(A_\gamma) = \{\lambda \in \mathbb{C} \mid \Delta(\lambda) = 0 \text{ for } \gamma \in [-\pi,\pi]\}$, and be denoted by $\lambda(A_\gamma)$. Then*

1. *$\lambda_n(A_\gamma) = \lambda_n(A_{-\gamma})$ for $n \in \mathbb{N}$ and $0 < \gamma < \pi$.*
2. *$\lambda_n(A_\alpha) \neq \lambda_m(A_\beta)$ for $n,m \in \mathbb{N}$ and $0 \leqslant \alpha,\beta \leqslant \pi$ with $\alpha \neq \beta$.*

Proof. At first we prove the case (1). Let $n \in \mathbb{N}$, $\gamma \in (0,\pi)$ and $\lambda_n(A_\gamma) \in \sigma_p(A_\gamma)$. From (4.2), $\Delta(\lambda) = 0$ if and only if $D(\lambda) = 2h\cos\gamma$. Hence $\lambda_n(A_\gamma)$ satisfies $D(\lambda_n(A_\gamma)) = 2h\cos\gamma$. Since $\cos(-\gamma) = \cos\gamma$ and $h > 0$, $D(\lambda_n(A_\gamma)) = D(\lambda_n(A_{-\gamma}))$ for $n \in \mathbb{N}$. We obtain $\lambda_n(A_\gamma) = \lambda_n(A_{-\gamma})$ for $n \in \mathbb{N}$.

Next we prove the case (2). Let $n,m \in \mathbb{N}$, $\alpha,\beta \in [0,\pi]$ with $\alpha \neq \beta$. From (4.2), $\lambda_n(A_\alpha) \in \sigma_p(A_\alpha)$, $\lambda_m(A_\beta) \in \sigma_p(A_\beta)$ satisfy

$$D(\lambda_n(A_\alpha)) = 2h\cos\alpha, \quad D(\lambda_m(A_\beta)) = 2h\cos\beta.$$

Since $\alpha,\beta \in [0,\pi]$ with $\alpha \neq \beta$, $\cos\alpha \neq \cos\beta$. Hence $D(\lambda_n(A_\alpha)) \neq D(\lambda_m(A_\beta))$. Consequently, $\lambda_n(A_\alpha) \neq \lambda_m(A_\beta)$ for $n,m \in \mathbb{N}$. □

LEMMA 4.2. (Corollary 1, P246, [12]) *Let T be a closed symmetric operator on a complex Hilbert space with finite defect indices (m,m), and T_1 and T_2 be self-adjoint extensions of T. If $\sigma(T_1) \cap (a,b) = \varnothing$, then $\sigma(T_2) \cap (a,b)$ consists of only isolated eigenvalues of total multiplicity $\leqslant m$.*

The eigenvalues of the S-L problems with the coupled boundary conditions are concerned in [3, 7]. And the simplicity of the eigenvalues of the S-L problem with the condition $h = 1$ is obtained in Theorem 3.4 of [3]. Here we use the similar method to prove that the eigenvalues, of the S-L problems with the condition $h > 0$, are simple.

LEMMA 4.3. *If $0<\gamma<\pi$ or $-\pi<\gamma<0$, γ is as in (1.4), then the eigenvalues of the S-L operators L are simple.*

Proof. From Theorem 3.2, $\sigma(A_\gamma)=\sigma_p(A_\gamma)$. Let $\lambda_n(A_0)\in\sigma(A_0)$ for some $n\in\mathbb{N}$. By Lemma 4.1, we can choose the eigenvalue $\lambda_m(A_\pi)\in\sigma(A_\pi)$ to be the first eigenvalue in $\sigma(A_\pi)$ to the right of $\lambda_n(A_0)$ and $\lambda_n(A_0)\neq\lambda_m(A_\pi)$. We show the monotonicity of $D(\lambda)$ in the interval $[\lambda_n(A_0),\lambda_m(A_\pi)]$ by a contradiction. Assume $D(\lambda)$ given in (4.1), is neither strictly increasing nor strictly decreasing in the interval $[\lambda_n(A_0),\lambda_m(A_\pi)]$. Then there exists an $\alpha\in(0,\pi)$ such that

$$D(\lambda)=2h\cos\alpha$$

has three solutions in $(\lambda_n(A_0),\lambda_m(A_\pi))$. That is, there are three points of $\sigma(A_\alpha)$ in $(\lambda_n(A_0),\lambda_m(A_\pi))$. On the other hand, no points of $\sigma(A_0)$, $\sigma(A_\pi)$ are in $(\lambda_n(A_0),\lambda_m(A_\pi))$. And the operator L for $\gamma=0$ and $\gamma=\pi$ are both self-adjoint operators. This is a contradiction from Lemma 4.2. Hence $D(\lambda)$ is strictly increasing or strictly decreasing in the interval $[\lambda_n(A_0),\lambda_m(A_\pi)]$. From Theorem 3.2, Lemma 4.2 and the equation (4.1), if $0<\gamma<\pi$ or $-\pi<\gamma<0$, then the eigenvalues of the S-L operators L are simple. $\square$

By Lemmas 2.4, 3.3, 4.3 and the spectral theorem for compact operator, we have

LEMMA 4.4. *Let $\lambda_1,\lambda_2,\lambda_3,\cdots,$ be the collection of all eigenvalues of the S-L operators L and let $\varphi_1(x),\varphi_2(x),\cdots$ be the corresponding normalized eigenfunctions. Then*

$$|\lambda_1|<|\lambda_2|<\cdots|\lambda_n|\cdots\to\infty.$$

And $\{\varphi_n;n\in\mathbb{N}\}$ is complete in H and

$$\langle\varphi_n,\varphi_m\rangle=\begin{cases}1, & n=m,\\ 0, & n\neq m.\end{cases}$$

LEMMA 4.5. *The S-L problems (1.1)–(1.4) is equivalent to the following integral equation*

$$y(x,\lambda)-\lambda\big(h\int_{-1}^{0}G(x,\xi)y(\xi)d\xi+\int_{0}^{1}G(x,\xi)y(\xi)d\xi\big)=0. \tag{4.3}$$

Proof. From Theorem 3.1 we know that

$$y(x,\lambda)=h\int_{-1}^{0}G(x,\xi)f(\xi)d\xi+\int_{0}^{1}G(x,\xi)f(\xi)d\xi \tag{4.4}$$

satisfies $-y''(x)+q(x)y(x)=f(x)$ and the CBC and TC (1.2)–(1.4). The nonhomogeneous differential equation (3.1) can be written in the form $-y''(x)+q(x)y(x)=\tilde{f}(x)$ where $\tilde{f}(x)=f(x)+\lambda y$. Then the equation has a solution

$$y(x,\lambda)=h\int_{-1}^{0}G(x,\xi)\tilde{f}(\xi)d\xi+\int_{0}^{1}G(x,\xi)\tilde{f}(\xi)d\xi, \tag{4.5}$$

which satisfies the CBC and TC (1.2), (1.3). If $f(x) \equiv 0$, then the corresponding homogeneous cases are the S-L problems (1.1)–(1.4). Consequently the problem is equivalent to

$$y(x,\lambda) - \lambda\big(h\int_{-1}^{0} G(x,\xi)y(\xi)d\xi + \int_{0}^{1} G(x,\xi)y(\xi)d\xi\big) = 0. \tag{4.6}$$

□

THEOREM 4.6. *Let* $\{\lambda_n : n = 1,2,3,\cdots\}$ *denote the eigenvalues of the S-L problems (1.1)–(1.4) and* $\varphi_n(x)$ *be the corresponding normalized eigenfunction. Then*

$$G(x,\xi) = -\sum_{n=1}^{\infty} \frac{\varphi_n(x)\overline{\varphi_n(\xi)}}{\lambda_n}.$$

Proof. Suppose λ_n be the eigenvalue of the S-L problems (1.1)–(1.4) and $\varphi_n(x)$ be the corresponding normalized eigenfunction. Let $P(x,\xi) = G(x,\xi) + \sum_{n=1}^{\infty} \frac{\varphi_n(x)\overline{\varphi_n(\xi)}}{\lambda_n}$, then $P(x,\xi)$ is continuous and symmetric. We assume $P(x,\xi) \neq 0$. Then by the Fredholm integral equation, there is a number $\tilde{\lambda}$ and a function $\tilde{y}(x) \neq 0$ in H such that

$$\tilde{y}(x) = \tilde{\lambda}\big(h\int_{-1}^{0} P(x,\xi)\tilde{y}(\xi)d\xi + \int_{0}^{1} P(x,\xi)\tilde{y}(\xi)d\xi\big). \tag{4.7}$$

By Lemma 4.5

$$\varphi_n(x) - \lambda_n\big(h\int_{-1}^{0} G(x,\xi)\varphi_n(\xi)d\xi + \int_{0}^{1} G(x,\xi)\varphi_n(x)(\xi)d\xi\big) = 0. \tag{4.8}$$

Putting $G(x,\xi) = P(x,\xi) - \sum_{n=1}^{\infty} \frac{\varphi_n(x)\overline{\varphi_n(\xi)}}{\lambda_n}$ in the equation (4.8) and through some calculations, we obtain

$$h\int_{-1}^{0} P(x,\xi)\varphi_n(\xi)d\xi + \int_{0}^{1} P(x,\xi)\varphi_n(\xi)d\xi = 0. \tag{4.9}$$

Next we prove $\langle \tilde{y},\ \varphi_n\rangle = 0$ and $\tilde{y}$ is an eigenfunction. In accordance with (4.7) and (4.9), it leads to

$$\begin{aligned}
\langle \tilde{y}, \varphi_n\rangle =& h\int_{-1}^{0} \tilde{y}(x)\overline{\varphi_n(x)}dx + \int_{0}^{1} \tilde{y}(x)\overline{\varphi_n(x)}dx \\
=& \tilde{\lambda}h\int_{-1}^{0}\Big(h\int_{-1}^{0} P(x,\xi)\tilde{y}(\xi)d\xi + \int_{0}^{1} P(x,\xi)\tilde{y}(\xi)d\xi\Big)\overline{\varphi_n(x)}dx \\
&+ \tilde{\lambda}\int_{0}^{1}\Big(h\int_{-1}^{0} P(x,\xi)\tilde{y}(\xi)d\xi + \int_{0}^{1} P(x,\xi)\tilde{y}(\xi)d\xi\Big)\overline{\varphi_n(x)}dx \\
=& \tilde{\lambda}h\int_{-1}^{0}\Big(h\int_{-1}^{0} P(x,\xi)\overline{\varphi_n(x)}dx + \int_{0}^{1} P(x,\xi)\overline{\varphi_n(x)}dx\Big)\tilde{y}(\xi)d\xi \\
&+ \tilde{\lambda}\int_{0}^{1}\Big(h\int_{-1}^{0} P(x,\xi)\overline{\varphi_n(x)}dx + \int_{0}^{1} P(x,\xi)\overline{\varphi_n(x)}dx\Big)\tilde{y}(\xi)d\xi = 0.
\end{aligned}$$

And by (4.7) we have

$$
\begin{aligned}
\tilde{y}(x)-&\tilde{\lambda}\big(h\int_{-1}^{0}G(x,\xi)\tilde{y}(\xi)d\xi+\int_{0}^{1}G(x,\xi)\tilde{y}(\xi)d\xi\big)\\
&=\tilde{y}(x)-\tilde{\lambda}\big(h\int_{-1}^{0}\big(P(x,\xi)-\sum_{n=1}^{\infty}\frac{\varphi_n(x)\overline{\varphi_n(\xi)}}{\lambda_n}\big)\tilde{y}(\xi)d\xi+\int_{0}^{1}\big(P(x,\xi)\\
&\quad-\sum_{n=1}^{\infty}\frac{\varphi_n(x)\overline{\varphi_n(\xi)}}{\lambda_n}\big)\tilde{y}(\xi)d\xi\big)\\
&=\tilde{y}(x)-\tilde{\lambda}\Big(\big(h\int_{-1}^{0}P(x,\xi)\tilde{y}(\xi)d\xi+\int_{0}^{1}P(x,\xi)\tilde{y}(\xi)d\xi\big)-\sum_{n=1}^{\infty}\frac{\varphi_n(x)}{\lambda_n}\langle\tilde{y},\varphi_n\rangle\Big)\\
&=\tilde{y}(x)-\tilde{\lambda}\Big(\big(h\int_{-1}^{0}P(x,\xi)\tilde{y}(\xi)d\xi+\int_{0}^{1}P(x,\xi)\tilde{y}(\xi)d\xi\big)=0.
\end{aligned}
$$

This implies that $\tilde{y}$ is the eigenfunction of the S-L problems (1.1)–(1.4) by Lemma 4.5. Thus from $\langle\tilde{y},\varphi_n\rangle=0$ and the completeness of the eigenfunctions, it leads to $\tilde{y}=0$. Consequently $P(x,\xi)=0$. We complete the proof. □

At last, we will prove the modified Parseval equality, i.e. the Parseval equality in the associated Hilbert space H, holds.

THEOREM 4.7. *Let $f\in H$, then the modified Parseval equality holds, namely*

$$\|f\|_H^2=\sum_{n=1}^{\infty}c_n^2(f), \tag{4.10}$$

where $\|f\|_H^2=\langle f,f\rangle$ and

$$c_n(f)=h\int_{-1}^{0}f(x)\overline{\varphi_n(x)}dx+\int_{0}^{1}f(x)\overline{\varphi_n(x)}dx. \tag{4.11}$$

Proof. Let $\tilde{C}_0^{\infty}$ be the set of all functions defined by

$$f(x)=\begin{cases} f_1(x),\ x\in[-1,0),\\ f_2(x),\ x\in(0,1],\end{cases}$$

where $f_1\in C_0^{\infty}[-1,0)$ and $f_2\in C_0^{\infty}(0,1]$. Obviously, $\tilde{C}_0^{\infty}\subset H$. And it is easy to verify $\tilde{C}_0^{\infty}$ is dense in H. At first we prove (4.10) holds for $f\in\tilde{C}_0^{\infty}$. Denote $g(x)=-f''(x)+q(x)f$. Then by Lemma 4.5 and Theorem 4.6

$$
\begin{aligned}
f(x)=&h\int_{-1}^{0}G(x,\xi)g(\xi)d\xi+\int_{0}^{1}G(x,\xi)g(\xi)d\xi\\
&=-\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\varphi_n(x)\Big(h\int_{-1}^{0}\overline{\varphi_n(\xi)}g(\xi)d\xi+\int_{0}^{1}\overline{\varphi_n(\xi)}g(\xi)d\xi\Big).
\end{aligned}
$$

Multiplying by $\overline{\varphi_m(x)}$ and integrating it, we have

$$\begin{aligned} &h\int_{-1}^{0}\overline{\varphi_m(x)}f(x)dx+\int_{0}^{1}\overline{\varphi_m(x)}f(x)dx \\ &=-\frac{1}{\lambda_m}\left(h\int_{-1}^{0}\overline{\varphi_m(\xi)}g(\xi)d\xi+\int_{0}^{1}\overline{\varphi_m(\xi)}g(\xi)d\xi\right). \end{aligned}$$

Then for $f\in\tilde{C}_0^{\infty}$

$$f(x)=\sum_{n=1}^{\infty}c_n(f)\varphi_n(x), \tag{4.12}$$

where $c_n(f)=\langle f,\varphi_n\rangle=h\int_{-1}^{0}f(x)\overline{\varphi_n(x)}dx+\int_{0}^{1}f(x)\overline{\varphi_n(x)}dx$. Thus for $f\in\tilde{C}_0^{\infty}$

$$\|f\|_H^2=\sum_{n=1}^{\infty}c_n^2(f). \tag{4.13}$$

Next we prove (4.10) holds for all $f\in H$. since $\tilde{C}_0^{\infty}$ is dense in H ([1]), there exists a sequence $\{f_k\}_{k\in\mathbb{N}}\subset\tilde{C}_0^{\infty}$ converging to f in H, we will prove $\sum_{n=1}^{\infty}c_n^2(f)<\infty$ and $\lim_{k\to\infty}\sum_{n=1}^{\infty}c_n^2(f_k)=\sum_{n=1}^{\infty}c_n^2(f)$. By the Cauchy-Schwartz inequality $|c_n(f_k)-c_n(f)|=|\langle f_k-f,\varphi_n\rangle|\leqslant\|f_k-f\|_H$. This implies $\lim_{k\to\infty}c_n(f_k)=c_n(f)$. Since $\sum_{n=1}^{\infty}\big(c_n(f_k)-c_n(f_m)\big)^2$ $=\sum_{n=1}^{\infty}c_n^2(f_k-f_m)=\|f_k-f_m\|_H^2$, so

$$\sum_{n=1}^{N}\big(c_n(f_k)-c_n(f_m)\big)^2\leqslant\|f_k-f_m\|_H^2. \tag{4.14}$$

Let $k\to\infty$, then $\sum_{n=1}^{N}\big(c_n(f)-c_n(f_m)\big)^2\leqslant\|f-f_m\|_H^2$. Letting $N\to\infty$ we have

$$\sum_{n=1}^{\infty}\big(c_n(f)-c_n(f_m)\big)^2\leqslant\|f-f_m\|_H^2. \tag{4.15}$$

Then by the Minkowski inequality

$$\begin{aligned} \sum_{n=1}^{\infty}c_n^2(f)&=\sum_{n=1}^{\infty}\Big(c_n(f)-c_n(f_m)+c_n(f_m)\Big)^2 \\ &\leqslant\left(\left(\sum_{n=1}^{\infty}\big(c_n(f)-c_n(f_m)\big)^2\right)^{1/2}+\left(\sum_{n=1}^{\infty}c_n^2(f_m)\right)^{1/2}\right)^2<\infty \end{aligned}$$

and by the Hölder's inequality

$$
\begin{aligned}
|\sum_{n=1}^{\infty} c_n^2(f) - \sum_{n=1}^{\infty} c_n^2(f_k)| =& |\sum_{n=1}^{\infty} \big(c_n(f) - c_n(f_k)\big)\big(c_n(f) + c_n(f_k)\big)| \\
&\leqslant \Big(\sum_{n=1}^{\infty} \Big(c_n(f) - c_n(f_k)\Big)^2\Big)^{1/2} \Big(\sum_{n=1}^{\infty} \Big(c_n(f) + c_n(f_k)\Big)^2\Big)^{1/2} \to 0
\end{aligned}
$$

as $k \to \infty$. This means that $\lim\limits_{k\to\infty} \sum_{n=1}^{\infty} c_n^2(f_k) = \sum_{n=1}^{\infty} c_n^2(f)$.

Since $f_k \to f$ in H as $k \to \infty$, $\lim\limits_{k\to\infty} \|f_k\|_H = \|f\|_H$. We obtain

$$
\|f\|_H^2 = \lim_{k\to\infty} \|f_k\|_H^2 = \lim_{k\to\infty} \sum_{n=1}^{\infty} c_n^2(f_k) = \sum_{n=1}^{\infty} c_n^2(f). \tag{4.16}
$$

This completes the proof. □

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