

## THE NORM OF BACKWARD DIFFERENCE OPERATOR $\Delta^{(n)}$ ON CERTAIN SEQUENCE SPACES

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*Abstract.* Let  $p \geq 1$  and  $n$  be a non-negative integer and  $A = (a_{m,k})_{m,k \geq 0}$  be a non-negative matrix. In this paper the norm of backward difference operators  $\Delta^{(n)}$  and  $\Delta^{(-n)}$  from the sequence space  $l_p$  into the certain sequence space  $A_p$  are computed, where  $A_p$  is the space of all real sequences  $x = (x_k)_{k=0}^\infty$  such that

$$\sum_{m=0}^{\infty} \left| \sum_{k=0}^{\infty} a_{m,k} x_k \right|^p < \infty.$$

Moreover, the results are applied for well known matrices such as Cesàro matrix of order  $n$  and Hilbert and also new matrices which are introduced in this study.

### 1. Introduction

Let  $p \geq 1$  and  $\omega$  denote the set of all real-valued sequences. Any vector subspace of  $\omega$  is called a sequence space. The classical space  $l_p$  is the set of all real sequences  $x = (x_k)_{k=0}^\infty \in \omega$  such that

$$\|x\|_p = \left( \sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} < \infty.$$

Let  $A = (a_{m,k})_{m,k \geq 0}$  be a matrix. We define the matrix domain  $A_p$  by

$$\begin{aligned} A_p &= \{ x = (x_k) : Ax \in l_p \} \\ &= \left\{ x = (x_k) : \sum_{m=0}^{\infty} \left| \sum_{k=0}^{\infty} a_{m,k} x_k \right|^p < \infty \right\}, \end{aligned} \tag{1.1}$$

which is a sequence space. The new sequence space  $A_p$  generated by the limitation matrix  $A$  from a sequence space  $l_p$  can be the expansion or the contraction and or the overlap of the original space  $l_p$  [3].

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The matrix domain which plays an important role to construct a new sequence space of classical space  $l_p$ , has been studied by several authors. For instance, the matrix domains of the difference operator are investigated in [1, 3, 14, 15, 16] and the matrix domains of fractional difference operator are introduced in [2, 10, 11, 12, 13]. In these works topological properties, inclusion relations, duals and matrix transformations of these spaces are investigated, but the norm of matrix operators on these matrix domains are not studied.

Although the norm of matrix operators on the sequence space  $l_p$  have computed by many mathematicians such as Hardy, Bennett and Borwein [4, 6, 7, 8, 9], the problem of finding the norm of operators on matrix domains has not studied extensively. The authors recently computed norm of operators on some matrix domains [17]. In this present paper, we try to solve this problem for backward difference operator from  $l_p$  into  $A_p$ .

The semi-norm on the matrix domain  $A_p$ ,  $\|\cdot\|_{A_p}$ , is defined by

$$\|x\|_{A_p} = \left( \sum_{m=0}^{\infty} \left| \sum_{k=0}^{\infty} a_{m,k} x_k \right|^p \right)^{\frac{1}{p}}.$$

Note that this function will be not a norm, since if  $x = (1, -1, 0, 0, \dots)$  and the matrix  $A$  is defined such that  $a_{0,0} = a_{0,1} = 1$  and the remaining entries be zero, then  $\|x\|_{A_p} = 0$  while  $x \neq 0$ . Consider that  $A_p = l_p$  and  $\|\cdot\|_{A_p} = \|\cdot\|_p$ , for  $A = I$ .

Throughout this paper, we suppose that  $n$  is an arbitrary non-negative integer and  $\binom{-1}{0} = 1$  and  $\binom{n}{k} = 0$  for  $k > n \geq 0$ . The backward difference operators  $\Delta^{(n)} = (\delta_{k,j}^{(n)})$  of order  $n$  and  $\Delta^{(-n)} = (\delta_{k,j}^{(-n)})$  of order  $-n$  are defined as below, respectively,

$$\delta_{k,j}^{(n)} = \begin{cases} (-1)^{k-j} \binom{n}{k-j} & j \leq k \leq n + j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_{k,j}^{(-n)} = \begin{cases} \binom{n+k-j-1}{k-j} & 0 \leq j \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $\Delta^{(n)} = \Delta^{(-n)} = I$ , when  $n = 0$  and  $I$  is the identity matrix.

In this paper, we consider the inequalities of the forms

$$\|\Delta^{(n)}x\|_{A_p} \leq U\|x\|_p, \quad \|\Delta^{(-n)}x\|_{A_p} \leq V\|x\|_p,$$

for all sequence  $x \in l_p$ . The constants  $U$  and  $V$  are not depending on  $x$ , and the norms of  $\Delta^{(n)}$  and  $\Delta^{(-n)}$  are the smallest possible values of  $U$  and  $V$ , respectively. Note that in the above inequalities, we choose the matrix domains  $A_p$  which satisfy boundedness of the operators  $\Delta^{(n)}$  and  $\Delta^{(-n)}$ .

We use the notation  $\|\cdot\|_{A_p}$  for the norm of operators from  $l_p$  into  $A_p$ , and  $\|\cdot\|_p$  for the norm of operators from  $l_p$  into itself.

In this study, we focus on computing the norm of operator  $\Delta^{(n)}$  from  $l_p$  into  $A_p$ , for  $p = 1$  in Section 2 and for  $p > 1$  in Section 3. Moreover the norm of operator  $\Delta^{(-n)}$  from  $l_p$  into  $A_p$  is considered in Section 4.

### 2. The norm of operator $\Delta^{(n)}$ from $l_1$ into $A_1$

In this section, we try to solve the problem of finding norm of operator  $\Delta^{(n)}$  from  $l_1$  into  $A_1$ , where  $A$  are Cesàro, Hilbert, identity and backward difference matrices. We may begin with the following theorem which is essential in the study.

**THEOREM 2.1.** *Let  $A = (a_{k,i})_{k,i \geq 0}$  be a matrix and  $B = (b_{k,i})_{k,i \geq 0}$  be a lower triangular matrix. If  $M = \sup_j u_j < \infty$  where*

$$u_j = \sum_{k=0}^{\infty} |(AB)_{k,j}|,$$

for  $j = 0, 1, \dots$ , then  $B$  is a bounded operator from  $l_1$  into  $A_1$  and

$$\|B\|_{A_1} = M.$$

*Proof.* Let  $x$  be a sequence in  $l_1$ . We have

$$\begin{aligned} \|Bx\|_{A_1} &= \sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} \sum_{i=0}^j a_{k,j} b_{j,i} x_i \right| = \sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} a_{k,i} b_{i,j} x_j \right| \\ &= \sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} (AB)_{k,j} x_j \right| \leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} |(AB)_{k,j}| |x_j| \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |(AB)_{k,j}| |x_j| = \sum_{j=0}^{\infty} u_j |x_j| \leq M \|x\|_1, \end{aligned}$$

which says that  $\|B\|_{A_1} \leq M$ . Let  $m$  be a non-negative integer. We take  $x = e_m$  which  $e_m$  denotes the sequence having 1 in place  $m$  and 0 elsewhere, then  $\|x\|_1 = 1$  and  $\|Bx\|_{A_1} = u_m$ . Hence

$$u_m = \frac{\|Bx\|_{A_1}}{\|x\|_1} \leq \|B\|_{A_1},$$

and  $M = \sup_m u_m \leq \|B\|_{A_1}$ . Therefore we have the desired result.  $\square$

In the following, we investigate the norms of Cesàro matrix of order 1,  $C^1 = (c_{k,j})$ , which is defined by

$$c_{k,j} = \begin{cases} \frac{1}{k+1} & 0 \leq j \leq k \\ 0 & j > k. \end{cases}$$

To do this, the following two lemmas are needed.

**LEMMA 2.2.** *If  $n \in \mathbb{N}$ , then*

$$\sum_{j=0}^m (-1)^j \binom{n}{j} = \begin{cases} (-1)^m \binom{n-1}{m} & m < n \\ 0 & m = n. \end{cases}$$

*Proof.* Use the identity  $\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}$  for  $j \geq 1$ , for  $m < n$  and note that the left hand side of the equality is the summation of the coefficients of binomial  $(1 - z)^n$  for  $m = n$ .  $\square$

LEMMA 2.3. *If  $n \in \mathbb{N}$ , then*

$$\binom{n-1}{0} + \frac{1}{2} \binom{n-1}{1} + \dots + \frac{1}{n} \binom{n-1}{n-1} = \frac{2^n - 1}{n}.$$

*Proof.* By integrating from 0 to 1 of both sides of the identity

$$(1 + z)^{n-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} z^j,$$

the proof is obvious.  $\square$

THEOREM 2.4. *Let  $C^1$  be the Cesàro matrix of order 1. Then  $\Delta^{(n)}$  is a bounded operator from  $l_1$  into  $C_1^1$  and*

$$\|\Delta^{(n)}\|_{C_1^1} = \frac{2^n - 1}{n}.$$

*Proof.* According to above theorem and Lemma 2.2, we deduce that

$$\begin{aligned} u_j &= \sum_{k=j}^{\infty} \left| (C^1 \Delta^{(n)})_{k,j} \right| = \sum_{k=j}^{n+j} \left| \sum_{i=j}^k c_{k,i} \delta_{i,j}^{(n)} \right| \\ &= \sum_{k=j}^{n+j} \left| \sum_{i=j}^k \frac{(-1)^{i-j}}{k+1} \binom{n}{i-j} \right| = \sum_{k=j}^{n+j} \frac{1}{k+1} \left| \sum_{i=0}^{k-j} (-1)^i \binom{n}{i} \right| \\ &= \frac{1}{j+1} \binom{n-1}{0} + \frac{1}{j+2} \binom{n-1}{1} + \dots + \frac{1}{j+n} \binom{n-1}{n-1}. \end{aligned}$$

So by Lemma 2.3

$$\|\Delta^{(n)}\|_{C_1^1} = \sup_j u_j = u_0 = \frac{2^n - 1}{n}. \quad \square$$

Consider the Hilbert matrix  $H = (h_{j,k})$  whose entries are  $h_{j,k} = \frac{1}{j+k+1}$  for all  $j, k \geq 0$ . For the next theorem, we need the definition of  $\beta$  function and the following lemma

$$\beta(m, n) = \int_0^1 z^{m-1} (1 - z)^{n-1} dz,$$

where  $m, n \in \mathbb{N}$ .

LEMMA 2.5. For  $n \in \mathbb{N}$

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \frac{1}{j+m} = \int_0^1 z^{m-1} (1-z)^n dz = \beta(m, n+1).$$

*Proof.* By using identity

$$(1-z)^n = \sum_{j=0}^n (-1)^j \binom{n}{j} z^j,$$

and multiplying both sides in term  $z^{m-1}$  also integrating from 0 to 1, we get the result.  $\square$

THEOREM 2.6. If  $H$  is the Hilbert matrix, then  $\Delta^{(n)}$  is a bounded operator from  $l_1$  into  $H_1$  and

$$\|\Delta^{(n)}\|_{H_1} = \frac{1}{n}.$$

*Proof.* By the notation of Theorem 2.1 and Lemma 2.5

$$\begin{aligned} u_j &= \sum_{k=0}^{\infty} \left| (H\Delta^{(n)})_{k,j} \right| = \sum_{k=0}^{\infty} \left| \sum_{i=j}^{n+j} h_{k,i} \delta_{i,j}^{(n)} \right| \\ &= \sum_{k=0}^{\infty} \sum_{i=j}^{n+j} (-1)^{i-j} \binom{n}{i-j} \frac{1}{k+i+1} = \sum_{k=0}^{\infty} \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{1}{k+i+j+1} \\ &= \sum_{k=0}^{\infty} \int_0^1 z^{k+j} (1-z)^n dz = \int_0^1 \sum_{k=0}^{\infty} z^{k+j} (1-z)^n dz \\ &= \int_0^1 \frac{z^j}{1-z} (1-z)^n dz = \beta(j+1, n), \end{aligned}$$

hence  $u_j = \beta(j+1, n)$ . Since the function  $\beta(j, n)$  is decreasing with respect to  $j$  for all  $n$ , so

$$\|\Delta^{(n)}\|_{H_1} = \sup_j u_j = u_0 = \beta(1, n) = \frac{1}{n}. \quad \square$$

THEOREM 2.7. The backward difference operator  $\Delta^{(n)}$  is a bounded operator from  $l_1$  into  $l_1$  and

$$\|\Delta^{(n)}\|_1 = 2^n.$$

*Proof.* According to notations of Theorem 2.1 for identity matrix, we obtain

$$u_j = \sum_{k=j}^{n+j} \left| (-1)^{k-j} \binom{n}{k-j} \right| = \sum_{k=0}^n \binom{n}{k} = 2^n.$$

So

$$\|\Delta^{(n)}\|_1 = \sup_j u_j = 2^n. \quad \square$$

**THEOREM 2.8.** *The difference operator  $\Delta^{(n)}$  is a bounded operator from  $l_1$  into  $\Delta_1^{(m)}$  and*

$$\|\Delta^{(n)}\|_{\Delta_1^{(m)}} = 2^{n+m}.$$

*Proof.* It is easy, so we omit the proof.  $\square$

### 3. Upper bounds of the operator $\Delta^{(n)}$ from $l_p$ into $A_p$

The purpose of this section is to find the norm of operator  $\Delta^{(n)}$  from  $l_p$  space into  $C_p^n$  and  $H_p$  spaces. To do this, we need the Schur’s Theorem and a lemma which are essential in the study.

**THEOREM 3.1.** ([9], Theorem 275) *Let  $p > 1$  and  $T = (t_{m,k})$  be a matrix operator with  $t_{m,k} \geq 0$  for all  $m, k$ . Suppose that  $K, R$  are two strictly positive numbers such that*

$$\sum_{m=0}^{\infty} t_{m,k} \leq K \quad \text{for all } k, \quad \sum_{k=0}^{\infty} t_{m,k} \leq R \quad \text{for all } m,$$

(bounds for column and row sums respectively). Then

$$\|T\|_p \leq R^{1/p^*} K^{1/p},$$

where  $p^*$  is the conjugate of  $p$  i.e.  $\frac{1}{p} + \frac{1}{p^*} = 1$ .

The above theorem is known as Schur’s theorem.

**LEMMA 3.2.** *Let  $p \geq 1$  and  $A = (a_{m,k})$  be a matrix and  $T = (t_{m,k})$  be a lower triangular matrix. If  $AT$  is a bounded operator on  $l_p$ , then  $T$  will be a bounded operator from  $l_p$  into  $A_p$  and*

$$\|T\|_{A_p} = \|AT\|_p.$$

*Proof.* For every  $x \in l_p$ ,

$$\begin{aligned} \|Tx\|_{A_p}^p &= \sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} \sum_{i=0}^j a_{k,j} t_{j,i} x_i \right|^p = \sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} a_{k,i} t_{i,j} x_j \right|^p \\ &= \sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} (AT)_{k,j} x_j \right|^p = \|ATx\|_p^p, \end{aligned}$$

so the proof is finished.  $\square$

Let the sequences  $a^n = (a_j^n)_{j=0}^\infty$  be defined by

$$a_j^n = \binom{n+j-1}{j}, \tag{3.1}$$

these sequences play an essential role in this study, hence we bring the first four of these sequences in below:

$$\begin{aligned} a^0 &: 1 \ 0 \ 0 \ 0 \ \dots, \\ a^1 &: 1 \ 1 \ 1 \ 1 \ \dots, \\ a^2 &: 1 \ 2 \ 3 \ 4 \ \dots, \\ a^3 &: 1 \ 3 \ 6 \ 10 \ \dots. \end{aligned}$$

One can note that the relation  $a_j^{n+1} = \sum_{k=0}^j a_k^n$  is hold for these sequences which is stated in the following lemma.

LEMMA 3.3. *We have*

$$\sum_{k=0}^j \binom{n+j-k-1}{j-k} = \sum_{k=0}^j \binom{n+k-1}{k} = \binom{n+j}{j}.$$

*Proof.* The proof is obvious.  $\square$

Also the next useful lemma shows that the above sequences  $a^n$  are the coefficients of the binomial  $(1-z)^{-n}$ .

LEMMA 3.4. *For  $|z| < 1$ , we have*

$$(1-z)^{-n} = \sum_{j=0}^\infty a_j^n z^j = \sum_{j=0}^\infty \binom{n+j-1}{j} z^j.$$

*Proof.* By differentiating  $n-1$  times of the identity  $(1-z)^{-1} = \sum_{j=0}^\infty z^j$ , we get the result.  $\square$

If  $(a_k)$  is a non-negative sequence with  $a_0 > 0$  and  $A_j = a_0 + a_1 + \dots + a_j$ , the Nörlund matrix  $N_a = (a_{j,k})$  is defined as follows:

$$a_{j,k} = \begin{cases} \frac{a_{j-k}}{A_j} & 0 \leq k \leq j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

The Cesàro matrix of order  $n$ ,  $C^n = (c_{j,k}^n)$ , is the Nörlund matrix  $N_{a^n}$  with the sequence  $a^n$  as in (3.1). So

$$c_{j,k}^n = \begin{cases} \frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}} & 0 \leq k \leq j \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Note that  $C^0$  and  $C^1$  are the well known identity and Cesàro matrices, respectively.

Hardy in [8] has proved that  $C^n$  is a bounded operator on  $l_p$  and

$$\|C^n\|_p = \frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)}, \tag{3.4}$$

where  $p > 1$ . In particular for  $n = 1$ ,  $\|C^1\|_p = p^*$ .

The sequence space associated with  $C^n$ , according to relation (1.1), will be called  $C_p^n$ . So

$$C_p^n = \left\{ x = (x_k) : \sum_{k=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{j=0}^k \binom{n+j-k-1}{j-k} x_j \right|^p < \infty \right\}.$$

For proving our main theorem in this section, we need the following combinatoric lemma.

LEMMA 3.5. For  $j = 1, 2, \dots$ , we have

$$\sum_{k=0}^j (-1)^k \binom{n}{k} \binom{n+j-k-1}{j-k} = 0.$$

*Proof.* By using Lemma 3.4 and the following identity

$$\begin{aligned} 1 &= (1-z)^n(1-z)^{-n} \\ &= \sum_{j=0}^n (-1)^j \binom{n}{j} z^j \sum_{j=0}^{\infty} \binom{n+j-1}{j} z^j \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^j (-1)^k \binom{n}{k} \binom{n+j-k-1}{j-k} z^j \\ &= 1 + \sum_{j=1}^{\infty} \sum_{k=0}^j (-1)^k \binom{n}{k} \binom{n+j-k-1}{j-k} z^j, \end{aligned}$$

the result is obvious.  $\square$

THEOREM 3.6. Suppose that  $p > 1$  and  $C^n$  is the Cesàro matrix of order  $n$ . Then  $\Delta^{(n)}$  is a bounded operator from  $l_p$  into  $C_p^n$  and

$$\|\Delta^{(n)}\|_{C_p^n} = 1.$$

*Proof.* If  $B = C^n \Delta^{(n)}$ , we have

$$\begin{aligned} b_{t,s} &= (C^n \Delta^{(n)})_{t,s} = \sum_{j=s}^t C_{t,j}^n \Delta_{j,s}^{(n)} \\ &= \frac{1}{\binom{t+n}{t}} \sum_{j=s}^t (-1)^{j-s} \binom{n}{j-s} \binom{n+t-j-1}{t-j} \\ &= \frac{1}{\binom{t+n}{t}} \sum_{j=0}^{t-s} (-1)^j \binom{n}{j} \binom{n+t-j-s-1}{t-j-s}. \end{aligned}$$



Now if  $t = s$ , then  $t - s = 0$  hence  $j = 0$  and  $b_{s,t} = \frac{1}{\binom{t+n}{t}}$ . Also if  $t > s$ , then  $u = t - s \geq 1$  hence by using Lemma 3.5

$$b_{s,t} = \frac{1}{\binom{t+n}{t}} \sum_{j=0}^u (-1)^j \binom{n}{j} \binom{n+u-j-1}{u-j} = 0.$$

So

$$b_{s,t} = \begin{cases} \frac{1}{\binom{t+n}{t}} & s = t \\ 0 & s \neq t, \end{cases}$$

and by applying Lemma 3.2, we have  $\|\Delta^{(n)}\|_{C_p^n} = \|B\|_p$ . Since by Theorem 3.1 for  $B$ ,  $R \leq 1$  and  $C \leq 1$ , we obtain  $\|\Delta^{(n)}\|_{C_p^n} \leq 1$ . Now let  $x = e_1$ , we have  $\|e_1\|_p = 1$  and  $\|\Delta^{(n)}e_1\|_{C_p^n} = 1$ , so  $\|\Delta^{(n)}\|_{C_p^n} = 1$ .  $\square$

#### 4. Upper bounds of the operator $\Delta^{(-n)}$ from $l_p$ into $A_p$

In this section, we introduce four type of sequence spaces  $H_p^n$ ,  $E_p^n$ ,  $D_p^{[m,n]}$  and  $B_p^n$  and will find the norm of operator  $\Delta^{(-n)}$  from  $l_p$  into these spaces.

**THEOREM 4.1.** ([9], Theorem 323) *Let  $p > 1$  and  $H$  be the Hilbert matrix. Then  $H$  is a bounded operator on  $l_p$  and*

$$\|H\|_p = \pi \operatorname{csc}(\pi/p).$$

**THEOREM 4.2.** *Suppose that  $p > 1$  and the matrix  $H^n$  is defined by*

$$h_{j,k}^n = \frac{n!}{\prod_{i=0}^n (j+k+1+i)}, \quad (\text{for } j, k = 0, 1, \dots). \tag{4.1}$$

Then  $\Delta^{(-n)}$  is a bounded operator from  $l_p$  into  $H_p^n$  and

$$\|\Delta^{(-n)}\|_{H_p^n} = \pi \operatorname{csc}(\pi/p).$$

*Proof.* According to Lemma 3.2 and Theorem 4.1, it is sufficient to prove  $H^n \Delta^{(-n)} = H$ . By Lemma 3.4, we have

$$\begin{aligned} (H^n \Delta^{(-n)})_{k,m} &= \sum_{j=m}^{\infty} h_{k,j}^n \delta_{j,m}^{(-n)} \\ &= \sum_{j=m}^{\infty} \binom{n+j-m-1}{j-m} \frac{n!}{(j+k+1)(j+k+2)\cdots(j+k+n+1)} \\ &= \sum_{j=0}^{\infty} \binom{n+j-1}{j} \frac{(j+k+m)!n!}{(j+m+k+n+1)!} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=0}^{\infty} \binom{n+j-1}{j} \beta(j+m+k+1, n+1) \\
 &= \sum_{j=0}^{\infty} \binom{n+j-1}{j} \int_0^1 z^{j+m+k} (1-z)^n dz \\
 &= \int_0^1 \sum_{j=0}^{\infty} \binom{n+j-1}{j} z^j z^{m+k} (1-z)^n dz \\
 &= \int_0^1 (1-z)^{-n} z^{m+k} (1-z)^n dz \\
 &= \frac{1}{k+m+1} = h_{k,m}. \quad \square
 \end{aligned}$$

Note that for  $n = 0$ , we have  $H^n = H$ .

In the following, we define matrix  $E^n = (e^n_{j,k})$  by

$$e^n_{j,k} = \begin{cases} \frac{(-1)^{j-k}}{j+1} \binom{n-1}{j-k} & k \leq j \leq k+n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

For computing the norm of operator  $\Delta^{(-n)}$  from  $l_p$  into sequence space  $E^n_p$ , we need the following lemma.

LEMMA 4.3. For  $j = 0, 1, 2, \dots$  and  $n \in \mathbb{N}$ , we have

$$\sum_{k=0}^j (-1)^{j-k} \binom{n-1}{j-k} \binom{n+k-1}{k} = 1.$$

*Proof.* Let  $|z| < 1$ . By using identity

$$\begin{aligned}
 \sum_{j=0}^{\infty} z^j &= (1-z)^{-1} = (1-z)^{-n} (1-z)^{n-1} \\
 &= \sum_{j=0}^{\infty} \binom{n+j-1}{j} z^j \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} z^j \\
 &= \sum_{j=0}^{\infty} \sum_{k=0}^j (-1)^{j-k} \binom{n-1}{j-k} \binom{n+k-1}{k} z^j,
 \end{aligned}$$

the proof is trivial.  $\square$

THEOREM 4.4. Let the matrix  $E^n$  be defined as in (4.2). Then  $\Delta^{(-n)}$  is a bounded operator from  $l_p$  into  $E^n_p$  and  $\|\Delta^{(-n)}\|_{E^n_p} = p^*$ .

*Proof.* It is sufficient to prove  $E^n \Delta^{(-n)} = C^1$  where  $C^1$  is the Cesàro matrix. By Lemma 4.3

$$\begin{aligned} (E^n \Delta^{(-n)})_{j,m} &= \sum_{k=m}^j e_{j,k}^n \delta_{k,m}^{(-n)} \\ &= \frac{1}{j+1} \sum_{k=m}^j (-1)^{j-k} \binom{n-1}{j-k} \binom{n+k-m-1}{k-m} \\ &= \frac{1}{j+1} \sum_{k=0}^{j-m} (-1)^{j-m-k} \binom{n-1}{j-m-k} \binom{n+k-1}{k} \\ &= \frac{1}{j+1} = C_{j,m}^1. \end{aligned}$$

Hence from Lemma 3.2 and relation 3.4, we conclude the results.  $\square$

By the sequences  $a^n$  in relation (3.1), we define a lower triangular matrix  $D^{(m,n)} = (d_{j,k})$  with

$$\begin{aligned} d_{j,k} &= \begin{cases} \frac{a_{j-k}^m}{a_j^{n+m+1}} & 0 \leq k \leq j, \\ 0 & k > j, \end{cases} \\ &= \begin{cases} \frac{\binom{m+j-k-1}{j-k}}{\binom{n+m+j}{j}} & 0 \leq k \leq j, \\ 0 & k > j, \end{cases} \end{aligned} \quad (4.3)$$

where  $m$  and  $n$  are non-negative integers.

Note that for  $m = n = 0$ ,  $D^{[0,0]}$  is the identity matrix, for  $m = 1$ ,  $n = 0$ ,  $D^{[1,0]}$  is the Cesàro matrix, and also for  $m = 0, 1$  and  $n = 2$ , we have

$$D^{[0,2]} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & 0 & \cdots \\ 0 & 0 & \frac{1}{6} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad D^{[1,2]} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (4.4)$$

In sequel, we need the following lemma.

LEMMA 4.5. For non-negative integers  $n$ ,  $m$  and  $j$ , we have

$$\sum_{k=0}^j \binom{n+k-1}{k} \binom{m+j-k-1}{j-k} = \binom{n+m+j-1}{j}.$$

*Proof.* Let  $|z| < 1$ . From the identities

$$\begin{aligned} (1-z)^{-n} (1-z)^{-m} &= \sum_{j=0}^{\infty} \binom{n+j-1}{j} z^j \sum_{j=0}^{\infty} \binom{m+j-1}{j} z^j \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^j \binom{n+k-1}{k} \binom{m+j-k-1}{j-k} z^j, \end{aligned}$$

and

$$(1 - z)^{-n}(1 - z)^{-m} = (1 - z)^{-(n+m)} = \sum_{j=0}^{\infty} \binom{n+m+j-1}{j} z^j,$$

we obtain the claim.  $\square$

Now, we are ready to obtain the norm of operator  $\Delta^{(-n)}$  from  $l_p$  into  $D_p^{[m,n]}$ .

**THEOREM 4.6.** *Suppose that  $D^{[m,n]}$  is defined as in (4.3). Then  $\Delta^{(-n)}$  is a bounded operator from  $l_p$  into  $D_p^{[m,n]}$  and*

$$\|\Delta^{(-n)}\|_{D_p^{[m,n]}} = \frac{\Gamma(n+m+1)\Gamma(1/p^*)}{\Gamma(n+m+1/p^*)}.$$

In particular,  $\|I\|_{D_p^{[1,0]}} = p^*$ .

*Proof.* By using Lemma 4.5

$$\begin{aligned} (D^{[m,n]}\Delta^{(-n)})_{j,i} &= \sum_{k=i}^j d_{j,k} \delta_{k,i}^{(-n)} \\ &= \frac{1}{\binom{n+m+j}{j}} \sum_{k=i}^j \binom{n+k-i-1}{k-i} \binom{m+j-k-1}{j-k} \\ &= \frac{1}{\binom{n+m+j}{j}} \sum_{k=0}^{j-i} \binom{n+k-1}{k} \binom{m+j-i-k-1}{j-i-k} \\ &= \frac{\binom{n+m+j-i-1}{j-i}}{\binom{n+m+j}{j}} = C_{j,i}^{n+m}, \end{aligned}$$

so  $D^{[m,n]}\Delta^{(-n)} = C^{n+m}$ . Now by applying Lemma 3.2 and relation 3.4, we deduce the result.  $\square$

**EXAMPLE 4.7.** For both matrices in relation (4.4), we have

$$\|\Delta^{-2}\|_{D_p^{[0,2]}} = \frac{2p^{*2}}{p^*+1}, \quad \|\Delta^{-2}\|_{D_p^{[1,2]}} = \frac{6p^{*3}}{(2p^*+1)(p^*+1)}.$$

Bennett in Theorem 11.5 from [4] investigated the following inequality

$$\|x\|_{C_p^1} \leq \|x\|_{H_p} \leq \frac{\pi}{p^*} \csc(\pi/p) \|x\|_{C_p^1}, \tag{4.5}$$

for all  $x \in l_p$ . Similarly, we have the following inequality.

COROLLARY 4.8. *If  $p > 1$ , then*

$$\|x\|_{C_p^1} \leq \|\Delta^{(-n)}x\|_{H_p^n} \leq \frac{\pi}{p^*} \csc(\pi/p) \|x\|_{C_p^1},$$

for all  $x \in l_p$ .

*Proof.* By Lemma 3.2 and Theorem 4.2, we have

$$\|\Delta^{(-n)}x\|_{H_p^n} = \|H^n \Delta^{(-n)}x\|_p = \|Hx\|_p = \|x\|_{H_p}.$$

Hence relation (4.5) completes the proof.  $\square$

Bennett used the factorization  $H = C^t B$ , to prove the right hand side of relation (4.5), where  $B$  is given by

$$B_{j,k} = \frac{(j+1)}{(j+k+1)(j+k+2)}, \tag{4.6}$$

and  $C^t$  is the Copson matrix.

THEOREM 4.9. ([5], Proposition 2) *If  $p > 1$  and the matrix  $B$  is defined by (4.6), then  $B$  is a bounded operator on  $l_p$  and*

$$\|B\|_p = \frac{\pi}{p} \csc(\pi/p^*).$$

For  $H^n$  which is defined as in (4.1), we have a similar factorization of the form  $H^n = C^t B^n$ , where  $B^n$  is given by

$$b_{j,k}^n = \frac{(n+1)!(j+k)!(j+1)}{(j+k+n+2)!}. \tag{4.7}$$

If  $n = 0$  in (4.7), then  $B^n = B$ .

THEOREM 4.10. *Suppose that  $p > 1$  and the matrix  $B^n$  is defined by (4.7). Then  $\Delta^{(-n)}$  is a bounded operator from  $l_p$  into  $B_p^n$  and*

$$\|\Delta^{(-n)}\|_{B_p^n} = \frac{\pi}{p} \csc(\pi/p^*).$$

*Proof.* According to Lemma 3.2 and Theorem 4.9, it is sufficient to prove  $B^n \Delta^{(-n)} = B$ .

$$\begin{aligned} (B^n \Delta^{(-n)})_{k,m} &= \sum_{j=m}^{\infty} b_{k,j}^n \delta_{j,m}^{(-n)} = \sum_{j=m}^{\infty} \binom{n+j-m-1}{j-m} \frac{(n+1)!(j+k)!(k+1)}{(j+k+n+2)!} \\ &= (k+1) \sum_{j=0}^{\infty} \binom{n+j-1}{j} \beta(j+m+k+1, n+2) \end{aligned}$$

$$\begin{aligned}
&= (k+1) \sum_{j=0}^{\infty} \binom{n+j-1}{j} \int_0^1 z^{j+m+k} (1-z)^{n+1} dz \\
&= (k+1) \int_0^1 (1-z)^{-n} z^{m+k} (1-z)^{n+1} dz \\
&= \frac{(k+1)}{(m+k+1)(m+k+2)} = b_{k,m}. \quad \square
\end{aligned}$$

## REFERENCES

- [1] C. AYDIN AND F. BAŞAR, *Some topological and geometric properties of the domain of the generalized difference matrix  $B(r; s)$  in the sequence space  $l(p)$* , Thai J. Math. **12** (1) (2014), 113–132.
- [2] P. BALIARSINGH, *Some new difference sequence spaces of fractional order and their dual spaces*, Appl. Math. Comput. **219** (18) (2013), 9737–9742.
- [3] F. BAŞAR, *Summability Theory and Its Applications*, Bentham Science Publishers, e-books, Monographs, İstanbul, 2012.
- [4] G. BENNETT, *Factorizing the classical inequalities*, Mem. Amer. Math. Soc. **576** (1996), 1–130.
- [5] G. BENNETT, *Lower bounds for matrices*, Linear Algebra Appl. **82** (1986), 81–98.
- [6] D. BORWEIN, *Nörlund operators on  $l_p$* , Canad. Math. Bull. **36** (1993), 8–14.
- [7] D. BORWEIN AND F. P. CASS, *Nörlund matrices as bounded operators on  $l_p$* , Arch. Math. **42** (1984), 464–469.
- [8] G. H. HARDY, *An inequality for Hausdorff means*, J. London Math. Soc. **18** (1943), 46–50.
- [9] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities*, 2nd edition, Cambridge University press, Cambridge, 2001.
- [10] U. KADAK, *Generalized weighted invariant mean based on fractional difference operator with applications to approximation theorems for functions of two variables*, Results Math. **72** (3) (2017), 1181–1202.
- [11] U. KADAK AND P. BALIARSINGH, *On certain Euler difference sequence spaces of fractional order and related dual properties*, J. Nonlinear Sci. Appl. **8** (2015), 997–1004.
- [12] M. KIRIŞCI AND F. BAŞAR, *Some new sequence spaces derived by the domain of generalized difference matrix*, Comput. Math. Appl. **60** (5) (2010), 1299–1309.
- [13] M. KIRIŞCI AND U. KADAK, *The method of almost convergence with operator of the form fractional order and applications*, J. Nonlinear Sci. Appl. **10** (2017), 828–824.
- [14] H. KIZMAZ, *On certain sequence spaces I*, Canad. Math. Bull. **25** (2) (1981), 169–176.
- [15] H. KIZMAZ, *On certain sequence spaces II*, Internat. J. Math. Math. Sci. **18** (2) (1995), 721–724.
- [16] E. MALKOWSKY AND V. RAKOČEVIĆ, *An introduction into the theory of sequence spaces and measures of non-compactness*, Zbornik Radova, Matematički Institut SANU, Belgrade, **9** (17) (2000), 143–243.
- [17] H. ROOPAEI AND D. FOROUTANNIA, *The norm of matrix operators on Cesàro weighted sequence*, Linear and Multilinear Algebra, 2018, doi:10.1080/03081087.2017.1415293.

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