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HYPONORMALITY OF TOEPLITZ OPERATORS WITH POLYNOMIAL SYMBOLS ON THE VECTOR VALUED BERGMAN SPACE

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Abstract. We arrive at a necessary condition and a sufficient condition for the hyponormal block Toeplitz operator T_{F+G^*} on the vector valued Bergman space $L_a^2(\mathbb{D},\mathbb{C}^n)$, where F and G are matrix valued analytic polynomials. Both of them are also new for the Bergman space $L_a^2(\mathbb{D})$.

1. Introduction

In the complex plane $\mathbb{C}$, let $\mathbb{T}$ be the unit circle and $\mathbb{D}$ be the open unit disk with the normalized Lebesgue area measure dA. Denote by $L^\infty(\mathbb{D})$ and $h^\infty(\mathbb{D})$ the space of essential bounded measurable functions and the space of all bounded harmonic functions on $\mathbb{D}$, respectively. Let $L^2(\mathbb{D})$ be the space of the square integrable functions on $\mathbb{D}$ with respect to dA and $L_a^2(\mathbb{D})$ be the place of all analytic functions in $L^2(\mathbb{D})$. We denote the space of vector valued square integrable functions on $\mathbb{D}$ by $L^2(\mathbb{D},\mathbb{C}^n) = L^2(\mathbb{D})\otimes\mathbb{C}^n$ and the vector valued Bergman space on $\mathbb{D}$ by $L_a^2(\mathbb{D},\mathbb{C}^n) = L_a^2(\mathbb{D})\otimes\mathbb{C}^n$ respectively, where $\otimes$ means the Hilbert space tensor product.

Let $M_{n\times n}$ be the set of all $n\times n$ complex matrices and $L^\infty(\mathbb{D})\otimes M_{n\times n}$ be the space of matrix valued essential bounded Lebesgue measurable functions on $\mathbb{D}$. For $\Phi(z) = [\varphi_{ij}(z)]_{n\times n} \in L^\infty(\mathbb{D})\otimes M_{n\times n}$, the block Toeplitz operator T_Φ and block Hankel operator H_Φ with matrix symbol $\Phi(z)$ on $L_a^2(\mathbb{D},\mathbb{C}^n)$ are defined by

$$T_\Phi h = P(\Phi h) \text{ and } H_\Phi h = (I-P)(\Phi h), \quad \forall h\in L_a^2(\mathbb{D},\mathbb{C}^n),$$

where P is the orthogonal projection from $L^2(\mathbb{D},\mathbb{C}^n)$ onto $L_a^2(\mathbb{D},\mathbb{C}^n)$. Since $L^\infty(\mathbb{D},\mathbb{C}^n) = L^\infty(\mathbb{D})\oplus\cdots\oplus L^\infty(\mathbb{D})$ and $L_a^2(\mathbb{D},\mathbb{C}^n) = L_a^2(\mathbb{D})\oplus\cdots\oplus L_a^2(\mathbb{D})$, the block Toeplitz operator T_Φ and the block Hankel operator H_Φ have the following matrix representations:

$$T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & \cdots & T_{\varphi_{1n}} \\ \vdots & & \vdots \\ T_{\varphi_{n1}} & \cdots & T_{\varphi_{nn}} \end{bmatrix} \text{ and } H_\Phi = \begin{bmatrix} H_{\varphi_{11}} & \cdots & H_{\varphi_{1n}} \\ \vdots & & \vdots \\ H_{\varphi_{n1}} & \cdots & H_{\varphi_{nn}} \end{bmatrix},$$

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where

$$\Phi(z) = \begin{bmatrix} \varphi_{11}(z) & \cdots & \varphi_{1n}(z) \\ \vdots & & \vdots \\ \varphi_{n1}(z) & \cdots & \varphi_{nn}(z) \end{bmatrix},$$

$T_{\varphi_{ij}}$ is the Toeplitz operator with symbol $\varphi_{ij} \in L^\infty(D)$ and $H_{\varphi_{ij}}$ is the Hankel operator with symbol $\varphi_{ij} \in L^\infty(D)$ $(1 \leqslant i, j \leqslant n)$ on the Bergman space $L_a^2(\mathbb{D})$, respectively.

A bounded linear operator A on a Hilbert space is called hyponormal if $A^*A - AA^*$ is a positive operator. There is an extensive literature on hyponormal Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ (see [2, 5, 7, 9, 13] for example). Some necessary conditions and some sufficient conditions for the hyponormality of Toeplitz operators with certain symbols on the Bergman space can be found in [1, 6, 8, 11, 12]. Further, the corresponding problem has been studied in many other spaces. Some characterizations of hyponormal Toeplitz operators on the vector valued Hardy space can be seen in [3, 4].

In this paper, we consider the hyponormality of the block Toeplitz operator T_{F+G^*} on the vector valued Bergman space $L_a^2(\mathbb{D}, \mathbb{C}^n)$, where $F(z) = \sum_{k=0}^N A_k z^{m+k}$ and $G(z) = \sum_{k=0}^N B_k z^{m+k}$ with $A_k, B_k \in M_{n\times n}$, $m, N \geqslant 1$. Recall that G^* is the adjoint of the matrix G. Denote by $M_j = \sum_{k=0}^{N-j}(m+k)(m+k+j)(A_{k+j}A_k^* - B_{k+j}B_k^*)$ for $1 \leqslant j \leqslant N$. Inspired by the work in [6, 8, 11], we use the properties of the matrices $\{M_j\}_{j=1}^N$ to study the hyponormality of Block Toeplitz operators T_{F+G^*} on $L_a^2(\mathbb{D}, \mathbb{C}^n)$.

2. A necessary condition

Our goal in this section is to show a necessary condition for the Toeplitz operators to be hyponormal with matrix valued harmonic polynomial symbols on $L_a^2(\mathbb{D}, \mathbb{C}^n)$. Firstly, we recall some useful results.

THEOREM 1. ([10]) *Let $\Phi(z) \in h^\infty(\mathbb{D}) \otimes M_{n\times n}$. If T_Φ on $L_a^2(\mathbb{D}, \mathbb{C}^n)$ is hyponormal, then $\Phi^*(z)\Phi(z) = \Phi(z)\Phi^*(z)$ on $\mathbb{D}$.*

THEOREM 2. ([10]) *Suppose that $F(z)$ and $G(z)$ are vector valued analytic polynomials. If T_{F+G^*} on $L_a^2(\mathbb{D}, \mathbb{C}^n)$ is hyponormal, then $F'(z)\ [F'(z)]^* - G'(z)[G'(z)]^*$ is a positive semi-definite matrix for all $z \in \mathbb{T}$.*

THEOREM 3. *Suppose that $\Phi(z) = F(z) + G^*(z) \in h^\infty(\mathbb{D}) \otimes M_{n\times n}$, where F and G are vector valued analytic functions. Then T_Φ is hyponormal on $L_a^2(\mathbb{D}, \mathbb{C}^n)$ if and only if $H_{F^*}^* H_{F^*} - H_{G^*}^* H_{G^*} \geqslant 0$ on $L_a^2(\mathbb{D}, \mathbb{C}^n)$ and $\Phi^*(z)\Phi(z) = \Phi(z)\Phi^*(z)$ on $\mathbb{D}$.*

Proof. A direct calculation shows that

$$\begin{aligned} T_\Phi^* T_\Phi - T_\Phi T_\Phi^* &= T_\Phi^* T_\Phi - T_{\Phi^*\Phi} + (T_{\Phi\Phi^*} - T_\Phi T_\Phi^*) + T_{\Phi^*\Phi - \Phi\Phi^*} \\ &= -H_\Phi^* H_\Phi + H_{\Phi^*}^* H_{\Phi^*} + T_{\Phi^*\Phi - \Phi\Phi^*} \\ &= H_{F^*}^* H_{F^*} - H_{G^*}^* H_{G^*} + T_{\Phi^*\Phi - \Phi\Phi^*}. \end{aligned}$$

Combining Theorem 1, we get the desired result. □

Let $F(z)=\sum_{k=0}^{N}A_kz^{m+k}$ and $G(z)=\sum_{k=0}^{N}B_kz^{m+k}$ with $A_k,B_k\in M_{n\times n}$, $m,N\geqslant 1$. It is clear that there exist polynomials f_{ij} and g_{ij} such that $F(z)=[f_{ij}(z)]_{n\times n}$ and $G(z)=[g_{ij}(z)]_{n\times n}$. Denote by $M_j=\sum_{k=0}^{N-j}(m+k)(m+k+j)(A_{k+j}A_k^*-B_{k+j}B_k^*)$ for $1\leqslant j\leqslant N$. Then

$$\begin{aligned}
&F'(z)[F'(z)]^*-G'(z)[G'(z)]^*\\
=&\sum_{k=0}^{N}(m+k)^2(A_kA_k^*-B_kB_k^*)\\
&+2Re\sum_{j=1}^{N}\sum_{k=0}^{N-j}(m+k)(m+k+j)(A_kA_{k+j}^*-B_kB_{k+j}^*)\overline{z}^j,\ \forall z\in\mathbb{T}.
\end{aligned}$$

By Theorem 2, if T_{F+G^*} is hyponormal, then

$$\langle\sum_{k=0}^{N}(m+k)^2(A_kA_k^*-B_kB_k^*)e,\ e\rangle_{\mathbb{C}^n}+2Re\sum_{j=1}^{N}\langle M_je,e\rangle_{\mathbb{C}^n}\overline{z}^j\geqslant 0$$

for $e\in\mathbb{C}^n$ and $z\in\mathbb{T}$. Integral the above inequality of $z=e^{i\theta}$ with respect to θ from 0 to 2π, it follows that $\sum_{k=0}^{N}(m+k)^2(A_kA_k^*-B_kB_k^*)$ is a positive semi-definite matrix. In general, there does not exist complex number $\overline{z}\in\mathbb{T}$ such that

$$\overline{z}^j\langle M_j\,e,e\rangle_{\mathbb{C}^n}=-|\langle M_j\,e,e\rangle_{\mathbb{C}^n}|$$

for any $1\leqslant j\leqslant N$. In the following, we consider the relationship between the above positive semi-definite matrix and the matrix M_j $(1\leqslant j\leqslant N)$.

THEOREM 4. *Let $F(z)=\sum_{k=0}^{N}A_kz^{m+k}$ and $G(z)=\sum_{k=0}^{N}B_kz^{m+k}$ with $A_k,B_k\in M_{n\times n}$, $m,N\geqslant 1$. If T_{F+G^*} is hyponormal on $L_a^2(\mathbb{D},\mathbb{C}^n)$, then the following statements hold for any $e\in\mathbb{C}^n$.*

(1) $\langle\sum\limits_{k=0}^{N}(m+k)^2(A_kA_k^*-B_kB_k^*)e,e\rangle_{\mathbb{C}^n}\geqslant\frac{2}{N+1}\sum\limits_{j=1}^{N}|\langle M_j\,e,e\rangle_{\mathbb{C}^n}|$, *if* $N\geqslant 3$ *;*

(2) $\langle\sum\limits_{k=0}^{N}(m+k)^2(A_kA_k^*-B_kB_k^*)e,e\rangle_{\mathbb{C}^n}\geqslant\frac{2}{N}\sum\limits_{j=1}^{N}|\langle M_j\,e,e\rangle_{\mathbb{C}^n}|$, *if* $1\leqslant N\leqslant 2$.

Proof. For every $e=(e_1,\cdots,e_n)^T\in\mathbb{C}^n$, let

$$c_j=\begin{cases}-\frac{\langle M_je,\ e\rangle_{\mathbb{C}^n}}{|\langle M_je,e\rangle_{\mathbb{C}^n}|}, & \text{if } \langle M_je,e\rangle_{\mathbb{C}^n}\neq 0\\ 1, & \text{if } \langle M_je,e\rangle_{\mathbb{C}^n}=0\end{cases}.$$

For $0<a<1$, u_l is defined as follows:

$$u_l=\begin{cases} 1, & \text{if} \quad l=0 \\ \frac{\prod_{s=1}^{i} c_s c_{(N+1)-s}}{c_{(N+1)-i}} a^{2i-1}, & \text{if} \quad l=(i-1)(N+1)+i,\ 1\leqslant i\leqslant [\frac{N+1}{2}] \\ (\prod_{s=1}^{i} c_s c_{(N+1)-s}) a^{2i}, & \text{if} \quad l=i(N+1),\ 1\leqslant i\leqslant [\frac{N}{2}] \\ 0, & \text{otherwise} \end{cases}.$$

Therefore,

(i) $u_0\overline{u}_1=\overline{c_1}a$;

(ii) $u_{(i-1)(N+1)}\overline{u}_{(i-1)(N+1)+i}=\overline{c_i}a^{4i-3}$, when $2\leqslant i\leqslant [\frac{N+1}{2}]$;

(iii) $u_{(i-1)(N+1)+i}\overline{u}_{i(N+1)}=\overline{c_{(N+1)-i}}\,a^{4i-1}$, when $1\leqslant i\leqslant [\frac{N}{2}]$.

Moreover, $u_l\overline{u}_{l+j}\neq 0$ for any $l\in\mathbb{Z}_+$ and $1\leqslant j\leqslant N$ if and only if one of the above cases holds.

For every integer $p\geqslant 1$, we have $\widehat{h}(z)=\sum_{l=0}^{\infty}u_l z^{l+p}\in L_a^2(\mathbb{D})$. In fact,

$$\|\widehat{h}(z)\|^2\leqslant\sum_{l=0}^{\infty}|u_l|^2=1+a^2+a^4+\cdots+a^{2N}=\frac{1-a^{2(N+1)}}{1-a^2}<+\infty.$$

Denote $h=e\cdot\widehat{h}=(e_1\widehat{h},\cdots,e_n\widehat{h})^T=(h_1,h_2,\cdots,h_n)^T$. Here, M^T means the transpose of the M. It is easy to see that $h\in L_a^2(\mathbb{D},\mathbb{C}^n)$. By Theorem 3, we have

$$\begin{aligned}
0&\leqslant\langle(H_{F^*}^*H_{F^*}-H_{G^*}^*H_{G^*})h,h\rangle_{L_a^2(\mathbb{D},\mathbb{C}^n)}\\
&=\sum_{k=0}^{N}\langle[(A_kH_{\overline{z}^{m+k}}^*)(A_k^*H_{\overline{z}^{m+k}})-(B_kH_{\overline{z}^{m+k}}^*)(B_k^*H_{\overline{z}^{m+k}})]h,h\rangle_{L_a^2(\mathbb{D},\mathbb{C}^n)}\\
&\quad+2\mathrm{Re}\sum_{j=1}^{N}\sum_{k=0}^{N-j}\langle[(A_{k+j}H_{\overline{z}^{m+k+j}}^*)(A_k^*H_{\overline{z}^{m+k}})-(B_{k+j}H_{\overline{z}^{m+k+j}}^*)(B_k^*H_{\overline{z}^{m+k}})]h,h\rangle_{L_a^2(\mathbb{D},\mathbb{C}^n)}\\
&=\sum_{k=0}^{N}\sum_{l=0}^{\infty}\sum_{q=0}^{\infty}\langle(A_kA_k^*-B_kB_k^*)e,e\rangle_{\mathbb{C}^n}\langle H_{\overline{z}^{m+k}}(u_lz^{l+p}),H_{\overline{z}^{m+k}}(u_qz^{q+p})\rangle_{L_a^2(\mathbb{D})}\\
&\quad+2\mathrm{Re}\sum_{j=1}^{N}\sum_{k=0}^{N-j}\sum_{l=0}^{\infty}\sum_{q=0}^{\infty}\langle(A_{k+j}A_k^*-B_{k+j}B_k^*)e,e\rangle_{\mathbb{C}^n}\\
&\quad\times\langle H_{\overline{z}^{m+k}}(u_lz^{l+p}),\ H_{\overline{z}^{m+k+j}}(u_qz^{q+p})\rangle_{L_a^2(\mathbb{D})}\\
&=\sum_{k=0}^{N}\sum_{l=0}^{\infty}\langle(A_kA_k^*-B_kB_k^*)\,e,e\rangle_{\mathbb{C}^n}u_l\overline{u_l}\langle H_{\overline{z}^{m+k}}z^{p+l},H_{\overline{z}^{m+k}}z^{p+l}\rangle_{L_a^2(\mathbb{D})}\\
&\quad+2\mathrm{Re}\sum_{j=1}^{N}\sum_{k=0}^{N-j}\big[\langle\,(\,A_{k+j}A_k^*-B_{k+j}B_k^*)\,e,e\rangle_{\mathbb{C}^n}\\
&\quad\times\sum_{l=0}^{\infty}\big(\,u_l\,\overline{u_{l+j}}\,\langle H_{\overline{z}^{m+k}}z^{p+l},H_{\overline{z}^{m+k+j}}z^{p+l+j}\,\rangle_{L_a^2(\mathbb{D})}\,\big)\,\big].
\end{aligned}$$

The last equality comes from the fact that $\{H_{\overline{z}^m}(z^k)\}$ and $\{H_{\overline{z}^s}(z^l)\}$ are orthogonal in $L^2(\mathbb{D})$ for $k-m\neq l-s$, and $\langle H^*_{\overline{z}^{m_2}}H_{\overline{z}^{m_1}}z^{k+m_1-m_2},z^k\rangle_{L^2_a(\mathbb{D})}>0$, $\forall k\geqslant\max\{0,m_2-m_1\}$.

Divide the both sides of the above inequality by $\|H_{\overline{z}^m}z^p\|^2_{L^2_a(\mathbb{D})}$ and let $p\to+\infty$. Note that

$$\lim_{p\to+\infty}\frac{\langle H_{\overline{z}^{m+k}}z^{p+l},H_{\overline{z}^{m+k}}z^{p+l}\rangle_{L^2_a(\mathbb{D})}}{\|H_{\overline{z}^m}z^p\|^2_{L^2_a(\mathbb{D})}}=\frac{(m+k)^2}{m^2}$$

and

$$\lim_{p\to+\infty}\frac{\langle H_{\overline{z}^{m+k}}z^{p+l},H_{\overline{z}^{m+k+j}}z^{p+l+j}\rangle_{L^2_a(\mathbb{D})}}{\|H_{\overline{z}^m}z^p\|^2_{L^2_a(\mathbb{D})}}=\frac{(m+k)(m+k+j)}{m^2}.$$

The details can be found in [11]. Therefore,

$$\begin{aligned}&\langle\sum_{k=0}^{N}(m+k)^2(\,A_kA_k^*-B_kB_k^*)\,e,e\rangle_{\mathbb{C}^n}\Big(\sum_{l=0}^{\infty}|u_l|^2\Big)\\ \geqslant&-2\mathrm{Re}\sum_{j=1}^{N}\sum_{l=0}^{\infty}u_l\overline{u_{l+j}}\langle M_je,e\rangle_{\mathbb{C}^n}.\end{aligned}$$

Combining conditions (ii) and (iii) of u_l, it is easy to check that

$$\begin{aligned}&\langle\sum_{k=0}^{N}(m+k)^2(A_kA_k^*-B_kB_k^*)\,e,e\rangle_{\mathbb{C}^n}\\ \geqslant&\,2\sum_{j=1}^{[\frac{N+1}{2}]}a^{4(j-1)+1}\frac{1-a^2}{1-a^{2(N+1)}}|\langle M_je,e\rangle_{\mathbb{C}^n}|\\ &+2\sum_{j=1}^{[\frac{N}{2}]}a^{4j-1}\frac{1-a^2}{1-a^{2(N+1)}}|\langle M_{(N+1)-j}e,e\rangle_{\mathbb{C}^n}|.\end{aligned}$$

Letting $a\to1$, we get the desired conclusion.

Now we prove (2). Let

$$v_l=\begin{cases}(c_1c_N)^k\,a^{2k}, & l=k(N+1)\\ c_1^{k+1}c_N^ka^{2k+1}, & l=k(N+1)+1\\ 0, & \text{otherwise}\end{cases}$$

with $0<a<1$ and $k=0,1,\cdots$. Direct calculations imply that if $v_l\overline{v_{l+j}}\neq0$, then $v_l\overline{v_{l+j}}=\overline{c_j}a^{2kN+(2j-1)}$ for $1\leqslant j\leqslant N$. Replacing u_l by v_l, we set $\widetilde{h}(z)=\sum_{l=0}^{\infty}v_lz^{l+p}\in L^2_a(\mathbb{D})$ for $p\geqslant1$. Then $\|\widetilde{h}(z)\|^2\leqslant\sum_{l=0}^{\infty}|u_l|^2=\frac{1+a^2}{1-a^4}<+\infty$. As in the proof of (1), we get

$$\begin{aligned}
&\langle \sum_{k=0}^{N}(m+k)^2(A_kA_k^* - B_kB_k^*)\, e,e\rangle_{\mathbb{C}^n} \sum_{l=0}^{\infty}|u_l|^2 \\
&\geqslant 2\sum_{j=1}^{N}\sum_{k=0}^{+\infty} a^{2kN+(2j-1)}a^{2kN+2j-1}|\langle M_j e,e\rangle_{\mathbb{C}^n}| \\
&= 2\sum_{j=1}^{N} a^{2j-1}\frac{1-a^2}{1-a^{2N}}|\langle M_j e,e\rangle_{\mathbb{C}^n}|.
\end{aligned}$$

The rest of the proof is similar to the proof of (1). □

In the case of $n=1$, we have the following Corollary.

COROLLARY 1. *Given* $f(z)=\sum_{k=0}^{N}a_kz^{m+k}$ *and* $g(z)=\sum_{k=0}^{N}b_kz^{m+k}$ *with* $m,N\geqslant 1$. *Let* $M_j=\sum_{p=0}^{N-j}(m+p)(m+p+j)(a_{p+j}\overline{a_p}-b_{p+j}\overline{b_p})$. *If* $T_{f+\overline{g}}$ *is hyponormal on* $L_a^2(\mathbb{D})$, *then*

(1) $\sum_{i=0}^{N}(m+i)^2(|a_i|^2-|b_i|^2)\geqslant \frac{2}{N+1}\sum_{j=1}^{N}|M_j|$, *if* $N\geqslant 3$;

(2) $\sum_{i=0}^{N}(m+i)^2(|a_i|^2-|b_i|^2)\geqslant \frac{2}{N}\sum_{j=1}^{N}|M_j|$, *if* $1\leqslant N\leqslant 2$.

3. A sufficient condition

The following Theorem is our main result in this section. As a corollary, we also get a sufficient condition for the hyponormal Toeplitz operators with polynomial symbols on $L_a^2(\mathbb{D})$.

THEOREM 5. *Suppose that* $\Phi(z)=F(z)+G^*(z)\in h^\infty(\mathbb{D})\otimes M_{n\times n}$, *that* $F(z)=\sum_{k=0}^{N}A_kz^{m+k}$ *and* $G(z)=\sum_{k=0}^{N}B_kz^{m+k}$ *with* $A_k,B_k\in M_{n\times n}$, $m\geqslant 1$ *and* $N\geqslant 1$. *If the following conditions hold, then* T_{F+G^*} *is hyponormal on* $L_a^2(\mathbb{D},\mathbb{C}^n)$.

(1) $\Phi^*(z)\Phi(z)=\Phi(z)\Phi^*(z)$ *on* $\mathbb{D}$;

(2) $B_jB_j^*-A_jA_j^*$ *is positive semi-definite;*

(3) $|\langle M_j\, e_1,e_1\rangle_{\mathbb{C}^n}||\langle M_j\, e_2,e_2\rangle_{\mathbb{C}^n}|\geqslant|\langle M_j\, e_1,e_2\rangle_{\mathbb{C}^n}|^2$;

(4) $\mathrm{Re}\big[\langle(A_{k+j}A_k^*-B_{k+j}B_k^*)e_1,e_2\rangle_{\mathbb{C}^n}\overline{\langle(A_{k+j+l}A_{k+l}^*-B_{k+j+l}B_{k+l}^*)e_1,e_2\rangle_{\mathbb{C}^n}}\big]\geqslant 0$;

(5) $\langle\sum_{i=0}^{N}(m+i)^2(A_iA_i^*-B_iB_i^*)e,e\rangle_{\mathbb{C}^n}\geqslant 2\sum_{j=1}^{N}|\langle M_je,e\rangle_{\mathbb{C}^n}|$,

for all $e_1,e_2,e\in\mathbb{C}^n$, $1\leqslant j\leqslant N$, $0\leqslant k\leqslant N-j$ *and* $0\leqslant l\leqslant N-(k+j)$.

Proof. Let $h(z)=\sum\limits_{p=0}^{\infty}e_pz^p\in L_a^2(\mathbb{D},\mathbb{C}^n)$ with $\sum\limits_{p=0}^{\infty}\frac{\langle e_p,e_p\rangle_{\mathbb{C}^n}}{p+1}<\infty$. It is easy to check that

$$
\begin{aligned}
&\langle (H_{F^*}^*H_{F^*}-H_{G^*}^*H_{G^*})h,h\rangle_{L_a^2(\mathbb{D},\mathbb{C}^n)}\\
&=\sum_{p=0}^{\infty}\sum_{k=0}^{N}\langle (A_kA_k^*-B_kB_k^*)e_p,e_p\rangle_{\mathbb{C}^n}\ \|H_{\overline{z}^{m+k}}z^p\|^2_{L_a^2(\mathbb{D})}\\
&\quad+2\mathrm{Re}\sum_{p=0}^{\infty}\sum_{j=1}^{N}\sum_{k=0}^{N-j}\langle (A_{k+j}A_k^*-B_{k+j}B_k^*)e_p,e_{p+j}\rangle_{\mathbb{C}^n}\langle H_{\overline{z}^{m+k}}z^p,H_{\overline{z}^{m+k+j}}z^{p+j}\rangle_{L_a^2(\mathbb{D})}\\
&=\sum_{p=0}^{\infty}\sum_{k=0}^{N}\langle (A_kA_k^*-B_kB_k^*)e_p,e_p\rangle_{\mathbb{C}^n}\ \|H_{\overline{z}^{m+k}}z^p\|^2_{L_a^2(\mathbb{D})}\\
&\quad+2\sum_{p=0}^{\infty}\sum_{j=1}^{N}\mathrm{Re}\Big[\sum_{k=0}^{N-j}\langle (A_{k+j}A_k^*-B_{k+j}B_k^*)e_p,e_{p+j}\rangle_{\mathbb{C}^n}\langle H_{\overline{z}^{m+k}}z^p,H_{\overline{z}^{m+k+j}}z^{p+j}\rangle_{L_a^2(\mathbb{D})}\Big]\\
&\geqslant\sum_{p=0}^{\infty}\sum_{k=0}^{N}\langle (A_kA_k^*-B_kB_k^*)e_p,e_p\rangle_{\mathbb{C}^n}\|H_{\overline{z}^{m+k}}z^p\|^2_{L_a^2(\mathbb{D})}\\
&\quad-2\sum_{p=0}^{\infty}\sum_{j=1}^{N}\Big|\sum_{k=0}^{N-j}\langle (A_{k+j}A_k^*-B_{k+j}B_k^*)e_p,e_{p+j}\rangle_{\mathbb{C}^n}\langle H_{\overline{z}^{m+k}}z^p,H_{\overline{z}^{m+k+j}}z^{p+j}\rangle_{L_a^2(\mathbb{D})}\Big|.
\end{aligned}
$$

For non-negative integers m,k,j, we have

$$
\begin{aligned}
&\langle H^*_{\overline{z}^{m+j+k}}H_{\overline{z}^{m+k}}z^p,z^{p+j}\rangle_{L_a^2(\mathbb{D})}\\
&=\begin{cases}
\frac{(m+k)(m+j+k)(p+m+j+1)}{m(m+j)(k+m+j+p+1)}\langle H^*_{\overline{z}^{m+j}}H_{\overline{z}^{m}}z^p,z^{p+j}\rangle_{L_a^2(\mathbb{D})}, & \text{if}\quad p\geqslant m+k\\
\frac{(p+1)(p+j+1)(m+p+j+1)}{m(m+j)(k+m+j+p+1)}\langle H^*_{\overline{z}^{m+j}}H_{\overline{z}^{m}}z^p,z^{p+j}\rangle_{L_a^2(\mathbb{D})}, & \text{if}\ \ m\leqslant p<m+k\,.\\
\frac{(p+m+j+1)}{(p+m+j+k+1)}\langle H^*_{\overline{z}^{m+j}}H_{\overline{z}^{m}}z^p,z^{p+j}\rangle_{L_a^2(\mathbb{D})}, & \text{if}\quad 0\leqslant p<m
\end{cases}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\langle H_{\overline{z}^{m+k}}z^p,H_{\overline{z}^{m+k+j}}z^{p+j}\rangle_{L_a^2(\mathbb{D})}\\
&\leqslant\frac{(m+k+j)(m+k)}{m(m+j)}\langle H_{\overline{z}^{m}}z^p,H_{\overline{z}^{m+j}}z^{p+j}\rangle_{L_a^2(\mathbb{D})}\\
&\leqslant\frac{(m+k+j)(m+k)}{m^2}\ \|H_{\overline{z}^{m}}z^p\|_{L_a^2(\mathbb{D})}\|H_{\overline{z}^{m}}z^{p+j}\|_{L_a^2(\mathbb{D})},
\end{aligned}
$$

for $m,k,j=1,2,\cdots$ and $p=0,1,2,\cdots$.

Recall $M_j=\sum_{k=0}^{N-j}(m+k)(m+k+j)(A_{k+j}A_k^*-B_{k+j}B_k^*)$ for $1\leqslant j\leqslant N$. Condition (4) implies that,

$$
\begin{aligned}
&\Big|\sum_{k=0}^{N-j}\langle (A_{k+j}A_k^*-B_{k+j}B_k^*)e_p,e_{p+j}\rangle_{\mathbb{C}^n}\ \langle H_{\overline{z}^{m+k}}z^p,H_{\overline{z}^{m+k+j}}z^{p+j}\rangle_{L_a^2(\mathbb{D})}\Big|\\
&\leqslant\frac{1}{m^2}\Big|\langle M_j\ e_p,\ e_{p+j}\rangle_{\mathbb{C}^n}\ \Big|\|H_{\overline{z}^{m}}z^p\|_{L_a^2(\mathbb{D})}\|H_{\overline{z}^{m}}z^{p+j}\|_{L_a^2(\mathbb{D})}.
\end{aligned}
$$

To finish the proof, we only need to prove that

$$\begin{aligned}&\sum_{p=0}^{\infty}\sum_{k=0}^{N}\langle(A_kA_k^*-B_kB_k^*)e_p,e_p\rangle_{\mathbb{C}^n}\ \|H_{\overline{z}^{m+k}}z^p\|^2_{L_a^2(\mathbb{D})}\\ \geqslant\ &\frac{2}{m^2}\sum_{p=0}^{\infty}\sum_{j=1}^{N}\left|\langle M_j\ e_p,\ e_{p+j}\rangle_{\mathbb{C}^n}\ \right|\|H_{\overline{z}^m}z^p\|_{L_a^2(\mathbb{D})}\|H_{\overline{z}^m}z^{p+j}\|_{L_a^2(\mathbb{D})}.\end{aligned}$$

Note that $\zeta_i(k)=\dfrac{\|H_{\overline{z}^{m+i}}z^k\|^2_{L_a^2(\mathbb{D})}}{\|H_{\overline{z}^m}z^k\|^2_{L_a^2(\mathbb{D})}}\,(i=1,2,\cdots,N)$ is increasing and $\zeta_i(k)\to\frac{(m+i)^2}{m^2}$ as $k\to+\infty$. So $\zeta_i(k)\leqslant\dfrac{(m+i)^2}{m^2}$. Combining conditions (2) and (5), we have

$$\begin{aligned}&\frac{\sum_{k=0}^{N}\langle(A_kA_k^*-B_kB_k^*)e_p,e_p\rangle_{\mathbb{C}^n}\ \|H_{\overline{z}^{m+k}}z^p\|^2_{L_a^2(\mathbb{D})}}{\|H_{\overline{z}^m}z^p\|^2_{L_a^2(\mathbb{D})}}\\ \geqslant\ &\frac{1}{m^2}\langle\sum_{k=0}^{N}(m+k)^2(A_kA_k^*-B_kB_k^*)e_p,e_p\rangle_{\mathbb{C}^n}\\ \geqslant\ &\frac{2}{m^2}\sum_{j=1}^{N}|\langle M_je_p,e_p\rangle_{\mathbb{C}^n}|\geqslant 0.\end{aligned}$$

Let $\eta(p)=\left(\sum_{k=0}^{N}\langle(A_kA_k^*-B_kB_k^*)e_p,e_p\rangle_{\mathbb{C}^n}\ \|H_{\overline{z}^{m+k}}z^p\|^2_{L_a^2(\mathbb{D})}\right)^{1/2}$ for $p\geqslant 0$. Then

$$\frac{\eta^2(p)}{\|H_{\overline{z}^m}z^p\|^2_{L_a^2(\mathbb{D})}}\geqslant\frac{2}{m^2}\sum_{k=1}^{N}|\langle M_ke_p,e_p\rangle_{\mathbb{C}^n}|$$

and

$$\frac{\eta^2(p+j)}{\|H_{\overline{z}^m}z^{p+j}\|^2_{L_a^2(\mathbb{D})}}\geqslant\frac{2}{m^2}\sum_{k=1}^{N}|\langle M_ke_{p+j},e_{p+j}\rangle_{\mathbb{C}^n}|.$$

It follows that

$$\begin{aligned}&\eta(p)\eta(p+j)\\ \geqslant\ &\frac{2}{m^2}\ \Big(\sum_{k=1}^{N}|\langle M_ke_p,e_p\rangle_{\mathbb{C}^n}|\Big)^{\frac12}\ \Big(\sum_{k=1}^{N}|\langle M_ke_{p+j},e_{p+j}\rangle_{\mathbb{C}^n}\ |\Big)^{\frac12}\\ &\times\|H_{\overline{z}^m}z^p\|_{L_a^2(\mathbb{D})}\|H_{\overline{z}^m}z^{p+j}\|_{L_a^2(\mathbb{D})}.\end{aligned}\tag{1}$$

Thus

$$
\begin{aligned}
\sum_{p=0}^{\infty}\eta^2(p) = & \frac{1}{2}\eta^2(0)+\frac{1}{2}\sum_{j=1}^{N-1}\left(\frac{\sum_{k=j+1}^{N}|\langle M_k e_j, e_j\rangle_{\mathbb{C}^n}|}{\sum_{k=1}^{N}|\langle M_k e_j, e_j\rangle_{\mathbb{C}^n}|}\eta^2(j)\right) \\
& +\frac{1}{2}\sum_{p=0}^{\infty}\sum_{j=1}^{N}\left(\frac{|\langle M_j e_p, e_p\rangle_{\mathbb{C}^n}|}{\sum_{k=1}^{N}|\langle M_k e_p, e_p\rangle_{\mathbb{C}^n}|}\eta^2(p)\right. \\
& \left.+\frac{|\langle M_j e_{p+j}, e_{p+j}\rangle_{\mathbb{C}^n}|}{\sum_{k=1}^{N}|\langle M_k e_{p+j}, e_{p+j}\rangle_{\mathbb{C}^n}|}\eta^2(p+j)\right) \\
\geqslant & \sum_{p=0}^{\infty}\sum_{j=1}^{N}\frac{|\langle M_j e_p, e_p\rangle_{\mathbb{C}^n}|^{\frac{1}{2}}|\langle M_j e_{p+j}, e_{p+j}\rangle_{\mathbb{C}^n}|^{\frac{1}{2}}\eta(p)\eta(p+j)}{\left(\sum_{k=1}^{N}|\langle M_k e_p, e_p\rangle_{\mathbb{C}^n}|\right)^{\frac{1}{2}}\left(\sum_{k=1}^{N}|\langle M_k e_{p+j}, e_{p+j}\rangle_{\mathbb{C}^n}|\right)^{\frac{1}{2}}} \\
\geqslant & 2\sum_{p=0}^{\infty}\sum_{j=1}^{N}\frac{1}{m^2}\left|\langle M_j e_p, e_{p+j}\rangle_{\mathbb{C}^n}\right|\|H_{\overline{z}^m}z^p\|_{L^2_a(\mathbb{D})}\|H_{\overline{z}^m}z^{p+j}\|_{L^2_a(\mathbb{D})}.
\end{aligned}
$$

The last inequality follows from the inequality (1) and the condition (3). Hence we finish the proof. □

In particular, when $n=1$, we have the following result.

COROLLARY 2. *Given $f(z)=\sum_{k=0}^{N}a_k z^{m+k}$ and $g(z)=\sum_{k=0}^{N}b_k z^{m+k}$ with $m,N\geqslant 1$. Let $M_j=\sum_{p=0}^{N-j}(m+p)(m+p+j)(a_{p+j}\overline{a_p}-b_{p+j}\overline{b_p})$. Suppose $|b_k|\geqslant|a_k|$ for $1\leqslant k\leqslant N$ and $\mathrm{Re}\left[(a_{k+j}\overline{a_k}-b_{k+j}\overline{b_k})(\overline{a_{k+j+l}}a_{k+l}-\overline{b_{k+j+l}}b_{k+l})\right]\geqslant 0$ with $1\leqslant j\leqslant N-k$ and $0\leqslant l\leqslant N-(k+j)$. Then $T_{f+\overline{g}}$ is hyponormal, if*

$$
\sum_{i=0}^{N}(m+i)^2(|a_i|^2-|b_i|^2)\geqslant 2\sum_{j=1}^{N}|M_j|.
$$

EXAMPLE 1. Let $F(z)+G^*(z)=$

$$
\begin{bmatrix} z+z^2+z^3 & 4z+\frac{1}{2}z^2+\frac{1}{6}z^3 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} \overline{z}+\overline{z}^2+\overline{z}^3 & 2\overline{z}+\frac{9}{8}\overline{z}^2+\frac{1}{3}\overline{z}^3 \\ 0 & 0 \end{bmatrix}.
$$

Then T_{F+G^*} is hyponormal on $L^2_a(\mathbb{D},\mathbb{C}^2)$.

EXAMPLE 2. Consider the polynomial $\varphi(z)=\frac{1}{3}\overline{z}^3+\frac{9}{8}\overline{z}^2+2\overline{z}+4z+\frac{1}{2}z^2+\frac{1}{6}z^3$. On the Bergman space, Corollary 2 shows that T_φ is a hyponormal operator.

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