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USING Q–CALCULUS TO STUDY LDLT FACTORIZATION OF A CERTAIN VANDERMONDE MATRIX

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Abstract. We use tools from q-calculus to study LDL^T decomposition of the Vandermonde matrix V_q with entries $v_{i,j} = q^{ij}$. We prove that the matrix L is given as a product of diagonal matrices and the lower triangular Toeplitz matrix T_q with elements $t_{i,j} = 1/(q;q)_{i-j}$, where $(z;q)_k$ is the q-Pochhammer symbol. We investigate some properties of the matrix T_q, in particular, we compute explicitly the inverse of this matrix.

1. Introduction and main results

Let us consider a Vandermonde matrix

$$V_q := \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & q & q^2 & q^3 & \dots \\ 1 & q^2 & q^4 & q^6 & \dots \\ 1 & q^3 & q^6 & q^9 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

of size $n \times n$. In the special case when $q = e^{-2\pi \mathrm{i}/n}$, this matrix is called *the discrete Fourier transform matrix*. Explicit matrix factorizations of the discrete Fourier transform matrix are very important, since they are often used in various versions of the Fast Fourier Transform algorithm [5]. Motivated by this connection, in this note we plan to study the LDL^T factorization of the matrix V_q and to investigate the properties of the factors appearing in such decomposition. The tools and techniques, which are used to prove our results, come from q-calculus.

First, let us present several definitions and notation. In what follows, we assume that $n \in \mathbb{N}$ and $q \in \mathbb{C}$. We define the q-Pochhammer symbol

$$(z;q)_n := (1-z)(1-zq)\cdots(1-zq^{n-1}), \quad n \geqslant 1, \tag{1}$$

and $(z;q)_0 := 1$. We will denote by I the $n \times n$ identity matrix. The following matrices of size $n \times n$ will be used frequently in this paper: a lower-triangular Toeplitz matrix

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$T_q = \{t_{i,j}\}_{0 \leqslant i,j \leqslant n-1}$ defined by $t_{i,j} = 1/(q;q)_{i-j}$ if $i \geqslant j$, and a diagonal matrix $P_q = \{p_{i,j}\}_{0 \leqslant i,j \leqslant n-1}$ having elements $p_{i,i} = (q;q)_i$, or, more explicitly,

$$T_q := \begin{bmatrix} 1 & 0 & 0 & 0 \dots \\ \frac{1}{(q;q)_1} & 1 & 0 & 0 \dots \\ \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 & 0 \dots \\ \frac{1}{(q;q)_3} & \frac{1}{(q;q)_2} & \frac{1}{(q;q)_1} & 1 \dots \\ \vdots & \vdots & \vdots & \vdots \ddots \end{bmatrix}, \qquad P_q := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & (q;q)_1 & 0 & 0 & \dots \\ 0 & 0 & (q;q)_2 & 0 & \dots \\ 0 & 0 & 0 & (q;q)_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that the matrices T_q and $T_{q^{-1}}$ are well-defined for all $q \in \mathbb{C} \setminus \mathscr{A}_n$, where the set $\mathscr{A}_n$ is given by

$$\mathscr{A}_n := \{q \in \mathbb{C} : q^j = 1 \text{ for some } j = 1, 2, \dots, n-1\}.$$

In our first result we identify explicitly the matrices appearing in the LDL^T factorization of the Vandermonde matrix V_q.

THEOREM 1. *Assume that $q \in \mathbb{C} \setminus \mathscr{A}_n$. Then $V_q = LDL^T$, where $L = P_q T_q (P_q)^{-1}$ and $D = \{d_{i,j}\}_{0 \leqslant i,j \leqslant n-1}$ is a diagonal matrix having elements $d_{i,i} = (-1)^i q^{i(i-1)/2}(q;q)_i$.*

In section 2 we give a very simple proof of Theorem 1 (our proof is based on the q-Binomial Theorem). Alternatively, one could derive this result starting from formulas (2.4) and (2.5) in the paper [4] by Oruc and Phillips, who use symmetric functions to study LU decomposition of general Vandermonde matrices.

REMARK 1. It is easy to see that the entries of the matrix $L = P_q T_q (P_q)^{-1}$ are given by

$$l_{i,j} = \frac{(q;q)_i}{(q;q)_j (q;q)_{i-j}}, \quad i \geqslant j. \tag{2}$$

This matrix is known in the literature as *the q-Pascal matrix* and it has appeared in [2, 3].

In our second result we present some properties of the Toeplitz matrix T_q, including an explicit formula for its inverse. First we define the following two matrices of size $n \times n$:

$$S := \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad D_q := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & q & 0 & 0 & \dots \\ 0 & 0 & q^2 & 0 & \dots \\ 0 & 0 & 0 & q^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3}$$

THEOREM 2. *Assume that $q \in \mathbb{C} \setminus \mathscr{A}_n$. Then*

(i) $(T_q)^{-1} = T_{q^{-1}}(I - S)^{-1} = D_{q^{-1}} T_{q^{-1}} D_q$;

(ii) for $m \in \mathbb{N}$ we have

$$T_q D_{q^{-m}} T_{q^{-1}} D_{q^m} = I + \sum_{j=1}^{m-1} \frac{(q^{1-m};q)_j}{(q;q)_j} S^j. \tag{4}$$

REMARK 2. Note that the matrix $H := (I-S)^{-1}$, which appears in item (i), is a lower triangular Toeplitz matrix having elements $h_{i,j} = 1$ if $i \geqslant j$ and $h_{i,j} = 0$ otherwise. Similarly, the matrix in the right-hand side of (4) is a lower-triangular Toeplitz matrix, having m non-zero diagonals: this matrix has coefficient 1 on the main diagonal and the coefficient $(q^{1-m};q)_j/(q;q)_j$ on the sub-diagonal number j, for $1 \leqslant j \leqslant m-1$.

2. Proofs

The only tool that will be needed for proving Theorems 1 and 2 is the q-Binomial Theorem (see [1] [Theorem 10.2.1]), which states that

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{j\geqslant 0} \frac{(a;q)_j}{(q;q)_j} z^j, \quad |q|<1,\ |z|<1. \tag{5}$$

Here $(z;q)_\infty := \prod_{l\geqslant 0}(1-zq^l)$ and it is clear that this infinite product converges for all $z\in\mathbb{C}$ and $|q|<1$. We also record here the following two corollaries of the q-Binomial Theorem, which will be needed later:

$$\frac{1}{(z;q)_\infty} = \sum_{j\geqslant 0} \frac{z^j}{(q;q)_j}, \quad |q|<1,\ |z|<1, \tag{6}$$

$$(z;q)_\infty = \sum_{j\geqslant 0} \frac{(-1)^j q^{j(j-1)/2}}{(q;q)_j} z^j, \quad |q|<1,\ z\in\mathbb{C}. \tag{7}$$

Proof of Theorem 1. Using formula (2) and considering an element (i,j) of the matrix LDL^T we see that formula $V_q = LDL^T$ is equivalent to the following identity: for any integers $i,j \geqslant 0$

$$q^{ij} = \sum_{k=0}^{\min(i,j)} \frac{(-1)^k q^{k(k-1)/2} (q;q)_i (q;q)_j}{(q;q)_k (q;q)_{i-k} (q;q)_{j-k}}. \tag{8}$$

We will prove the above identity by writing the Taylor series of the function

$$g(u,v) := \frac{(uv;q)_\infty}{(u;q)_\infty (v;q)_\infty}, \quad |u|<1,\ |v|<1,\ |q|<1,$$

in two different ways. First of all, from formula (5) we obtain

$$g(u,v) = \frac{1}{(v;q)_\infty} \times \frac{(uv;q)_\infty}{(u;q)_\infty} = \frac{1}{(v;q)_\infty} \sum_{i\geqslant 0} \frac{(v;q)_i}{(q;q)_i} u^i.$$

Using the fact that $(v;q)_i/(v;q)_\infty = 1/(q^i v;q)_\infty$ and expanding this expression in Taylor series in v via (6) we conclude that

$$g(u,v) = \sum_{i\geqslant 0}\sum_{j\geqslant 0} \frac{q^{ij}u^i v^j}{(q;q)_i (q;q)_j}. \tag{9}$$

On the other hand, we can obtain the series expansion of $g(u,v)$ by applying formulas (6) and (7) in the form

$$(uv;q)_\infty = \sum_{k\geqslant 0} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k} u^k v^k,$$
$$\frac{1}{(u;q)_\infty} = \sum_{l\geqslant 0} \frac{u^l}{(q;q)_l},$$
$$\frac{1}{(v;q)_\infty} = \sum_{m\geqslant 0} \frac{v^m}{(q;q)_m}.$$

We multiply the above three series expansions and obtain a Taylor series representation in the form

$$g(u,v) = \sum_{k\geqslant 0}\sum_{l\geqslant 0}\sum_{m\geqslant 0} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k (q;q)_l (q;q)_m} u^{k+l} v^{k+m}. \tag{10}$$

Comparing the coefficients in front of the term $u^i v^j$ in both formulas (9) and (10) gives us the desired result (8). □

Proof of Theorem 2. Let us prove the identity $T_q T_{q^{-1}} = (I-S)^{-1}$, which is equivalent to the first equality in item (i) (the second equality in (i) follows from formula (4) with $m=1$). The main idea of the proof is that the Toeplitz matrix T_q can be expressed in the following form

$$T_q = I + \sum_{j\geqslant 1} \frac{S^j}{(q;q)_j}, \tag{11}$$

where S is the matrix defined in (3). The above formula is easy to derive, given that for $1 \leqslant j \leqslant n-1$ the entries of the matrix S^j have value 1 on the sub-diagonal number j and value zero everywhere else. In particular, S^j is a zero matrix for $j \geqslant n$, thus the series in (11) terminates at $j = n-1$. Similarly, using the identity

$$(1/q;1/q)_j = (-1)^j q^{-j(j+1)/2} (q;q)_j, \tag{12}$$

and formula (11) we obtain

$$T_{q^{-1}} = I + \sum_{j\geqslant 1} \frac{(-1)^j q^{j(j-1)/2}}{(q;q)_j} (qS)^j. \tag{13}$$

Now, assume that $|q|<1$. Then formulas (6) and (11) give us

$$T_q = [(S;q)_\infty]^{-1} = (I-S)^{-1} \times (I-qS)^{-1} \times (I-q^2 S)^{-1} \times \cdots. \tag{14}$$

Similarly, formulas (7) and (13) give us

$$T_{q^{-1}} = (qS;q)_\infty = (I-qS)\times(I-q^2S)\times(I-q^3S)\times\cdots. \tag{15}$$

From the above two identities we see that all the terms $(I-q^iS)$ in the product $T_qT_{q^{-1}}$ are cancelled, except for the first term $(I-S)^{-1}$, thus we obtain $T_qT_{q^{-1}}=(I-S)^{-1}$ for $|q|<1$. We extend this result from $|q|<1$ to the general case $q\in\mathbb{C}\setminus\mathscr{A}_n$ by analytical continuation in q.

The proof of formula (4) uses the same ideas. Again, first we assume that $|q|<1$. From (12) we check that $D_{q^{-m}}T_{q^{-1}}D_{q^m}$ is a Toeplitz matrix of the form

$$D_{q^{-m}}T_{q^{-1}}D_{q^m} = I+\sum_{j\geqslant 1}\frac{(-1)^j q^{j(j-1)/2}}{(q;q)_j}(q^{1-m}S)^j = (q^{1-m}S;q)_\infty.$$

Using the above result and formula (14) we obtain

$$T_qD_{q^{-m}}T_{q^{-1}}D_{q^m} = [(S;q)_\infty]^{-1}\times(q^{1-m}S;q)_\infty = (q^{1-m}S;q)_{m-1}.$$

The desired desired result (4) follows by applying (5) and analytical continuation in q. □

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REFERENCES

[1] G. E. ANDREWS, R. ASKEY, AND R. ROY, *Special functions*, The University Press, Cambridge, 1999.

[2] H. ORUC, *Factorization of the Vandermonde matrix and its applications*, Applied Mathematics Letters **20** (9): 982–987, 2007.

[3] H. ORUC AND H. K. AKMAZ, *Symmetric functions and the Vandermonde matrix*, Journal of Computational and Applied Mathematics **172** (1): 49–64, 2004.

[4] H. ORUC AND G. M. PHILLIPS, *Explicit factorization of the Vandermonde matrix*, Linear Algebra and its Applications **315** (1–3): 113–123, 2000.

[5] C. VAN LOAN, *Computational Frameworks for the Fast Fourier Transform*, Society for Industrial and Applied Mathematics, 1992.

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