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ON SUPERCYCLICITY FOR ABELIAN SEMIGROUPS OF MATRICES ON $\mathbb{R}^n$

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Abstract. We give a complete characterization of supercylicity for abelian semigroups of matrices on $\mathbb{R}^n$, $n \geqslant 1$. We solve the problem of determining the minimal number of matrices over $\mathbb{R}$ which form a supercyclic abelian semigroup on $\mathbb{R}^n$. In particular, we show that no abelian semigroup generated by $\left[\frac{n-1}{2}\right]$ matrices on $\mathbb{R}^n$ can be supercyclic. ($[\,]$ denotes the integer part). This answers a question raised by the second author in [H. Marzougui, Monatsh. Math. 175 (2014), 401–410]. Furthemore, we show that supercyclicity and $\mathbb{R}_+$-supercyclicity are equivalent.

1. Introduction

Let $M_n(\mathbb{R})$ be the set of all square matrices over $\mathbb{R}$ of order $n \geqslant 1$ and let $\mathrm{GL}(n,\mathbb{R})$ be the group of invertible matrices of $M_n(\mathbb{R})$. Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$. By a sub-semigroup of $M_n(\mathbb{R})$, we mean a subset which is stable under multiplication and contains the identity matrix. For a vector $v \in \mathbb{R}^n$, we consider the orbit of G through v: $G(v) = \{Av : A \in G\} \subset \mathbb{R}^n$. The orbit $G(v) \subset \mathbb{R}^n$ is called *dense* (resp. *somewhere dense*) in $\mathbb{R}^n$ if $\overline{G(v)} = \mathbb{R}^n$ (resp. $\overline{G(v)}$ has non-empty interior), where $\overline{E}$ denotes the closure of a subset $E \subset \mathbb{R}^n$. We say that G is *hypercyclic* if there exists a vector $v \in \mathbb{R}^n$ such that $G(v)$ is dense in $\mathbb{R}^n$. In this case, v is called a *hypercyclic vector* for G. This definition generalizes the notion of hypercyclicity of a single operator to a semigroup of matrices. We refer the reader to the recent books [5] and to [11] and papers [1], [2], [4], [7], [8], [9], [15] for a thorough account on hypercyclicity. We say that G is *supercyclic* if there exists a vector $v \in \mathbb{R}^n$ such that $\mathbb{R}G := \{\lambda Av : A \in G,\ \lambda \in \mathbb{R}\}$ is dense in $\mathbb{R}^n$. In this case, v is called a *supercyclic vector* for G. For a single operator on a separable Banach space, the notion of supercyclicity was introduced by Hilden and Wallen [13]. Since then much research about supercyclicity has been done, we mention in particular [10], [12], [13]. Hilden and Wallen [13] proved in particular that on $\mathbb{C}^n$, $n \geqslant 2$, no operator can be supercyclic (see [14], see also [12], and [10] for another proof). In the trivial case $n = 1$, each non-zero matrix is supercyclic. In the real case, no operator can be supercyclic when $n \geqslant 3$ (see

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Corollary 4.10, see also [12]). However, if $n=2$, a rotation $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, with θ irrational, is supercyclic. It is clear that if an operator is hypercylic then it is supercyclic, but a supercyclic operator need not be hypercyclic.

For abelian semigroups of matrices on $\mathbb{C}^n$, supercyclicity was recently studied by the author [14] (see also [16]). In [14], the author asks whether there exist supercyclic abelian semigroups of matrices on $\mathbb{R}^n$ for $n \geqslant 3$. This paper can be viewed as a continuation of that work.

First, we give a general result answering the above question for any abelian subsemigroup of $M_n(\mathbb{R})$ by providing an effective way of checking that a given semigroup is supercyclic. Second, we prove that there is no supercyclic abelian semigroup in $\mathscr{K}_\eta(\mathbb{R})$, where η has length $r+2s$ (see the definition below) generated by $n-s$ matrices (see Theorem 4.4). Further, we show that the minimal number of matrices of $M_n(\mathbb{R})$ required to form a supercyclic abelian semigroup is $\left[\frac{n-1}{2}\right]+1$ (Theorem 4.9). This answers a question raised by the author in [14]. Third, we prove that supercyclicity and positive (or $\mathbb{R}_+$)-supercyclicity are equivalent (see Theorem 5.1).

This paper is organized as follows: In Section 2, we introduce the notations, definitions and we give some results about hypercyclicity that are needed throughout the paper. In Section 3, we prove Theorem 3.1. Section 4 is devoted to finitely generated abelian semigroups; we prove the Theorems 4.1, 4.4, 4.9 and Corollaries. In Section 5, we prove the equivalence between supercyclicity and positive supercyclicity.

2. Preliminaries

To state our main results, we need to introduce the following notations, definitions and some results on hypercyclicity. Set $\mathbb{N}$ be the set of non negative integers.

1) **The semigroup $\mathscr{K}_\eta(\mathbb{R})$.** Let $n\in\mathbb{N}$, $n\geqslant 1$ be fixed. Let $r,\ s\in\mathbb{N}$. By a partition of n, we mean a finite sequence of positive integers $\eta=(n_1,\dots,n_r;\ m_1,\dots,m_s)$ such that $\sum_{j=1}^{r} n_j + 2\sum_{j=1}^{s} m_j = n$. In particular, we have $r+2s\leqslant n$. The number $r+2s$ will be called the *length* of the partition. Given a partition $\eta=(n_1,\dots,n_r;\ m_1,\dots,m_s)$, we denote by:

- $\mathscr{K}_\eta(\mathbb{R}) := \mathbb{T}_{n_1}(\mathbb{R})\oplus\dots\oplus\mathbb{T}_{n_r}(\mathbb{R})\oplus\mathbb{B}_{m_1}(\mathbb{R})\oplus\dots\oplus\mathbb{B}_{m_s}(\mathbb{R})$, where
  - $\mathbb{T}_m(\mathbb{R})$ is the set of all $m\times m$ lower triangular matrices over $\mathbb{R}$ with only one eigenvalue, for each $m=1,2,\dots,n$
  - $\mathbb{B}_m(\mathbb{R})$ is the set of matrices of $M_{2m}(\mathbb{R})$ of the form

$$\begin{bmatrix} C & & & 0 \\ C_{2,1} & C & & \\ \vdots & \ddots & \ddots & \\ C_{m,1} & \dots & C_{m,m-1} & C \end{bmatrix}$$

for each $1\leqslant m\leqslant\frac{n}{2}$, where $C,\ C_{i,j}\in\mathbb{S}$, $2\leqslant i\leqslant m, 1\leqslant j\leqslant m-1$ and $\mathbb{S}$ is the semigroup of matrices over $\mathbb{R}$ of the form $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$.

In particular:

– $\mathscr{K}_\eta(\mathbb{R}) = \mathbb{T}_n(\mathbb{R})$ and $\eta = (n)$ if $r = 1,\ s = 0$.

– $\mathscr{K}_\eta(\mathbb{R}) = \mathbb{B}_m(\mathbb{R})$ and $\eta = (m)$, $n = 2m$ if $r = 0,\ s = 1$.

– $\mathscr{K}_\eta(\mathbb{R}) = \mathbb{B}_{m_1}(\mathbb{R}) \oplus \ldots \oplus \mathbb{B}_{m_s}(\mathbb{R})$ and $\eta = (m_1, \ldots, m_s)$ if $r = 0,\ s > 1$.

We denote by

• $\mathscr{K}_\eta^+(\mathbb{R}) := \mathbb{T}_{n_1}^+(\mathbb{R}) \oplus \ldots \oplus \mathbb{T}_{n_r}^+(\mathbb{R}) \oplus \mathbb{B}_{m_1}^*(\mathbb{R}) \oplus \ldots \oplus \mathbb{B}_{m_s}^*(\mathbb{R})$, where

– $\mathbb{T}_m^+(\mathbb{R})$ is the group of matrices of $\mathbb{T}_m(\mathbb{R})$ with all diagonal elements positive.

– $\mathbb{B}_m^*(\mathbb{R}) := \mathbb{B}_m(\mathbb{R}) \cap \mathrm{GL}(2m, \mathbb{R})$ is the group of invertible matrices of $\mathbb{B}_m(\mathbb{R})$.

We let

– $\mathbb{T}_m^*(\mathbb{R}) = \mathbb{T}_m(\mathbb{R}) \cap \mathrm{GL}(m, \mathbb{R})$ the group of invertible matrices of $\mathbb{T}_m(\mathbb{R})$.

– $\mathscr{K}_\eta^*(\mathbb{R}) := \mathscr{K}_\eta(\mathbb{R}) \cap \mathrm{GL}(n,\ \mathbb{R})$, it is a sub-semigroup of $\mathrm{GL}(n,\ \mathbb{R})$.

– $\mathscr{B}_0 = (e_1, \ldots, e_n)$ the canonical basis of $\mathbb{R}^n$.

– I_n the identity matrix on $\mathbb{R}^n$.

For a row vector $v \in \mathbb{R}^n$, we will be denoting by v^T the transpose of v. We let

• $u_\eta = [e_{\eta,1}, \ldots, e_{\eta,r}; f_{\eta,1}, \ldots, f_{\eta,s}]^T \in \mathbb{R}^n$, where for $1 \leqslant k \leqslant r$; $1 \leqslant l \leqslant s$,

– $e_{\eta,k} = [1, 0, \ldots, 0]^T \in \mathbb{R}^{n_k}$, $f_{\eta,l} = [1, 0, \ldots, 0]^T \in \mathbb{R}^{2m_l}$.

– $f_\eta^{(l)} = [0, \ldots, 0, f_1^{(l)}, \ldots, f_s^{(l)}]^T \in \mathbb{R}^n$, where for $1 \leqslant l, j \leqslant s$

$f_j^{(l)} = [0, \delta_{j,l}, 0, \ldots, 0]^T \in \mathbb{R}^{2m_j}$, ($\delta_{j,l}$ is the Kronecker symbol). Equivalently, $f_\eta^{(l)} = e_{t_l}$, where $t_l = \sum_{j=1}^{r} n_j + 2\sum_{j=1}^{l-1} m_j + 2,\ \ l = 1, \ldots, s$.

2) **Abelian sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$.** Let G be an abelian sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$, for some partition η of n. Consider the matrix exponential map $\exp : M_n(\mathbb{R}) \longrightarrow \mathrm{GL}(n, \mathbb{R})$ defined as $\exp(M) = e^M$.

We let:

• $\mathrm{g}_\eta := \exp^{-1}(G) \cap \mathscr{K}_\eta(\mathbb{R})$.

• $\mathrm{g}_\eta(u) := \{Bu :\ B \in \mathrm{g}_\eta\}$, $u \in \mathbb{R}^n$.

• $\mathrm{g}_\eta^2 = \exp^{-1}(G^2) \cap \mathscr{K}_\eta(\mathbb{R})$, where $G^2 = \{A^2 : A \in G\}$.

• $G^* = G \cap \mathrm{GL}(n, \mathbb{R})$, $G^{*2} = \{A^2 :\ A \in G^*\}$.

• *The index of* G. Each $M \in G^*$ can be written as $M = \mathrm{diag}(M_1, \ldots, M_r;\ \widetilde{M}_1, \ldots, \widetilde{M}_s) \in \mathscr{K}_\eta^*(\mathbb{R})$. Denote by μ_k the eigenvalues of M_k, $k = 1, \ldots, r$. We define the *index* of G to be

$$\mathrm{ind}(G) := \begin{cases} 0, \text{ if } r = 0 \\ \begin{cases} 1, & \text{if } \exists\, M \in G^* \text{ with } \mu_1 < 0 \\ 0, & \text{otherwise} \end{cases}, \text{ if } r = 1 \\ \mathrm{card}\{k \in \{1, \ldots, r\} :\ \exists M \in G^* \text{ with } \mu_k < 0 \text{ and } \mu_i > 0,\ \forall\, i \neq k\}, \\ \text{if } r \notin \{0,\ 1\}. \end{cases}$$

In particular,

– If $G^* \subset \mathscr{K}_\eta^+(\mathbb{R})$ with $r \neq 0$, then $\mathrm{ind}(G) = 0$.

– If $G^* \subset \mathbb{B}_m^*(\mathbb{R})$, then $\mathrm{ind}(G) = 0$ (since $r = 0$).

– As an example; let G be the semigroup generated by $A_1 = \mathrm{diag}(e^\pi, e^\pi)$, $A_2 = \begin{bmatrix} -1 & 0 \\ \pi & -1 \end{bmatrix}$ and $A_3 = e^{-\pi\sqrt{2}} \begin{bmatrix} 1 & 0 \\ -\pi\sqrt{3} & 1 \end{bmatrix}$.

We see that G is an abelian sub-semigroup of $\mathbb{T}_2^*(\mathbb{R})$ with $\eta = (2)$, $r = 1$ and $\operatorname{ind}(G) = 1$.

3) **The normal form of an abelian sub-semigroup of** $M_n(\mathbb{R})$. First recall the following proposition.

PROPOSITION 2.1. ([3], *Proposition 2.2*) *Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$. Then there exists a $P \in GL(n,\mathbb{R})$ such that $P^{-1}GP$ is an abelian sub-semigroup of $\mathcal{K}_\eta(\mathbb{R})$, for some partition η of n.*

Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$, $n \geqslant 1$. Then, following Proposition 2.1, let $P \in \mathrm{GL}(n,\mathbb{R})$ such that $P^{-1}GP \subset \mathcal{K}_\eta(\mathbb{R})$ for some partition η of n. Given an integer $t \leqslant n$, we shall say that the semigroup G has "a normal form of length t" if G has a normal form in $\mathcal{K}_\eta(\mathbb{R})$, for some partition $\eta = (n_1,\dots,n_r;\ m_1,\dots,m_s)$ with length $t = r + 2s$. For such a choice of matrix P, we define the *index* of G to be $\operatorname{ind}(G) := \operatorname{ind}(P^{-1}GP)$. It is clear that this definition does not depend on P.

4) **Some results on hypercyclicity**. The following theorems characterize the hypercyclicity and the existence of somewhere dense orbit of any abelian semigroup of matrices on $\mathbb{R}^n$.

THEOREM 2.2. ([3], Theorem 1.1) *Let G be an abelian sub-semigroup of $\mathcal{K}_\eta(\mathbb{R})$, $n \geqslant 1$, where η has length $r + 2s$.*

1. *The following properties are equivalent:*
   - *(i) G has a somewhere dense orbit,*
   - *(ii) $G(u_\eta)$ is somewhere dense in $\mathbb{R}^n$,*
   - *(iii) $\mathrm{g}_\eta(u_\eta)$ is an additive sub-semigroup dense in $\mathbb{R}^n$.*
2. *Assume that G is generated by p matrices $A_1,\dots,A_p$ ($p \geqslant 1$) and let $B_1,\dots,B_p \in \mathrm{g}_\eta$ such that $A_1^2 = e^{B_1},\dots,A_p^2 = e^{B_p}$. Then G has a somewhere dense orbit in $\mathbb{R}^n$ if and only if $\sum_{k=1}^{p} \mathbb{N}B_k u_\eta + \sum_{l=1}^{s} 2\pi\mathbb{Z} f_\eta^{(l)}$ is dense in $\mathbb{R}^n$.*

THEOREM 2.3. ([3], Theorem 1.4) *Let G be an abelian sub-semigroup of $\mathcal{K}_\eta(\mathbb{R})$, $n \geqslant 1$, where η has length $r + 2s$.*

1. *The following properties are equivalent:*
   - *(i) G is hypercyclic,*
   - *(ii) $G(u_\eta)$ is dense in $\mathbb{R}^n$,*
   - *(iii) $\mathrm{g}_\eta(u_\eta)$ is an additive sub-semigroup dense in $\mathbb{R}^n$ and $\operatorname{ind}(G) = r$.*
2. *Assume that G is generated by p matrices $A_1,\dots,A_p$ ($p \geqslant 1$) and let $B_1,\dots,B_p \in \mathrm{g}_\eta$ such that $A_1^2 = e^{B_1},\dots,A_p^2 = e^{B_p}$. Then G is hypercyclic if and only if $\sum_{k=1}^{p} \mathbb{N}B_k u_\eta + \sum_{l=1}^{s} 2\pi\mathbb{Z} f_\eta^{(l)}$ is dense in $\mathbb{R}^n$ and $\operatorname{ind}(G) = r$.*

3. Supercyclic abelian sub-semigroups of $\mathcal{K}_\eta(\mathbb{R})$

The aim of this section is to prove the following theorem.

THEOREM 3.1. *Let $n \in \mathbb{N}$, $n \geqslant 1$ and let G be an abelian sub-semigroup of $\mathcal{K}_\eta(\mathbb{R})$, where η has length $r+2s$. Then the following are equivalent:*

(i) G is supercyclic,

(ii) u_η is a supercyclic vector for G,

(iii) $\mathrm{g}_\eta^2(u_\eta)+\mathbb{R}u_\eta$ is dense in $\mathbb{R}^n$ and $\mathrm{ind}(G)=r$.

We denote by

- $G' = \mathbb{R}G := \{\lambda A:\ \lambda \in \mathbb{R},\ A \in G\}$. It is an abelian semigroup of matrices on $\mathbb{R}^n$.
- $\mathrm{g}'_\eta = \exp^{-1}(G') \cap \mathcal{K}_\eta(\mathbb{R})$.

LEMMA 3.2. *We have* $(\mathrm{g}'_\eta)^2 = (\mathrm{g}_\eta)^2 + \mathbb{R}I_n$.

Proof. Let $A \in (\mathrm{g}'_\eta)^2$. Then $e^A = (cB)^2$ for some $c \in \mathbb{R}^*$ and $B \in G$. Set $c^2 = e^\alpha$ for some $\alpha \in \mathbb{R}$. Then $e^{-\alpha I_n + A} = B^2$ and so $-\alpha I_n + A \in \exp^{-1}(G^2)$. As $A \in \mathcal{K}_\eta(\mathbb{R})$, so is $-\alpha I_n + A$ and hence $A \in (\mathrm{g}_\eta)^2 + \mathbb{R}I_n$. Conversely, let $A = \alpha I_n + B$, where $B \in (\mathrm{g}_\eta)^2$ and $\alpha \in \mathbb{R}$. As $B \in \mathcal{K}_\eta(\mathbb{R})$, then so is A. Moreover, we have $e^A = e^\alpha e^B$. Since $e^B \in G^2$, so $e^A \in (G')^2$ and thus $A \in (\mathrm{g}'_\eta)^2$. □

LEMMA 3.3. ([3], Corollary 5.4) *Let G be an abelian sub-semigroup of $\mathcal{K}^*_\eta(\mathbb{R})$. Then G has a somewhere dense orbit if and only if so does G^2.*

LEMMA 3.4. *$\mathrm{g}'_\eta(u_\eta)$ is an additive sub-semigroup dense in $\mathbb{R}^n$ if and only if $(\mathrm{g}'_\eta)^2(u_\eta)$ is.*

Proof. This follows from Theorem 2.2 and Lemma 3.3. □

The proofs of the Lemmas 3.5 and 3.6 below are straightforward.

LEMMA 3.5. *Let G be an abelian sub-semigroup of $\mathcal{K}_\eta(\mathbb{R})$. Then G is supercyclic if and only if G' is hypercyclic.*

LEMMA 3.6. *We have* $\mathrm{ind}(G') = \mathrm{ind}(G)$.

LEMMA 3.7. ([3], Proposition 4.1) *Let G be an abelian sub-semigroup of $\mathcal{K}_\eta(\mathbb{R})$ and let $u \in \mathbb{R}^n$. Then $G^*(u)$ is somewhere dense (resp. dense) in $\mathbb{R}^n$ if and only if $G(u)$ is.*

LEMMA 3.8. ([3], Proposition 4.5) *Let G be an abelian sub-semigroup of $\mathcal{K}^*_\eta(\mathbb{R})$, where η has length $r+2s$. Then the following properties are equivalent:*

(i) $\overline{G(u_\eta)} = \mathbb{R}^n$,

(ii) $\overline{G(u_\eta)}$ *has non-empty interior and* $\mathrm{ind}(G) = r$.

LEMMA 3.9. *Let G be an abelian sub-semigroup of $\mathcal{K}_\eta(\mathbb{R})$, where η has length $r+2s$. Set $G' = \mathbb{R}G$. The following properties are equivalent:*

(i) G' *has a somewhere dense orbit,*

(ii) $\overline{G'(u_\eta)}$ *has non-empty interior,*

(iii) $(\mathrm{g}_\eta)^2(u_\eta) + \mathbb{R}u_\eta$ *is an additive sub-semigroup dense in* $\mathbb{R}^n$.

Proof. The proof follows from Theorem 2.2, Lemmas [3.2–3.5] and Lemma 3.7. □

Proof of Theorem 3.1. The proof follows from Theorem 2.3, Lemmas [3.5–3.9]. □

4. On finitely generated abelian supercyclic semigroup

THEOREM 4.1. *Let $n \in \mathbb{N}$, $n \geqslant 1$ and let G be an abelian sub-semigroup of $\mathcal{K}_\eta(\mathbb{R})$, where η has length $r+2s$. Assume that G is generated by p matrices $A_1,\dots,A_p$ ($p \geqslant 1$) and let $B_1,\dots,B_p \in \mathrm{g}_\eta$ such that $A_1^2 = e^{B_1},\dots,A_p^2 = e^{B_p}$. Then G is supercyclic if and only if $\sum\limits_{k=1}^{p} \mathbb{N}B_k u_\eta + \sum\limits_{l=1}^{s} 2\pi\mathbb{Z} f_\eta^{(l)} + \mathbb{R}u_\eta$ is dense in $\mathbb{R}^n$ and* $\mathrm{ind}(G) = r$.

The proof needs the following lemma.

LEMMA 4.2. ([3], Proposition 4.6) *Let G be an abelian sub-semigroup of $\mathcal{K}_\eta^+(\mathbb{R})$ and let $B_1,\dots,B_p \in \mathcal{K}_\eta(\mathbb{R})$ $(p \geqslant 1)$ such that $e^{B_1},\dots,e^{B_p}$ generate G. We have that $\mathrm{g}_\eta(u_\eta) = \sum\limits_{k=1}^{p} \mathbb{N}B_k u_\eta + \sum\limits_{l=1}^{s} 2\pi\mathbb{Z} f_\eta^{(l)}$.*

Proof of Theorem 4.1. The proof follows from Theorem 3.1, Lemmas 3.7 and 4.2. □

COROLLARY 4.3. *Let $n \in \mathbb{N}$, $n \geqslant 1$ and let G be an abelian sub-semigroup of $\mathcal{K}_\eta(\mathbb{R})$, where η has length $r+2s$.*

(1) If G is supercyclic, then it is hypercyclic if and only if it has a somewhere dense orbit.

(2) If G is supercyclic and generated by p matrices $A_1,\dots,A_p$ ($p \geqslant 1$) such that $A_1^2 = e^{B_1},\dots,A_p^2 = e^{B_p}$, where $B_1,\dots,B_p \in \mathrm{g}_\eta$, then it is hypercyclic if and only if $\sum\limits_{k=1}^{p} \mathbb{N}B_k u_\eta + \sum\limits_{l=1}^{s} 2\pi\mathbb{Z} f_\eta^{(l)}$ is dense in $\mathbb{R}^n$.

Proof. If G is supercyclic, then by Theorem 3.1, $\text{ind}(G) = r$ and so Corollary 4.3 follows from Theorems 2.2 and 2.3. □

THEOREM 4.4. *Let $n \geqslant 1$ be an integer and let G be an abelian sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$, for some partition η of n of length $r+2s$. If G is generated by $n-s-1$ matrices in $\mathscr{K}_\eta(\mathbb{R})$, it is not supercyclic.*

LEMMA 4.5. *Let $n, s \in \mathbb{N}$ such that $n \geqslant 2$ and $1 \leqslant s < n$. Let $H := \sum_{k=1}^{n-s-1} \mathbb{N}v_k + \sum_{k=1}^{s} \mathbb{Z}e_k + \mathbb{R}v$ with $v_k \in \mathbb{R}^n$, $1 \leqslant k \leqslant n-s-1$ and $v \in \mathbb{R}^n$. Then H is nowhere dense in $\mathbb{R}^n$.*

Proof. Let E be the $\mathbb{R}$-vector space generated by $(v_1, \dots, v_{n-s-1}, e_1, \dots, e_s)$. One has $H \subset E + \mathbb{R}v$. We distinguish two cases.

Case 1 : $\dim E \leqslant n-2$. In this case, $\dim(E + \mathbb{R}v) \leqslant n-1$ and so H is nowhere dense in $\mathbb{R}^n$.

Case 2 : $\dim E = n-1$.

– If $v \in E$ then $\dim(E + \mathbb{R}v) = n-1$ and so H is nowhere dense in $\mathbb{R}^n$.

– If $v \notin E$ then $E + \mathbb{R}v = \mathbb{R}^n$, thus $(v_1, \dots, v_{n-s-1}, e_1, \dots, e_s, v)$ is a basis of $\mathbb{R}^n$. Assume that H is somewhere dense in $\mathbb{R}^n$. Then there exists a vector

$$w = \sum_{k=1}^{n-s-1} \alpha_k v_k + \sum_{k=1}^{s} \beta_k e_k + \gamma v$$

with $\alpha_k \in \mathbb{R} \backslash \mathbb{Q}$, $1 \leqslant k \leqslant n-s-1$, $\beta_k \in \mathbb{R}$, $1 \leqslant k \leqslant s$ and $\gamma \in \mathbb{R}$ such that

$$w = \lim_{l \to +\infty} \sum_{k=1}^{n-s-1} m_{l,k} v_k + \sum_{k=1}^{s} s_{l,k} e_k + \lambda_l v,$$

where $m_{l,k} \in \mathbb{N}$, $1 \leqslant k \leqslant n-s-1$, $s_{l,k} \in \mathbb{Z}$, $1 \leqslant k \leqslant s$ and $\lambda_l \in \mathbb{R}$. Therefore, $\lim_{l \to +\infty} m_{l,k} = \alpha_k$ for every $1 \leqslant k \leqslant n-s-1$. This implies that $\alpha_k \in \mathbb{N}$, a contradiction. □

Proof of Theorem 4.4. Let $A_1, \dots, A_{n-s-1}$ be matrices in $\mathscr{K}_\eta(\mathbb{R})$ that generate G and let $B_1, \dots, B_{n-s-1} \in \mathrm{g}_\eta$ such that $A_1^2 = e^{B_1}, \dots, A_{n-s-1}^2 = e^{B_{n-s-1}}$.

Define $\mathscr{B}_0 \backslash (e_{t_1}, \dots, e_{t_s}) := (e_{i_{s+1}}, \dots, e_{i_n})$, where $e_{t_l} = f_\eta^{(l)}$, $1 \leqslant l \leqslant s$ (see page 3) and define the matrix S by

$$Se_k = \begin{cases} 2\pi f_\eta^{(k)}, & \text{if } 1 \leqslant k \leqslant s, \\ e_{i_k}, & \text{if } s+1 \leqslant k \leqslant n. \end{cases}$$

We see that $S \in \mathrm{GL}(n; \mathbb{R})$. Write $S^{-1}u_\eta = v$ and $S^{-1}B_k u_\eta = v_k$, $1 \leqslant k \leqslant n-s-1$. We let $H := \sum_{k=1}^{n-s-1} \mathbb{N}v_k + \sum_{k=1}^{s} \mathbb{Z}e_k + \mathbb{R}v$. Then we have that

$$S(H) = \sum_{k=1}^{n-s-1} \mathbb{N}B_k u_\eta + \sum_{l=1}^{s} 2\pi\mathbb{Z}f_\eta^{(l)} + \mathbb{R}u_\eta$$

By Lemma 4.5, H is nowhere dense in $\mathbb{R}^n$ and thus so is $S(H)$. We conclude by Theorem 4.1 that G is not supercyclic. □

PROPOSITION 4.6. *For any $n \in \mathbb{N}$, $n \geqslant 1$, $r,s \in \mathbb{N}$, and any partition η of n of length $r+2s$. there exist $n-s$ matrices in $\mathscr{K}^*_{\eta'}(\mathbb{R})$, where η' is a partition of n of length $1+r+2s$ or $r+2s$, that generate a supercyclic abelian semigroup.*

LEMMA 4.7. ([15], Theorem 1.5) *Let $n \in \mathbb{N}$, $n \geqslant 1$ and $r,s \in \mathbb{N}$. Then for any partition η of n of length $r+2s$, there exist $n-s+1$ matrices in $\mathscr{K}^*_{\eta}(\mathbb{R})$ that generate a hypercyclic abelian semigroup.*

Proof of Proposition 4.6. Set $\eta = (n_1,\dots,n_r;\ m_1,\dots,m_s)$. If $n=1$, then $r=1$, $s=0$ and $n_1=1$. So it is obvious that every $a \in \mathbb{R}^*$ generate a supercyclic semigroup of $\mathbb{R}$. Assume that $n \geqslant 2$. We distinguish two cases:

Case 1: $r \neq 0$.

– If $n_i \geqslant 2$, for some $1 \leqslant i \leqslant r$, say for example $n_1 \geqslant 2$, then $\eta_0 := (n_1 - 1,\dots,n_r;\ m_1,\dots,m_s)$ is a partition of $n-1$ of length $r+2s$. By Lemma 4.7, there exist $(n-1)-s+1 = n-s$ matrices $A'_1,\dots,A'_{n-s}$ in $\mathscr{K}^*_{\eta_0}(\mathbb{R})$ that generate a hypercyclic abelian semigroup G'. Set $A_j = \begin{bmatrix} 1 & O \\ O & A'_j \end{bmatrix}$, $j=1,\dots,n-s$ and let G be the semigroup generated by $A_1,\dots,A_{n-s}$. It is clear that G is an abelian semigroup of $\mathscr{K}^*_{\eta'}(\mathbb{R})$, where $\eta' = (1,n_1-1,\dots,n_r;\ m_1,\dots,m_s)$ is a partition of n of length $1+r+2s$.

Let $x' \in \mathbb{R}^{n-1}$ so that $G'x'$ is dense in $\mathbb{R}^{n-1}$ and set $x = [1,x']^T$. We shall prove that x is a supercyclic vector for G:

Let $y = [y_1,y']^T$ with $y_1 \in \mathbb{R}^*$ and $y' \in \mathbb{R}^{n-1}$. Then there exist sequences $\varphi_1(k),\dots,\varphi_{n-s}(k)$ of integers such that

$$\lim_{k\to+\infty} (A'_1)^{\varphi_1(k)} \dots (A'_{n-s})^{\varphi_{n-s}(k)} x' = y_1^{-1} y'.$$

Then we have $[y_1,y']^T = \lim_{k\to+\infty} y_1 A_1^{\varphi_1(k)} \dots A_{n-s}^{\varphi_{n-s}(k)} x$. Therefore $y \in \overline{\mathbb{R}Gx}$. We conclude that $\mathbb{R}^* \times \mathbb{R}^{n-1} \subset \overline{\mathbb{R}Gx}$ and hence $\overline{\mathbb{R}Gx} = \mathbb{R}^n$.

– If $n_i = 1$, for all $1 \leqslant i \leqslant r$, then $\eta_0 = (1,\dots,1;\ m_1,\dots,m_s)$ is a partition of $n-1$ of length $r-1+2s$. Then by Lemma 4.7, there exist $(n-1)-s+1 = n-s$ matrices $A'_1,\dots,A'_{n-s}$ in $\mathscr{K}^*_{\eta_0}(\mathbb{R})$ that generate a hypercyclic abelian semigroup G'. Set $A_j = \begin{bmatrix} 1 & O \\ O & A'_j \end{bmatrix}$, $j=1,\dots,n-s$ and let G be the semigroup generated by $A_1,\dots,A_{n-s}$. Then G is an abelian semigroup of $\mathscr{K}^*_{\eta'}(\mathbb{R})$, where $\eta' = (1,1,\dots,1;\ m_1,\dots,m_s)$ is a partition of n of length $r+2s$. Hence we prove similarly that G is supercyclic.

Case 2: $r=0$. Then $n = 2(m_1+\dots+m_s)$, $s \neq 0$ and $\eta_0 = (2m_1-1,m_2,\dots,m_s)$ is a partition of $n-1$ of length $1+2(s-1)$. By Lemma 4.7, there exist $(n-1)-s+1 = n-s$ matrices $A'_1,\dots,A'_{n-s}$ in $\mathscr{K}^*_{\eta_0}(\mathbb{R})$ that generate a hypercyclic abelian semigroup G'. Set $A_j = \begin{bmatrix} 1 & O \\ O & A'_j \end{bmatrix}$, $j=1,\dots,n-s$ and let G be the semigroup generated by $A_1,\dots,A_{n-s}$. It is clear that G is an abelian semigroup of $\mathscr{K}^*_{\eta'}(\mathbb{R})$, where $\eta' = (1,2m_1-1,m_2,\dots,m_s)$ is a partition of n of length $2+2(s-1)$.

Let $x' \in \mathbb{R}^{n-1}$ so that $G'x'$ is dense in $\mathbb{R}^{n-1}$ and set $x = [1,x']^T$. By the same way as above, x is a supercyclic vector for G. □

COROLLARY 4.8. *The minimum number of trigonalizable matrices of $M_n(\mathbb{R})$ that generate a supercyclic abelian semigroup is n.*

Proof. Let G be an abelian semigroup generated by trigonalizable matrices of $M_n(\mathbb{R})$. Then by Proposition 2.1, we may assume that G is an abelian sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$, for some partition $\eta = (n_1,\dots,n_r)$ of n (in this case $s=0$). If G is generated by $n-1$ matrices, then by Theorem 4.4, G is not supercyclic. Furthermore, by Proposition 4.6, there exist n matrices in $\mathscr{K}^*_{\eta'}(\mathbb{R})$, for some partition η' of n of length $1+r$ or r, generating a supercyclic abelian semigroup. Notice that these n matrices are in particular triangular. The proof is complete. □

The following theorem is, in some sense, best possible.

THEOREM 4.9. (Minimal generators) *Let $n \in \mathbb{N}$, $n \geqslant 1$. The minimum number of matrices of $M_n(\mathbb{R})$ that generate a supercyclic abelian semigroup is $\left[\frac{n-1}{2}\right]+1$.*

Proof. First, we prove that if G is generated by $[\frac{n-1}{2}]$ matrices of $M_n(\mathbb{R})$, then it is not supercyclic: By Proposition 2.1, we may assume that G is an abelian sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$, for some partition $\eta = (n_1,\dots,n_r;\ m_1,\dots,m_s)$ of n. If $r=0$, then $n = 2(m_1+\ldots+m_s)$, $2s \leqslant n$ and so $\left[\frac{n-1}{2}\right] \leqslant n-s-1$. If $r \neq 0$, then $1+2s \leqslant r+2s \leqslant n$ and so $\left[\frac{n-1}{2}\right] \leqslant n-s-1$. Therefore from Theorem 4.4, G is not supercyclic.

Second, we will show that there exist $\left[\frac{n-1}{2}\right]+1$ matrices of $M_n(\mathbb{R})$ that generate a supercyclic abelian semigroup. If n is even, then $n=2s$ and let $\eta = (m_1,\dots,m_s)$ with $m_i = 1$, $i = 1,\dots,s$. Then η is a partition of n of length $2s$. Then by Proposition 4.6, there exist $n-s = \frac{n}{2}$ matrices in $\mathscr{K}^*_{\eta'}(\mathbb{R})$, for some partition η' of length $2+2(s-1)$, generating a supercyclic abelian semigroup. If n is odd, then $n = 2s+1$ and let $\eta = (1;m_1,\dots,m_s)$ with $m_i = 1$, $i = 1,\dots,s$. Then η is a partition of n of length $1+2s$. Then by Proposition 4.6, there exist $n-s = \frac{n+1}{2}$ matrices in $\mathscr{K}^*_{\eta'}(\mathbb{R})$, for some partition η' of length $1+2s$, generating a supercyclic abelian semigroup. In either cases, there exist

$$\left[\frac{n-1}{2}\right]+1 = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

matrices of $M_n(\mathbb{R})$ generating a supercyclic abelian semigroup. As a result, Theorem 4.9 follows. □

COROLLARY 4.10. ([12], [13]) *For $n \geqslant 3$, no matrix on $\mathbb{R}^n$ is supercyclic.*

Proof. Since $n \geqslant 3$, so $1 \leqslant \left[\frac{n-1}{2}\right]$ and then the Corollary follows from Theorem 4.9. □

5. Positive supercyclicity

Let G be an abelian sub-semigroup of $M_n(\mathbb{R})$, it is called *positive supercyclic* or also $\mathbb{R}_+$*-supercyclic* if there exists $x \in \mathbb{R}^n$ such that $\mathbb{R}_+G(x) := \{\lambda Ax : A \in G,\ \lambda \in \mathbb{R}_+\}$ is dense in $\mathbb{R}^n$. This concept was introduced in [6] for one operator on a separable Banach space. Bermudez et al. [6] proved that if an operator T is $\mathbb{R}$-supercyclic, then in fact T is $\mathbb{R}_+$-supercyclic. Actually we prove that the same conclusion holds for any abelian semigroup of $M_n(\mathbb{R})$.

THEOREM 5.1. *Let $n \in \mathbb{N}$, $n \geqslant 1$ and let G be an abelian sub-semigroup of $M_n(\mathbb{R})$. Then the following are equivalent:*

(i) G is supercyclic,

(ii) G is $\mathbb{R}_+$-supercyclic.

Proof. By Proposition 2.1, we may assume that G is an abelian sub-semigroup of $\mathcal{K}_\eta(\mathbb{R})$, for some partition η of n of length $r+2s$. It is obvious that $(ii) \Rightarrow (i)$. Let us prove $(i) \Rightarrow (ii)$. Suppose that G is supercyclic. Then by Lemma 3.5, $G' := \mathbb{R}G$ is hypercyclic and so by Theorem 2.3, $G'(u_\eta)$ is dense in $\mathbb{R}^n$. We have $G'(u_\eta) = \mathbb{R}_+G(u_\eta) \cup \mathbb{R}_+G(-u_\eta)$. We distinguish two cases.

Case 1: $\mathbb{R}_+G(u_\eta)$ is nowhere dense. In this case, $\mathbb{R}_+G(-u_\eta)$ is dense in $\mathbb{R}^n$ and hence G is $\mathbb{R}_+$-supercyclic.

Case 2: $\mathbb{R}_+G(u_\eta)$ is somewhere dense. We have $\mathrm{ind}(G) = r = \mathrm{ind}(\mathbb{R}_+G)$ (since G is supercyclic). Furthermore, $\mathbb{R}_+G \subset \mathcal{K}_\eta(\mathbb{R})$, thus by Lemma 3.8, $\mathbb{R}_+G(u_\eta)$ is dense in $\mathbb{R}^n$ and so G is $\mathbb{R}_+$-supercyclic. The proof is complete. □

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