Operators and Matrices
Volume 12, Number 4 (2018), 903–931 doi:10.7153/oam-2018-12-55

MULTILINEAR MIXING OPERATORS AND LIPSCHITZ MIXING OPERATOR IDEALS

DAHMANE ACHOUR, ELHADJ DAHIA AND M. A. S. SALEH

(*Communicated by N.-C. Wong*)

Abstract. In [15], E. A. Sánchez Pérez introduced the class of $(s;q,\theta)$-mixing operators, as a generalization of $(s;q)$-mixing operators. We investigate analogous concepts here for the case of multilinear operators between Banach spaces and Lipschitz mappings between metric spaces, introducing the class of $(s,q;p_1,\ldots,p_m;\theta)$-mixing multilinear operators and the Lipschitz Banach ideal of (s,q,θ)-mixing mappings show that our approach provides a multilinear and Lipschitz extension of quotient theorem like the linear case. Several characterizations of these mappings are presented, especially, every Lipschitz (s,q)-mixing map is Lipschitz (s,q,θ)-mixing map and a result relies on the duality theory for (q,θ)-absolutely Lipschitz operators are given.

1. Introduction

Jarchow and Matter introduced in 1987 a general interpolation procedure for creating a new operator ideal from two given operator ideals (see [9]). Using this technique, Matter defined the operator ideal $\Pi_{q,\theta}$ of the (q,θ)-absolutely continuous linear operators where $1 \leqslant q < \infty$ and $0 \leqslant \theta < 1$. The resulting space must be understood as an ideally located in between absolutely q-summing linear operators and continuous linear operators, preserving some of the characteristic properties of the first class. The class of $(q;p_1,\ldots,p_m;\theta)$-absolutely continuous multilinear operators on Banach spaces has been defined and characterized by Dahia et al. in [8] as a natural multilinear extension of the ideal of (q,θ)-absolutely continuous linear operators for which the resulting vector space $\mathscr{L}^{\theta}_{as,(q;p_1,\ldots,p_m)}$ is a normed (Banach) multi-ideal. In [15], Sánchez Pérez introduced the interpolated operator ideal of (s,q,θ)-mixing linear operators that generalize the well known operator ideal of (q,p)-mixing operators [12].

This class of operators is characterized by interesting integral inequalities and by a certain splitting property (which explains the name "mixing"). More details on the (s,q,θ)-mixing operators can be found also in [16]. In this paper, we introduce and study the multilinear version of (s,q,θ)-mixing linear operators, that will be called $(s,q;p_1,\ldots,p_m;\theta)$-mixing multilinear operators. As far as we know that is a first attempt in this regard. We give some characterizations, for this class, by integral inequalities similar to linear case and we prove the quotient theorem for this class.

Mathematics subject classification (2010): Primary 47B10, Secondary 47L22, 46A32.
Keywords and phrases: Lipschitz operators, Lipschitz (s,q)-mixing map, (s,θ)-absolutely Lipschitz mappings, $(s;q,\theta)$-mixing operators, mixing multilinear operators, quotient theorem.

© Element, Zagreb

Also, we introduce the new space of a Lipschitz mixed $(s;q,\theta)$-summable sequences in pointed metric spaces obtaining in this the way a new Banach Lipschitz operator ideal of Lipschitz $(s;q,\theta)$-mixing mappings. The paper is structured into five sections. After the introductory one, in Section 2 we recall some notations, basic facts on sequence spaces and several basic definitions of some classes of linear operators between Banach spaces. In section 3, we extend to multilinear mappings the concept of $(s;q,\theta)$-mixing linear operators, and we show that our approach provides a multilinear extension of a quotient theorem for the linear case. In section 4, based on the definition of mixed $(s;q,\theta)$-summable sequences [15] and inspired by the Lipschitz version of mixed summable sequences [5, Section 4.2], we present the definition of Lipschitz mixed $(s;q,\theta)$-summable sequence of arbitrary pointed metric spaces and study its fundamental properties. We will have opportunities to use this space of sequences for introducing the classes of Lipschitz $(s;q,\theta)$-mixing maps and we characterize this class of non-linear operators by means of their summability properties. Composition property and some including results are given. In section 5, we establish the quotient Lipschitz theorem for the classes of $(s;q,\theta)$-mixing maps. As a consequence, we obtain that this class is a strong Lipschitz operator Banach ideal. Afterwards we prove that every Lipschitz (s,q)-mixing map is Lipschitz $(s;q,\theta)$-mixing. We end the paper with the characterization of Lipschitz $(s;q,\theta)$-mixing maps between metric spaces in terms of ideal norms of associated bounded linear operators between Chevet-Safer spaces.

2. Notations and preliminaries

The notations used in the paper are, in general, standard. Let $m \in \mathbb{N}$ and E_i, $(i = 1,\ldots,m)$, F be Banach spaces over $\mathbb{K}$, either $\mathbb{R}$ or $\mathbb{C}$ we will denote by $\mathscr{L}(E_1,\ldots,E_m; F)$ the Banach space of all continuous m-linear mappings from $E_1 \times \ldots \times E_m$ into F, under the norm

$$\|T\| = \sup_{x^i \in B_{E_i},\ 1 \leqslant i \leqslant m} \left\|T(x^1,\ldots,x^m)\right\|,$$

where B_{E_i} denotes the closed unit ball of $E_i (1 \leqslant i \leqslant m)$. Let now E be a Banach space and $1 \leqslant p \leqslant \infty$. The symbol $E^{\mathbb{N}}$ will denote the sequences with values in E. Let $\ell_p(E)$ be the Banach space of all absolutely p-summable sequences $x = (x_j)_j \in E^{\mathbb{N}}$ with the norm

$$\left\|x \middle| \ell_p(E)\right\| = (\sum_{j=1}^{\infty} \|x_j\|^p)^{\frac{1}{p}}.$$

We denote by $\ell_p^{\omega}(E)$ the Banach space of all weakly p-summable sequences $x = (x_j)_j$ in E with the norm

$$\left\|x \middle| \ell_p^{\omega}(E)\right\| = \sup_{\|\xi\|_{E^*} \leqslant 1} \left\|(\xi(x_j))_j \middle| \ell_p\right\|,$$

where E^* denotes the topological dual of E. If $p = \infty$ we are restricted to the case of bounded sequences in $\ell_\infty(E)$ we use the sup norm. If we take $E = \mathbb{K}$, then the spaces $\ell_p(\mathbb{K})$ and $\ell_p^{\omega}(\mathbb{K})$ coincide and we denote $\ell_p(\mathbb{K})$ by ℓ_p. If $1 < q \leqslant s \leqslant \infty$, we consider the real number $s^*(q)$ satisfying $\frac{1}{s^*(q)} + \frac{1}{s} = \frac{1}{q}$. A sequence $x = (x_j)_j$ of elements of

E is said to be mixed $(s;q)$-summable if there exists a sequence $\zeta = (\zeta_j)_j \in \ell_{s^*(q)}$ and a sequence $x^0 = (x_j^0)_j \in \ell_s^\omega(E)$ such that for all $j \in \mathbb{N}$ we have

$$x_j = \zeta_j \cdot x_j^0 \tag{2.1}$$

We denote by $\ell_{(s;q)}^m(E)$ the Banach space of all mixed $(s;q)$-summable sequences of elements of E with the norm

$$\mathfrak{m}_{(s;q)}(x) = \inf \left\| \zeta \middle| \ell_{s^*(q)} \right\| \left\| x^0 \middle| \ell_s^\omega(E) \right\|,$$

where the infimum is taken over all possible representations of x in the form (2.1). The relationships between the various sequence spaces are given by $\ell_q(E) \subset \ell_{(s;q)}^m(E) \subset \ell_q^\omega(E)$, with $\left\| x \middle| \ell_q^\omega(E) \right\| \leqslant \mathfrak{m}_{(s;q)}(x) \leqslant \left\| x \middle| \ell_q(E) \right\|$, for all $x \in \ell_q(E)$. We denote by $W(B_{E^*})$, the set of all regular Borel probability measures on B_{E^*} (with the weak star topology).

Let $1 < q \leqslant s \leqslant \infty$. An operator $u : E \longrightarrow F$ between Banach spaces is called (s,q)-mixing if there is a constant $C \geqslant 0$ such that for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in E$, the inequality

$$\mathfrak{m}_{(s;q)}((ux_j)_j) \leqslant C \cdot \left\| (x_j)_j \middle| \ell_q^\omega(E) \right\|$$

holds. The class of all (s,q)-mixing operators from E to F is denoted by $\mathfrak{M}_{(s;q)}(E,F)$. In this case, the (s,q)-mixing summing norm $\mathbf{M}_{(s;q)}(u)$ of u is the infimum of such constants C (see [12]). We recall the multilinear extension of the concept of (s,q)-mixing operators were introduced by C. A. S. Soares in [18].

DEFINITION 2.1. Let $0 < q \leqslant s < \infty$ and $0 < p_1, \ldots, p_m < \infty$ with $\frac{1}{q} \leqslant \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. An m-linear operator $T \in \mathscr{L}(E_1, \ldots, E_m; F)$ is $(s,q;p_1,\ldots,p_m)$-mixing if there exists a constant $C > 0$ such that

$$\mathfrak{m}_{(s,q)}((T(x_j^1, \ldots, x_j^m))_{j=1}^n) \leqslant C \prod_{i=1}^m \left\| (x_j^i)_{j=1}^n \middle| \ell_{p_i}^\omega(X) \right\| \tag{2.2}$$

for every $n \in \mathbb{N}$, $(x_j^i)_{j=1}^n \subset E_i$ $(i = 1, \ldots, m)$. In this case we define $\|T\|_{(s,q;p_1,\ldots,p_m)} = \inf\{C : \text{for all } C \text{ verifying the inequality}(2.2)\}$. We denote the class of all such mappings by $\mathscr{L}_{mx(s,q;p_1,\ldots,p_m)}(E_1, \ldots, E_m; F)$.

Now we recall the definition of (q,θ)-absolutely continuous linear operators by means of sequences (see [11, 10]). Let $1 \leqslant q < \infty$ and $0 \leqslant \theta < 1$. For $x = (x_j)_j \in E^{\mathbb{N}}$ we put

$$\delta_{q\theta}(x) = \sup_{\xi \in B_{E^*}} \left(\sum_{j=1}^{\infty} \left(|\xi(x_j)|^{1-\theta} \|x_j\|^\theta \right)^{\frac{p}{1-\theta}} \right)^{\frac{1-\theta}{p}}.$$

It is clear that if $\delta_{q\theta}(x) < \infty$ then $x \in \ell_{\frac{q}{1-\theta}}^\omega(E)$ with $\left\| x \middle| \ell_{\frac{q}{1-\theta}}^\omega(E) \right\| \leqslant \delta_{q\theta}(x)$.

DEFINITION 2.2. An operator $u : E \longrightarrow F$ between Banach spaces is (q,θ)-absolutely continuous, if there exists a constant $C \geqslant 0$ such that for any $(x_j)_{j=1}^n \subset E$, we have

$$\left\| (ux_j)_{j=1}^n \middle| \ell_{\frac{q}{1-\theta}}(E) \right\| \leqslant C \cdot \delta_{q\theta}((x_j)_{j=1}^n).$$

This class is denoted by $\Pi_{q,\theta}(E,F)$ and the infimum of all C by $\pi_{q,\theta}(u)$. The notion of $(s;q,\theta)$-mixing linear operators was introduced by E. A. Sánchez Pérez in [15]. Let $0 < q \leqslant s \leqslant \infty$, $0 \leqslant \theta < 1$ and $0 < s^*(q) \leqslant \infty$ such that $\frac{1}{s^*(q)} + \frac{1}{s} = \frac{1}{q}$. For all finite sequences $(x_j)_{j=1}^n \subset E$ we put

$$\mathfrak{m}_{(s;q,\theta)}\left((x_j)_{j=1}^n\right) = \inf\left\{ \left\| (\zeta_j)_{j=1}^n \middle| \ell_{\frac{s^*(q)}{1-\theta}} \right\| \delta_{s\theta}\left((x_j^0)_{j=1}^n\right) : x_j = \zeta_j x_j^0, j = 1,\dots,n \right\}.$$

The infimum is considered for all possible representations $x_j = \zeta_j x_j^0, j = 1,\dots,n$, with $(\zeta_j)_{j=1}^n \subset \mathbb{K}$ and $(x_j^0)_{j=1}^n \subset E$.

DEFINITION 2.3. The operator $u : E \longrightarrow F$, between Banach spaces, is $(s;q,\theta)$-mixing if there exists a constant $C \geqslant 0$ such that

$$\mathfrak{m}_{(s;q,\theta)}\left((ux_j)_{j=1}^n\right) \leqslant C \cdot \delta_{q\theta}((x_j)_{j=1}^n)$$

for every finite sequence $(x_j)_{j=1}^n$ in E. The infimum of all such constants C is represented by $\mathbf{M}_{(s;q)}^{\theta}(u)$.

The next results can be found in [15, Lemma 1.4 and Proposition 1.5]

PROPOSITION 2.4. *For every* $(x_j)_{j=1}^n \subset E$,

$$\mathfrak{m}_{(s;q,\theta)}((x_j)_{j=1}^n) = \sup_{\mu \in W(B_{E^*})} \left(\sum_{j=1}^n \left(\int_{B_{E^*}} \left|\langle x_j, \varphi\rangle\right|^s \left\|x_j\right\|^{\frac{\theta s}{1-\theta}} d\mu(\varphi) \right)^{\frac{q}{s}} \right)^{\frac{1-\theta}{q}}. \tag{2.3}$$

PROPOSITION 2.5. *The following statements are equivalent*

(i) $u : X \to Y$ *is* $(s;q,\theta)$*-mixing*

(ii) There is $C > 0$ *such that for every* $(x_j)_{j=1}^n \subset E$ *and every* $(\varphi_l)_{l=1}^k \subset F^*$*, the following inequality holds*

$$\left[\sum_{j=1}^n \left(\sum_{l=1}^k \left|\langle u(x_j), \varphi_l\rangle\right|^s \left\|ux_j\right\|^{\frac{\theta s}{1-\theta}} \right)^{\frac{q}{s}} \right]^{\frac{1-\theta}{q}} \leqslant C.\delta_{q\theta}\left((x_j)_{j=1}^n\right). \left\| (\varphi_l)_{l=1}^k \middle| \ell_s(F^*) \right\|^{1-\theta}. \tag{2.4}$$

Moreover, we have $\mathbf{M}_{(s;q)}^{\theta}(u) = \inf\{C > 0,\ C \text{ satisfies } (2.4)\}$.

Now we are going to introduce some concepts and notations for the Lipschitz case, let X, Y and Z be pointed metric spaces which have a special point designated by 0.

The set of all Lipschitz functions from X into Y that send the special point 0 to 0 will be denoted by $Lip(X,Y)$. For all $T \in Lip(X,Y)$ we put

$$Lip(T) = \inf\left\{C \geqslant 0 : d(T(x),T(x')) \leqslant Cd(x,x') \text{ for all } x,x' \in X\right\}.$$

The Banach space $Lip(X,\mathbb{R})$ of real-valued Lipschitz functions defined on X with the Lipschitz norm $Lip(\cdot)$ will be denoted by $X^{\#}$. Along with the paper we consider $B_{X^{\#}}$ endowed with the pointwise topology ($B_{X^{\#}}$ is a compact Hausdorff space in this topology). It is well known that $X^{\#}$ has a predual, namely the space of Arens and Eells Æ(X) (see [4] or [19] for more details on this space). A molecule on X is a scalar valued function m on X with finite support that satisfies $\sum_{x\in X} m(x) = 0$. We denote by $\mathscr{M}(X)$ the linear space of all molecules on X. For $x,x' \in X$ the molecule $m_{xx'}$ is defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A. For $m \in \mathscr{M}(X)$ we can write $m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j}$ for some suitable scalars λ_j, and we write $\|m\|_{\mathscr{M}(X)} = \inf\left\{\sum_{j=1}^{n}\left|\lambda_j\right| d(x_j,x'_j),\ m = \sum_{j=1}^{n}\lambda_j m_{x_j x'_j}\right\}$, where the infimum is taken over all representations of the molecule m. Denote by Æ(X) the completion of the normed space $(\mathscr{M}(X),\|.\|_{\mathscr{M}(X)})$. Sawashima [17] defined the Lipschitz dual $T^{\#}$ of $T \in Lip(X,Y)$ as the continuous linear operator $T^{\#} : Y^{\#} \longrightarrow X^{\#}$ given by $T^{\#}(g) := g \circ T$. In contrast to the linear case, where it was enough to consider sequences $(x_j)_j \in E^{\mathbb{N}}$, we require to consider sequences $\left((\sigma_j,x'_j,x''_j)\right)_{j\in\mathbb{N}}$ of triples $(\sigma_j,x'_j,x''_j) \in \mathbb{R}\times X\times X$. To simplify notation, let us write (σ,x',x'') for such a sequence. For a scalar sequence $\tau = (\tau_j)_j \subset \mathbb{R}\backslash\{0\}$ we will simply write $(\frac{\sigma}{\tau},x',x'')$ instead of $\left((\frac{\sigma_j}{\tau_j},x'_j,x''_j)\right)_{j\in\mathbb{N}}$. Let $1 \leqslant p < \infty$, the p-sequence set, denoted by $\ell_p(\mathbb{R}\times X\times X)$, is defined as

$$\ell_p^L(\mathbb{R}\times X\times X) = \left\{(\sigma_j,x',x'') \in \mathbb{R}\times X\times X : \sum_{j=1}^{\infty}\left|\sigma_j\right|^p d_X(x'_j,x''_j)^p < \infty\right\}.$$

We denote its strong p-norm by

$$\left\|(\sigma,x',x'')\middle|\ell_p^L\right\| = \left(\sum_{j=1}^{\infty}\left|\sigma_j\right|^p d_X(x'_j,x''_j)^p\right)^{\frac{1}{p}}.$$

Also the weak Lipschitz p-sequence set, denoted by $\ell_p^{L,\omega}(\mathbb{R}\times X\times X)$, is defined as

$$\ell_p^{L,\omega}(\mathbb{R}\times X\times X) = \left\{(\sigma_j,x',x'') \in \mathbb{R}\times X\times X : \sum_{j=1}^{\infty}\left|\sigma_j\right|^p\left|fx'_j - fx''_j\right|^p < \infty,\ \text{for all } f \in B_{X^{\#}}\right\}.$$

We denote its weak Lipschitz p-norm by

$$\left\|(\sigma,x',x'')\middle|\ell_p^{L,\omega}\right\| = \sup_{f\in B_{X^{\#}}}\left(\sum_{j=1}^{\infty}\left|\sigma_j\right|^p\left|fx'_j - fx''_j\right|^p\right)^{\frac{1}{p}}.$$

Observe that, since there is no linear structure on the set of triples (σ, x', x''), the above notions are not really norms. But because of the similarity with the usual norms of ℓ_p, we shall call them norms. In order to introduce the concept of Lipschitz $(s;q)$-mixing operator. The third author of this paper in, [14, Definition 3.1], defined the space of Lipschitz mixed $(s;q)$-summable sequence which is implicit in [5, Sec. 4.2]. A sequence $(\sigma, x', x'') \subset \mathbb{R} \times X \times X$ is called Lipschitz mixed $(s;q)$-summable if there exists a sequence $\tau \in \ell_{s^*(q)}$ such that $(\frac{\sigma}{\tau}, x', x'') \in \ell_s^{L,\omega}(\mathbb{R} \times X \times X)$. The space of all this sequence is denoted by $\mathfrak{M}_{(s;q)}^L(\mathbb{R} \times X \times X)$. Moreover, for a sequence $(\sigma, x', x'') \in \mathfrak{M}_{(s;q)}^L(\mathbb{R} \times X \times X)$ define

$$\mathfrak{m}_{(s;q)}^L(\sigma, x', x'') = \inf \left\| \tau \middle| \ell_{s^*(q)} \right\| \left\| \left(\frac{\sigma}{\tau}, x', x'' \right) \middle| \ell_s^{L,\omega} \right\|$$

where the infimum is taken over all sequences $\tau \in \ell_{s^*(q)}$.

DEFINITION 2.6. Let $0 < q \leqslant s \leqslant \infty$, a Lipschitz map $T \in Lip(X,Y)$ is called Lipschitz $(s;q)$-mixing operator if there is a constant $C \geqslant 0$ such that

$$\mathfrak{m}_{(s;q)}^L(\sigma, Tx', Tx'') \leqslant C \cdot \left\| (\sigma, x', x'') \middle| \ell_q^{L,\omega} \right\|$$

for arbitrary finite sequences $(\sigma, x', x'') \subset \mathbb{R} \times X \times X$.

Note that there is another definition of Lipschitz $(s;q)$-mixing operators equivalent to the above definition in when we have $1 \leqslant q \leqslant s < \infty$ (see [5, Corollary 4.3]).

3. $(s, q; p_1, \ldots, p_m; \theta)$-mixing multilinear operators

In this section we extend the definition of class of (s, q, θ)-mixing linear operators to the case of multilinear operators. In what follows, we consider the real numbers $0 \leqslant \theta < 1$ and $1 \leqslant q, s, p_1, \ldots, p_m < \infty$ such that $\frac{1}{q} \leqslant \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Before we study this class of multilinear operators, we recall the definition of $(q; p_1, \ldots, p_m; \theta)$-absolutely continuous multilinear operators were introduced by Dahia et al. in [8].

DEFINITION 3.1. A mapping $T \in \mathscr{L}(E_1, \ldots, E_m; F)$ is $(q; p_1, \ldots, p_m; \theta)$-absolutely continuous if there is a constant $C \geqslant 0$ such that for any $x_1^i, \ldots, x_n^i \in E_i$, $(1 \leqslant i \leqslant m)$ we have

$$\left\| \left(T\left(x_j^1, \ldots, x_j^m\right)\right)_{j=1}^n \middle| \ell_{\frac{q}{1-\theta}}(F) \right\| \leqslant C \prod_{i=1}^m \delta_{p_i \theta}((x_j^i)_{j=1}^n).$$

The smallest C satisfying the inequality above is indicated by $\|T\|_{\mathscr{L}^\theta_{as,(q;p_1,\ldots,p_m)}}$ and the class of these mappings by $\mathscr{L}^\theta_{as,(q;p_1,\ldots,p_m)}(E_1, \ldots, E_m; F)$ which is a Banach space with the norm $\|T\|_{\mathscr{L}^\theta_{as,(q;p_1,\ldots,p_m)}}$.

DEFINITION 3.2. A mapping $T \in \mathscr{L}(E_1,\ldots,E_m;F)$ is $(s,q;p_1,\ldots,p_m;\theta)$-mixing if there is a constant $C \geqslant 0$ such that for any $\left(x_j^i\right)_{j=1}^n \subset E_i \;(1 \leqslant i \leqslant m)$ we have

$$\mathfrak{m}_{(s;q,\theta)}((T(x_j^1,\ldots,x_j^m)_{j=1}^n) \leqslant C\prod_{i=1}^m \delta_{p_i\theta}((x_j^i)_{j=1}^n). \tag{3.1}$$

In this case we define

$$\|T\|_{mx(s,q;p_1,\ldots,p_m;\theta)} = \inf\{C : \text{for all } C \text{ verifying the inequality (3.1)}\}.$$

We denote this class of mappings by $\mathscr{L}^{\theta}_{mx(s,q;p_1,\ldots,p_m)}$.

For $\theta = 0$ we have

$$\mathscr{L}^{0}_{mx(s,q;p_1,\ldots,p_m)}(E_1,\ldots,E_m;F) = \mathscr{L}_{mx(s,q;p_1,\ldots,p_m)}(E_1,\ldots,E_m;F).$$

Notice that for $m=1$ we recover the class of linear (s,q,θ)-*mixing* operators. Unfortunately, we have no proof for the fact that the class $\mathscr{L}^{\theta}_{mx(s,q;p_1,\ldots,p_m)}$ is a Banach multi-ideal. However, we do not need in our paper of this result.

In the following we characterize the class of $(s,q;p_1,\ldots,p_m;\theta)$-mixing multilinear operators by means summability inequalities.

THEOREM 3.3. *The mapping $T \in \mathscr{L}(E_1,\ldots,E_m;F)$ is $(s,q;p_1,\ldots,p_m;\theta)$-mixing if and only if there is a constant $C \geqslant 0$ such that*

$$\left[\sum_{j=1}^n \left(\sum_{l=1}^k \left|\left\langle \varphi_l, T(x_j^1,\ldots,x_j^m)\right\rangle\right|^s \left\|T(x_j^1,\ldots,x_j^m)\right\|^{\frac{\theta s}{1-\theta}}\right)^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}} \tag{3.2}$$
$$\leqslant C\cdot\prod_{i=1}^m \delta_{p_i\theta}\left((x_j^i)_{j=1}^n\right)\cdot\left\|(\varphi_l)_{l=1}^k\middle|\ell_s(F^*)\right\|^{1-\theta}$$

for all $n,k \in \mathbb{N}$, $\left(x_j^i\right)_{j=1}^n \subset E_i \;(1 \leqslant i \leqslant m)$ and $(\varphi_l)_{l=1}^k \subset F^*$. In this particular case,

$$\|T\|_{mx(s,q;p_1,\ldots,p_m;\theta)} = \inf C$$

where the infimum is taken over all C satisfying (3.2).

Proof. We have two cases.

(i) *Case* $s=q$. Suppose that the conditions (3.2) is holds and take $k=1$,

$$\left(\sum_{j=1}^n \left|\left\langle \varphi, T(x_j^1,\ldots,x_j^m)\right\rangle\right|^q \left\|T(x_j^1,\ldots,x_j^m)\right\|^{\frac{\theta q}{1-\theta}}\right)^{\frac{1-\theta}{q}} \leqslant C\cdot\prod_{i=1}^m \delta_{p_i\theta}\left((x_j^i)_{j=1}^n\right)$$

for all $\varphi \in B_{F^*}$. Thus

$$\delta_{q\theta}\left((T(x_j^1,\ldots,x_j^m))_{j=1}^n\right) \leqslant C\cdot\prod_{i=1}^m \delta_{p_i\theta}\left((x_j^i)_{j=1}^n\right).$$

If we combine this inequality and (2.3) we obtain

$$\begin{aligned}
&\mathfrak{m}_{(s;q,\theta)}((T(x_j^1,\ldots,x_j^m)_{j=1}^n)\\
&\leqslant \sup_{\mu\in W(B_{F^*})}\left(\int_{B_{F^*}}\sup_{\varphi\in B_{F^*}}\sum_{j=1}^n\left|\left\langle\varphi,T(x_j^1,\ldots,x_j^m)\right\rangle\right|^q\left\|T(x_j^1,\ldots,x_j^m)\right\|^{\frac{\theta q}{1-\theta}}d\mu(\varphi)\right)^{\frac{1-\theta}{q}}\\
&=\delta_{q\theta}\left((T(x_j^1,\ldots,x_j^m))_{j=1}^n\right)\\
&\leqslant C\cdot\prod_{i=1}^m\delta_{p_i\theta}\left((x_j^i)_{j=1}^n\right).
\end{aligned}$$

Therefore, T is $(s,q;p_1,\ldots,p_m;\theta)$-mixing and $\|T\|_{mx(s,q;p_1,\ldots,p_m;\theta)}\leqslant C$. Conversely, suppose that $T\in\mathscr{L}^{\theta}_{mx(s,q;p_1,\ldots,p_m)}(E_1,\ldots,E_m;F)$. Given $(x_j^i)_{j=1}^n\subset E_i,(i=1,\ldots,m)$ and $(\varphi_l)_{l=1}^k\subset F^*$. If $T(x_j^1,\ldots,x_j^m)=\tau_jy_j,(1\leqslant j\leqslant n)$ where $(y_j)_{j=1}^n\subset F$ and $(\tau_j)_{j=1}^n\subset\mathbb{K}$, we have

$$\begin{aligned}
&\left[\sum_{j=1}^n\sum_{l=1}^k\left|\left\langle\varphi_l,T(x_j^1,\ldots,x_j^m)\right\rangle\right|^q\left\|T(x_j^1,\ldots,x_j^m)\right\|^{\frac{\theta q}{1-\theta}}\right]^{\frac{1-\theta}{q}}\\
&=\left[\sum_{j=1}^n\sum_{l=1}^k\left|\left\langle\varphi_l,\tau_jy_j\right\rangle\right|^q\left\|\tau_jy_j\right\|^{\frac{\theta q}{1-\theta}}\right]^{\frac{1-\theta}{q}}\\
&\leqslant\left(\sum_{l=1}^k\|\varphi_l\|^q\right)^{\frac{1-\theta}{q}}\sup_{\varphi\in B_{F^*}}\left[\sum_{j=1}^n\left|\left\langle\varphi,\tau_jy_j\right\rangle\right|^q\left\|\tau_jy_j\right\|^{\frac{\theta q}{1-\theta}}\right]^{\frac{1-\theta}{q}}\\
&\leqslant\left\|(\varphi_l)_{l=1}^k\Big|\ell_q(F^*)\right\|^{1-\theta}\left\|(\tau_j)_{j=1}^n\Big|\ell_\infty\right\|\delta_{q\theta}\left((y_j)_{j=1}^n\right).
\end{aligned}$$

Taking the Infimum on both sides over all possible representations of the form

$$T(x_j^1,\ldots,x_j^m)=\tau_jy_j,\ \ (j=1,\ldots,n)$$

we obtain

$$\begin{aligned}
&\left[\sum_{j=1}^n\sum_{l=1}^k\left|\left\langle\varphi_l,T(x_j^1,\ldots,x_j^m)\right\rangle\right|^q\left\|T(x_j^1,\ldots,x_j^m)\right\|^{\frac{\theta q}{1-\theta}}\right]^{\frac{1-\theta}{q}}\\
&\leqslant\left\|(\varphi_l)_{l=1}^k\Big|\ell_q(F^*)\right\|^{1-\theta}\cdot\mathfrak{m}_{(q;q,\theta)}(T(x_j^1,\ldots,x_j^m))_{j=1}^n)\\
&\leqslant\left\|(\varphi_l)_{l=1}^k\Big|\ell_q(F^*)\right\|^{1-\theta}\cdot\|T\|_{mx(q,q;p_1,\ldots,p_m;\theta)}\cdot\prod_{i=1}^m\delta_{p_i\theta}\left((x_j^i)_{j=1}^n\right).
\end{aligned}$$

Therefore $\inf C\leqslant\|T\|_{(s,q;p_1,\ldots,p_m;\theta)}$.

(ii) *Case* $s>q$. Assume that T is $(s,q;p_1,\ldots,p_m;\theta)$-mixing. Consider $0\neq\varphi_1,\ldots,\varphi_k\in F^*$. We define on B_{F^*} the probability measure $\nu=\sum_{l=1}^k\nu_l\delta_l$, where $\nu_l=$

$\frac{\|\varphi_l\|^s}{\left\|(\varphi_l)_{l=1}^k\middle|\ell_s(F^*)\right\|^s}, l=1,\ldots,k$ and δ_l is the Dirac measure at the point $\tilde{\varphi}_l = \frac{\varphi_l}{\|\varphi_l\|}$. For $(x_j^i)_{j=1}^n \subset E_j, (i=1,\ldots,m)$ we have

$$\begin{aligned}
&\int_{B_{F^*}} \left|\left\langle \varphi, T(x_j^1,\ldots,x_j^m)\right\rangle\right|^s d\nu(\varphi) \\
&= \sum_{l=1}^k \left|\left\langle \frac{\varphi_l}{\|\varphi_l\|}, T(x_j^1,\ldots,x_j^m)\right\rangle\right|^s \frac{\|\varphi_l\|^s}{\left\|(\varphi_l)_{l=1}^k\middle|\ell_s(F^*)\right\|^s} \\
&= \frac{1}{\left\|(\varphi_l)_{l=1}^k\middle|\ell_s(F^*)\right\|^s} \sum_{l=1}^k \left|\left\langle \varphi_l, T(x_j^1,\ldots,x_j^m)\right\rangle\right|^s.
\end{aligned}$$

From this equalities and (2.3) we get

$$\begin{aligned}
&\left[\sum_{j=1}^n \left(\sum_{l=1}^k \left|\left\langle \varphi_l, T(x_j^1,\ldots,x_j^m)\right\rangle\right|^s \left\|T(x_j^1,\ldots,x_j^m)\right\|^{\frac{\theta s}{1-\theta}}\right)^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}} \\
&= \left\|(\varphi_l)_{l=1}^k\middle|\ell_s(F^*)\right\|^{1-\theta} \left[\sum_{j=1}^n \left(\int_{B_{F^*}} \left|\left\langle \varphi, T(x_j^1,\ldots,x_j^m)\right\rangle\right|^s d\nu(\varphi) \left\|T(x_j^1,\ldots,x_j^m)\right\|^{\frac{\theta s}{1-\theta}}\right)^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}} \\
&\leqslant \left\|(\varphi_l)_{l=1}^k\middle|\ell_s(F^*)\right\|^{1-\theta} \cdot \mathfrak{m}_{(s;q,\theta)}((T(x_j^1,\ldots,x_j^m))_{j=1}^n) \\
&\leqslant \|T\|_{mx(s,q;p_1,\ldots,p_m;\theta)} \left\|(\varphi_l)_{l=1}^k\middle|\ell_s(F^*)\right\|^{1-\theta} \prod_{i=1}^m \delta_{p_i\theta}\left((x_j^i)_{j=1}^n\right)
\end{aligned}$$

and we obtain (3.2) with $\inf C \leqslant \|T\|_{mx(s,q;p_1,\ldots,p_m;\theta)}$.

Reciprocally, let us suppose that (3.2) is true. Given $\nu = \sum_{l=1}^k \nu_l \delta_l$ a discrete probability measure onto B_{F^*}. We have

$$\begin{aligned}
&\left[\sum_{j=1}^n \left(\int_{B_{F^*}} \left|\left\langle \varphi, T(x_j^1,\ldots,x_j^m)\right\rangle\right|^s \left\|T(x_j^1,\ldots,x_j^m)\right\|^{\frac{\theta s}{1-\theta}} d\nu(\varphi)\right)^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}} \\
&= \left[\sum_{j=1}^n \left(\sum_{l=1}^k \left|\left\langle \nu_l^{\frac{1}{s}} \varphi_l, T(x_j^1,\ldots,x_j^m)\right\rangle\right|^s \left\|T(x_j^1,\ldots,x_j^m)\right\|^{\frac{\theta s}{1-\theta}}\right)^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}} \\
&\leqslant C \prod_{i=1}^m \delta_{p_i\theta}\left((x_i^j)_{i=1}^n\right) \cdot \left\|(\nu_l^{\frac{1}{s}} \varphi_l)_{l=1}^k\middle|\ell_s(F^*)\right\|^{1-\theta} \\
&\leqslant C \prod_{i=1}^m \delta_{p_i\theta}\left((x_i^j)_{i=1}^n\right).
\end{aligned}$$

Since the discrete probability measures are dense in $W(B_{F^*})$ for the weak star topology defined by $C(B_{F^*})$, it follows that this inequality holds for all such $\nu \in W(B_{F^*})$ and

$(x_j^i)_{j=1}^n \subset E_j, (i=1,\ldots,m)$. Therefore, by (2.3) we obtain

$$\mathfrak{m}_{(s;q,\theta)}((T(x_j^1,\ldots,x_j^m)_{j=1}^n) \leqslant C\prod_{i=1}^m \delta_{p_i\theta}\left((x_i^j)_{i=1}^n\right).$$

This shows that T is $(s,q;p_1,\ldots,p_m;\theta)$-mixing and $\|T\|_{\mathscr{L}^{\theta}_{mx(s,q;p_1,\ldots,p_m)}} = \inf C$. $\square$

Now we are ready to present the main results of this section. In order to prove this theorem we need the following preliminary results. For a regular Borel probability measure ν on B_{F^*}, (with the weak star topology), we denote by i the isometric embedding $F \to C(B_{F^*})$ given by $i(y) = \langle y,\cdot\rangle$. For $y \in F$ consider the seminorm $\|\langle y,\cdot\rangle\|_{s,\theta} = \inf \sum_{l=1}^k \|y_l\|^\theta \left(\int_{B_{F^*}} |\langle y_l,\cdot\rangle|^s d\nu\right)^{\frac{1-\theta}{s}}$, the infimum computed over all decompositions of y as $y = \sum_{l=1}^k y_l$ in F. Following [8, 1] (see also [7, Section 2.1.3]) , let $L_{s,\theta}(\nu)$ be the completion of the quotient normed space $i(F)/\|\cdot\|_{s,\theta}^{-1}(0)$ of all classes of functions as $\langle y,\cdot\rangle \in i(F) \subset C(B_{F^*})$, $y \in F$, with the quotient norm $\|\cdot\|_{s,\theta}$.

THEOREM 3.4. *Let $0 \leqslant \theta < 1$ and $1 \leqslant r,s,q,p_1,\ldots,p_m < \infty$ such that $\frac{1}{q} = \frac{1}{p_1}+\ldots+\frac{1}{p_m}$ and $\frac{1}{q} = \frac{1}{s}+\frac{1}{r}$. For $T \in \mathscr{L}(E_1,\ldots,E_m;F)$, the following statements are equivalent*

a) The composition $u\circ T : E_1\times\ldots\times E_m \longrightarrow G$ is $(q;p_1,\ldots,p_m;\theta)$-absolutely continuous multilinear operators for any Banach space G and any (s,θ)-absolutely continuous operator $u : F \longrightarrow G$ with

$$\|u\circ T\|_{\mathscr{L}^{\theta}_{as,(q;p_1,\ldots,p_m)}} \leqslant C.\pi_{s,\theta}(u). \tag{3.3}$$

b) There is a constant $C \geqslant 0$ such that for any probability measure $\nu \in W(B_{F^*})$ there exists regular Borel probability measures $\mu_i \in W(B_{E_i^*}), 1 \leqslant i \leqslant m$ so that for all $(x^1,\ldots,x^m) \in E_1\times\ldots\times E_m$ the inequality

$$\inf\left\{\sum_{l=1}^k\left(\int_{B_{F^*}} |\langle\varphi,y_l\rangle|^s \|y_l\|^{\frac{\theta s}{1-\theta}} d\nu(\varphi)\right)^{\frac{1-\theta}{s}} : T(x^1,\ldots,x^m) = \sum_{l=1}^k y_l\right\} \tag{3.4}$$
$$\leqslant C.\prod_{i=1}^m \|x^i\|^\theta \left(\int_{B_{E_i^*}} |\langle\phi,x^i\rangle|^{p_i} d\mu_i(\phi)\right)^{\frac{1-\theta}{p_i}},$$

is valid.

c) T is $(s,q;p_1,\ldots,p_m;\theta)$-mixing with $\|T\|_{\mathscr{L}^{\theta}_{mx(s,q;p_1,\ldots,p_m)}} \leqslant C$.

Moreover, $\|T\|_{\mathscr{L}^{\theta}_{mx(s,q;p_1,\ldots,p_m)}} = \inf C$, where the infimum is taken over all constants C either a), b) or c).

Proof. $a) \Longrightarrow b)$ For each $\nu \in W(B_{F^*})$ we consider the operator $J_\nu : F \to L_{s,\theta}(\nu)$ given by $J_\nu(y) = [\langle y, \cdot \rangle]$, where $[\langle y, \cdot \rangle]$ is the equivalence class of $\langle y, \cdot \rangle$. In this case, J_ν is (s,θ)-absolutely continuous with $\pi_{s,\theta}(J_\nu) \leqslant 1$ (see [8, Lemma 3.4]), then by the hypothesis the m-linear mapping $J_\nu \circ T$ is $(q; p_1, \ldots, p_m; \theta)$-absolutely continuous with

$$\|J_\nu \circ T\|_{\mathscr{L}^\theta_{as(q;p_1,\ldots,p_m)}} \leqslant C.$$

By the domination theorem for the class $\mathscr{L}^\theta_{as(q;p_1,\ldots,p_m)}$ (see [8, Theorem 3.3]), there is regular Borel probability measures $\mu_i \in W(B_{E_i^*})$ such that for all $x^i \in E_i$, $1 \leqslant i \leqslant m$, we have

$$\begin{aligned}
&\left\| \left[\left\langle T(x^1, \ldots, x^m), \cdot \right\rangle \right] \right\|_{s,\theta} \\
&= \inf \left\{ \sum_{l=1}^{k} \left(\int_{B_{F^*}} |\langle \varphi, y_l \rangle|^s \|y_l\|^{\frac{\theta s}{1-\theta}} \, d\nu(\varphi) \right)^{\frac{1-\theta}{s}} : T(x^1, \ldots, x^m) = \sum_{l=1}^{k} y_l \right\} \\
&\leqslant C \prod_{i=1}^{m} \|x^i\|^\theta \left(\int_{B_{E_i^*}} \left| \left\langle \phi, x^i \right\rangle \right|^{p_i} d\mu_i(\phi) \right)^{\frac{1-\theta}{p_i}}.
\end{aligned}$$

$b) \Longrightarrow c)$ From (3.4) and by Hölder's inequality we have

$$\begin{aligned}
&\left(\sum_{j=1}^{n} \left(\int_{B_{F^*}} \left| \left\langle \varphi, T(x_j^1, \ldots, x_j^m) \right\rangle \right|^s \left\| T(x_j^1, \ldots, x_j^m) \right\|^{\frac{\theta s}{1-\theta}} d\mu(\varphi) \right)^{\frac{q}{s}} \right)^{\frac{1-\theta}{q}} \\
&\leqslant C \left[\sum_{j=1}^{n} \left(\prod_{i=1}^{m} \left(\int_{B_{E_i^*}} \left\| x_j^i \right\|^{\theta \frac{p_i}{1-\theta}} \left| \left\langle \phi, x_j^i \right\rangle \right|^{p_i} d\mu_i(\phi) \right)^{\frac{1-\theta}{p_i}} \right)^{\frac{q}{1-\theta}} \right]^{\frac{1-\theta}{q}} \\
&\leqslant C \prod_{i=1}^{m} \left(\sum_{j=1}^{n} \int_{B_{E_i^*}} \left\| x_j^i \right\|^{\theta \frac{p_i}{1-\theta}} \left| \left\langle \phi, x_j^i \right\rangle \right|^{p_i} d\mu_i(\phi) \right)^{\frac{1-\theta}{p_i}} \\
&\leqslant C \prod_{i=1}^{m} \delta_{p_i\theta} \left((x_j^i)_{j=1}^n \right)
\end{aligned}$$

for every $\nu \in W(B_{F^*}), n \in \mathbb{N}$ and $(x_j^i)_{j=1}^n \subset E_i, i = 1, \ldots, m$. By using the equality, (2.3) we obtain

$$\mathfrak{m}_{(s;q,\theta)}((T(x_j^1, \ldots, x_j^m)_{j=1}^n) \leqslant C \prod_{i=1}^{m} \delta_{p_i\theta} \left((x_j^i)_{j=1}^n \right).$$

Thus $T \in \mathscr{L}^\theta_{mx(s,q;p_1,\ldots,p_m)}(E_1, \ldots, E_m; F)$ and $\|T\|_{\mathscr{L}^\theta_{mx(s,q;p_1,\ldots,p_m)}} \leqslant C$.

$c) \Longrightarrow a)$ Let G a Banach space and let $u : F \longrightarrow G$ be a (s,θ)-absolutely continuous operator. By the Pietsch domination theorem concerning the (s,θ)-absolutely continuous operators [11, Theorem 4.1], there is a regular Borel probability measure

μ on B_{F^*} such that for all $(x_j^1,\ldots,x_j^m)\in E_1\times\ldots\times E_m, j=1,\ldots,n$ we have

$$\left\|u\left(T(x_j^1,\ldots,x_j^m)\right)\right\|\leqslant\pi_{s,\theta}(u)\left(\int_{B_{F^*}}\left|\left\langle\varphi,T(x_j^1,\ldots,x_j^m)\right\rangle\right|^s\left\|T(x_j^1,\ldots,x_j^m)\right\|^{\frac{\theta s}{1-\theta}}d\mu(\varphi)\right)^{\frac{1-\theta}{s}}.$$

Then by the hypothesis and (2.3) we get

$$\begin{aligned}&\left(\sum_{j=1}^n\left\|u\circ T(x_j^1,\ldots,x_j^m)\right\|^{\frac{q}{1-\theta}}\right)^{\frac{1-\theta}{q}}\\&\leqslant\pi_{s,\theta}(u)\left(\sum_{j=1}^n\left(\int_{B_{F^*}}\left|\left\langle\varphi,T(x_j^1,\ldots,x_j^m)\right\rangle\right|^s\left\|T(x_j^1,\ldots,x_j^m)\right\|^{\frac{\theta s}{1-\theta}}d\mu(\varphi)\right)^{\frac{q}{s}}\right)^{\frac{1-\theta}{q}}\\&\leqslant\pi_{s,\theta}(u).\mathfrak{m}_{(s;q,\theta)}((T(x_j^1,\ldots,x_j^m)_{j=1}^n)\\&\leqslant\pi_{s,\theta}(u).\left\|T\right\|_{\mathscr{L}^\theta_{mx(s,q;p_1,\ldots,p_m)}}\prod_{i=1}^m\delta_{p_i\theta}\left((x_j^i)_{j=1}^n\right).\end{aligned}$$

Hence $u\circ T\in\mathscr{L}^\theta_{as(q;p_1,\ldots,p_m)}(E_1,\ldots,E_m;G)$ and the inequality (3.3) holds. □

REMARK 3.5. Actually the equivalence between a) and c) in the above theorem asserts that the class of $(s,q;p_1,\ldots,p_m;\theta)$-mixing multilinear operators satisfy the quotient theorem

$$\mathscr{L}^\theta_{mx(s,q;p_1,\ldots,p_m)}=\Pi_{s,\theta}^{-1}\circ\mathscr{L}^\theta_{as(q;p_1,\ldots,p_m)}.$$

4. Lipschitz $(s;q,\theta)$-mixing maps

Throughout this section, $q,s,\theta,s^*(q)$ are real numbers such that $0<q\leqslant s\leqslant\infty$, $0\leqslant\theta<1$, $\frac{1}{s^*(q)}+\frac{1}{s}=\frac{1}{q}$ and X,Y are pointed metric spaces. The notion of Lipschitz mixed $(s;q,\theta)$-summable sequence can be constructed as follows. For all sequences $(\sigma,x',x'')\subset\mathbb{R}\times X\times X$ define

$$\delta^L_{s\theta}(\sigma,x',x'')=\sup_{f\in B_{X^\#}}\left[\sum_{j=1}^\infty\left(\left|\sigma_j\right|\left|f(x'_j)-f(x''_j)\right|^{1-\theta}d_X(x'_j,x''_j)^\theta\right)^{\frac{s}{1-\theta}}\right]^{\frac{1-\theta}{s}}$$

and

$$H^L_{s,\theta}(\mathbb{R}\times X\times X)=\left\{(\sigma,x',x'')\subset\mathbb{R}\times X\times X:\delta^L_{s\theta}(\sigma,x',x'')<\infty\right\}.$$

DEFINITION 4.1. A sequence $(\sigma,x',x'')\subset\mathbb{R}\times X\times X$ is called Lipschitz mixed $(s;q,\theta)$-summable, if there exists a sequence $\tau\in\ell_{\frac{s^*(q)}{1-\theta}}$ such that $(\frac{\sigma}{\tau},x',x'')\in H^L_{s,\theta}(\mathbb{R}\times X\times X)$. The space of all Lipschitz mixed $(s;q,\theta)$-summable sequences is denoted by $\mathfrak{M}^{L,\theta}_{(s;q)}(\mathbb{R}\times X\times X)$. Moreover, for a sequence $(\sigma,x',x'')\in\mathfrak{M}^{L,\theta}_{(s;q)}(\mathbb{R}\times X\times X)$ define

$$\mathfrak{m}^{L,\theta}_{(s;q)}(\sigma,x',x'')=\inf\left\|\tau\middle|\ell_{\frac{s^*(q)}{1-\theta}}\right\|\cdot\delta^L_{s\theta}\left(\frac{\sigma}{\tau},x',x''\right)\tag{4.1}$$

where the infimum is taken over all sequences $\tau \in \ell_{\frac{s^*(q)}{1-\theta}}$.

REMARK 4.2. Let (σ, x', x'') be an arbitrary sequence in $\mathbb{R} \times X \times X$.

(**1**) For the special case if $\theta = 0$, the class $\mathfrak{M}^{L,0}_{(s;q)}(\mathbb{R} \times X \times X)$ coincides with the class $\mathfrak{M}^{L}_{(s;q)}(\mathbb{R} \times X \times X)$.

(**2**) If $q = s$, then $\mathfrak{M}^{L,\theta}_{(q;q)}(\mathbb{R} \times X \times X) = H^L_{q,\theta}(\mathbb{R} \times X \times X)$ with

$$\mathfrak{m}^{L,\theta}_{(q;q)}(\sigma, x', x'') = \delta^L_{q\theta}(\sigma, x', x'').$$

(**3**) If $s = \infty$, then $\mathfrak{M}^{L,\theta}_{(\infty;q)}(\mathbb{R} \times X \times X) = \ell^L_{\frac{q}{1-\theta}}(\mathbb{R} \times X \times X)$ with

$$\mathfrak{m}^{L,\theta}_{(\infty;q)}(\sigma, x', x'') = \left\| (\sigma, x', x'') \middle| \ell^L_{\frac{q}{1-\theta}}(\mathbb{R} \times X \times X) \right\|.$$

Inspired by the analogous result of [15, Lemma 1.4] and the similar proof of [5, Proposition 4.2] we give an important characteristic.

PROPOSITION 4.3. *Let $0 < q < s < \infty$ and $0 \leqslant \theta < 1$. A sequence (σ, x', x'') is Lipschitz mixed $(s;q,\theta)$-summable if and only if*

$$\left[\sum_{j=1}^{\infty} \left[\int_{B_{X^\#}} |\sigma_j|^{\frac{s}{1-\theta}} \left| f(x'_j) - f(x''_j) \right|^s d_X(x'_j, x''_j)^{\frac{\theta s}{1-\theta}} \, d\mu(f) \right]^{\frac{q}{s}} \right]^{\frac{1-\theta}{q}} < \infty, \tag{4.2}$$

for every $\mu \in W(B_{X^\#})$. In this particular case

$$\sup_{\mu \in W(B_{X^\#})} \left[\sum_{j=1}^{\infty} \left[\int_{B_{X^\#}} |\sigma_j|^{\frac{s}{1-\theta}} \left| f(x'_j) - f(x''_j) \right|^s d_X(x'_j, x''_j)^{\frac{\theta s}{1-\theta}} \, d\mu(f) \right]^{\frac{q}{s}} \right]^{\frac{1-\theta}{q}}$$
$$= \mathfrak{m}^L_{(s;q,\theta)}(\sigma, x', x'').$$

Proof. Suppose that (σ, x', x'') is Lipschitz mixed $(s;q,\theta)$-summable sequence. Then there exists a sequence $\tau \in \ell_{\frac{s^*(q)}{1-\theta}}$ such that $(\frac{\sigma}{\tau}, x', x'') \in H^L_{s,\theta}(\mathbb{R} \times X \times X)$. Applying Hölder inequality we obtain

$$\left[\sum_{j=1}^{\infty}\left[\int_{B_{X^\#}}\left|\sigma_j\right|^{\frac{s}{1-\theta}}\left|f(x'_j)-f(x''_j)\right|^s d_X(x'_j,x''_j)^{\frac{\theta s}{1-\theta}}\, d\mu(f)\right]^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}}$$

$$=\left(\sum_{j=1}^{\infty}\left[\left|\tau_j\right|\left[\int_{B_{X^\#}}\left|\frac{\sigma_j}{\tau_j}\right|^{\frac{s}{1-\theta}}\left|f(x'_j)-f(x''_j)\right|^s d_X(x'_j,x''_j)^{\frac{\theta s}{1-\theta}}\, d\mu(f)\right]^{\frac{1-\theta}{s}}\right]^{\frac{q}{1-\theta}}\right)^{\frac{1-\theta}{q}}$$

$$\leqslant \left\|\tau\Big|\ell_{\frac{s^*(q)}{1-\theta}}\right\|\left(\sum_{j=1}^{\infty}\int_{B_{X^\#}}\left|\frac{\sigma_j}{\tau_j}\right|^{\frac{s}{1-\theta}}\left|f(x'_j)-f(x''_j)\right|^s d_X(x'_j,x''_j)^{\frac{\theta s}{1-\theta}}\, d\mu(f)\right)^{\frac{1-\theta}{s}}$$

$$\leqslant \left\|\tau\Big|\ell_{\frac{s^*(q)}{1-\theta}}\right\|\delta^L_{s\theta}\left(\frac{\sigma}{\tau},x',x''\right)<\infty.$$

Conversely, let a sequence (σ,x',x'') satisfy the condition (4.2). Define a number β as follow

$$\beta=\sup_{\mu\in W(B_{X^\#})}\left[\sum_{j=1}^{\infty}\left[\int_{B_{X^\#}}\left|\sigma_j\right|^{\frac{s}{1-\theta}}\left|f(x'_j)-f(x''_j)\right|^s d_X(x'_j,x''_j)^{\frac{\theta s}{1-\theta}}\, d\mu(f)\right]^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}}.$$

Then the number β is finite. Put $u=\frac{s^*(q)}{q}$ and $v=\frac{s}{q}$. Then $\frac{1}{u}+\frac{1}{v}=1$. We now consider the compact, convex subset

$$K=\left\{\xi=(\xi_j)_j:\sum_{j=1}^{\infty}\xi_j^u\leqslant\beta^{\frac{q}{1-\theta}}\text{ and }\xi_j\geqslant 0\right\}$$

of ℓ_u. Observe that the expression

$$\phi(\xi)=\sum_{j=1}^{\infty}(\xi_j+\varepsilon)^{-v}\cdot\int_{B_{X^\#}}\left|\sigma_j\right|^{\frac{s}{1-\theta}}\left|f(x'_j)-f(x''_j)\right|^s d_X(x'_j,x''_j)^{\frac{\theta s}{1-\theta}}d\mu(f),$$

where $\mu\in W(B_{X^\#})$, $\varepsilon>0$, defines a continuous convex function ϕ on K. Take the special family $\xi=(\xi_j)_j$ with

$$\xi_j=\left(\int_{B_{X^\#}}\left|\sigma_j\right|^{\frac{s}{1-\theta}}\left|f(x'_j)-f(x''_j)\right|^s d_X(x'_j,x''_j)^{\frac{\theta s}{1-\theta}}d\mu(f)\right)^{\frac{1}{u\cdot v}}.$$

Then $\xi\in K$ and $\phi(\xi)\leqslant\beta^{\frac{q}{1-\theta}}$. Since the collection $\mathfrak{F}$ of all functions ξ obtained in this way is concave, by Ky Fan's lemma, we can find $\xi^0=\left(\xi_j^0\right)_j\in K$ such that

$\phi(\xi^0) \leqslant \beta^{\frac{q}{1-\theta}}$ for all $\phi \in \mathfrak{F}$ simultaneously. In particular, considering the Dirac measure δ_f at a function $f \in B_{X^\#}$, we obtain

$$\sum_{j=1}^{\infty} \left(\xi_j^0+\varepsilon\right)^{-v} \left|\sigma_j\right|^{\frac{s}{1-\theta}} \left|f(x_j')-f(x_j'')\right|^s d_X(x_j',x_j'')^{\frac{\theta s}{1-\theta}} \leqslant \beta^{\frac{q}{1-\theta}}.$$

We put $\tau_\varepsilon = (\tau_j(\varepsilon))_j$ with $\tau_j(\varepsilon) = \left(\xi_j^0+\varepsilon\right)^{\frac{1-\theta}{q}}$. Then

$$\left\|\tau\middle|\ell_{\frac{s^*(q)}{1-\theta}}\right\| = \lim_{\varepsilon\to 0^+}\left\|\tau_\varepsilon\middle|\ell_{\frac{s^*(q)}{1-\theta}}\right\| = \lim_{\varepsilon\to 0^+}\left[\sum_{j=1}^{\infty}\tau_j(\varepsilon)^{\frac{s^*(q)}{1-\theta}}\right]^{\frac{1-\theta}{s^*(q)}} = \left[\sum_{j=1}^{\infty}(\xi_j^0)^{\frac{s^*(q)}{q}}\right]^{\frac{1-\theta}{s^*(q)}}$$
$$= \left[\sum_{j=1}^{\infty}(\xi_j^0)^u\right]^{\frac{1-\theta}{s^*(q)}} \leqslant \beta^{\frac{q}{s^*(q)}} = \beta^{\frac{1}{u}}$$

and for $f \in B_{X^\#}$,

$$\left[\sum_{j=1}^{\infty}\left|\frac{\sigma_j}{\tau_j}\right|^{\frac{s}{1-\theta}}\left|f(x_j')-f(x_j'')\right|^s d_X(x_j',x_j'')^{\frac{\theta s}{1-\theta}}\right]^{\frac{1-\theta}{s}}$$
$$= \lim_{\varepsilon\to 0^+}\left[\sum_{j=1}^{\infty}\left|\frac{\sigma_j}{\tau_j(\varepsilon)}\right|^{\frac{s}{1-\theta}}\left|f(x_j')-f(x_j'')\right|^s d_X(x_j',x_j'')^{\frac{\theta s}{1-\theta}}\right]^{\frac{1-\theta}{s}}$$
$$= \lim_{\varepsilon\to 0^+}\left[\sum_{j=1}^{\infty}\frac{\left|\sigma_j\right|^{\frac{s}{1-\theta}}}{\left(\xi_j^0+\varepsilon\right)^{v}}\left|f(x_j')-f(x_j'')\right|^s d_X(x_j',x_j'')^{\frac{\theta s}{1-\theta}}\right]^{\frac{1-\theta}{s}} \leqslant \beta^{\frac{q}{s}} = \beta^{\frac{1}{v}}.$$

Hence $\left\|\tau\middle|\ell_{\frac{s^*(q)}{1-\theta}}\right\| \delta^L_{s\theta}(\frac{\sigma}{\tau},x',x'') \leqslant \beta$. □

DEFINITION 4.4. Let $0<q\leqslant s\leqslant\infty$ and $0\leqslant\theta<1$. A Lipschitz map $T: X \longrightarrow Y$ between pointed metric spaces is called Lipschitz $(s;q,\theta)$-mixing if there is a constant $C \geqslant 0$ such that

$$\mathfrak{m}^{L,\theta}_{(s;q)}(\sigma, Tx', Tx'') \leqslant C\cdot\delta^L_{q\theta}(\sigma,x',x'') \tag{4.3}$$

for arbitrary finite sequences x', x'' in X and σ in $\mathbb{R}$. Let us denote by $\mathbf{M}^{L,\theta}_{(s;q)}(X,Y)$ the class of all Lipschitz $(s;q,\theta)$-mixing maps from X to Y. In such case, we put

$$\mathbf{m}^{L,\theta}_{(s;q)}(T) = \inf C,$$

where the infimum is taken over all constant C satisfying (4.3).

REMARK 4.5.

1. The linear space $\mathbf{M}_{(s;q)}^{L,\theta}(X,F)$ equipped with the norm $\mathbf{m}_{(s;q)}^{L,\theta}(\cdot)$ is a Banach space if $q \geqslant 1$ and a complete q-normed space if $0<q<1$.

2. If $\theta = 0$, then the class $\mathbf{M}_{(s;q)}^{L,\theta}(X,Y)$ coincides with the class $\Pi^L_{\left(\mathfrak{m}^L(s;q),q\right)}(X,Y)$ of Lipschitz $\left(\mathfrak{m}^L(s;q),q\right)$-summing mappings was introduced by the third author in [14, Definition 3.5] for $0<q\leqslant s\leqslant\infty$ and by J. A. Chávez-Domínguez in [5, Corollary 4.3] for $1\leqslant q<s<\infty$.

CONCLUDING REMARKS 4.6.

It is obvious that the Lipschitz $(s;p,\theta)-$mixing maps satisfy the ideal property, i.e.

$$\mathbf{m}_{(s;p)}^{L,\theta}(S\circ T\circ R)\leqslant Lip(S)\cdot\mathbf{m}_{(s;p)}^{L,\theta}(T)\cdot Lip(R)$$

whenever the composition makes sense.

- If $0<q<s\leqslant\infty$ and $0\leqslant\theta<1$, then $\ell^L_{\frac{q}{1-\theta}}(\mathbb{R}\times X\times X)\subset\mathfrak{M}_{(s;q)}^{L,\theta}(\mathbb{R}\times X\times X)$, with $\mathfrak{m}_{(s;q)}^{L,\theta}(\sigma,x',x'')\leqslant\left\|(\sigma,x',x'')\middle|\ell^L_{\frac{q}{1-\theta}}\right\|$, for every $(\sigma,x',x'')\in\ell^L_{\frac{q}{1-\theta}}(\mathbb{R}\times X\times X)$.

- If $0<q<\infty$ and $0\leqslant\theta<1$, then $\ell^L_{\frac{q}{1-\theta}}(\mathbb{R}\times X\times X)\subset\mathfrak{M}_{(q;q)}^{L,\theta}(\mathbb{R}\times X\times X)$, with $\mathfrak{m}_{(q;q)}^{L,\theta}(\sigma,x',x'')\leqslant\left\|(\sigma,x',x'')\middle|\ell^L_{\frac{q}{1-\theta}}\right\|$, for every $(\sigma,x',x'')\in\ell^L_{\frac{q}{1-\theta}}(\mathbb{R}\times X\times X)$.

- If $0<q\leqslant s\leqslant\infty$ and $0\leqslant\theta<1$, then $\mathfrak{M}_{(s;q)}^{L,\theta}(\mathbb{R}\times X\times X)\subset\ell^L_{\frac{s^*(q)}{1-\theta}}(\mathbb{R}\times X\times X)$. Moreover
$$\left\|(\sigma,x',x'')\middle|\ell^L_{\frac{s^*(q)}{1-\theta}}\right\|\leqslant\mathfrak{m}_{(s;q)}^{L,\theta}(\sigma,x',x'').$$

- If $0<q\leqslant s_1\leqslant s_2\leqslant\infty$ and $0\leqslant\theta<1$, then $\mathfrak{M}_{(s_2;q)}^{L,\theta}(\mathbb{R}\times X\times X)\subset\mathfrak{M}_{(s_1;q)}^{L,\theta}(\mathbb{R}\times X\times X)$. Moreover
$$\mathfrak{m}_{(s_1;q)}^{L,\theta}(\sigma,x',x'')\leqslant\mathfrak{m}_{(s_2;q)}^{L,\theta}(\sigma,x',x'')$$
for every $(\sigma,x',x'')\in\mathfrak{M}_{(s_2;q)}^{L,\theta}(\mathbb{R}\times X\times X)$.

The characterization of Lipschitz $(s;q,\theta)$-mixing maps is presented in the following theorem, it is somewhat inspired by analogous results in the linear theory and similar proof of [5, Theorem 4.1] we give an important characteristic.

THEOREM 4.7. *A Lipschitz map T from X to Y is Lipschitz $(s;q,\theta)$-mixing if and only if there is a constant $C \geqslant 0$ such that*

$$\left[\sum_{j=1}^{m}|\sigma_j|^{\frac{s}{1-\theta}}\left[\sum_{k=1}^{n}\left|\langle g_k,Tx'_j\rangle_{(Y^{\#},Y)}-\langle g_k,Tx''_j\rangle_{(Y^{\#},Y)}\right|^{s}d_Y(Tx'_j,Tx''_j)^{\frac{\theta s}{1-\theta}}\right]^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}} \leqslant C\cdot\delta^{L}_{q\theta}(\sigma,x',x'')\cdot\left\|(g_k)_{k=1}^{n}\right|\ell_s(Y^{\#})\right\|^{1-\theta} \tag{4.4}$$

for every $\sigma_1,\ldots,\sigma_m\in\mathbb{R}$; $x'_1,\ldots,x'_m,x''_1,\cdots,x''_m\in X$; $g_1,\ldots,g_n\in Y^{\#}$ and $m,n\in\mathbb{N}$. Moreover

$$\mathbf{m}^{L,\theta}_{(s;q)}(T)=\inf C.$$

Taking the infimum over all $C\geqslant 0$ verifying the above inequality.

Proof. Assume that T is a Lipschitz $(s;q,\theta)$-mixing map. Consider $g_1,\ldots,g_n\in Y^{\#}$ and define the discrete probability $\mu=\sum_{k=1}^{n}t_k\delta_k$, where $t_k=Lip(g_k)^s\cdot\left\|(g_k)_{h=1}^{n}\right|\ell_s(Y^{\#})\right\|^{-s}$ and δ_k denotes the Dirac measure at $b_k=\frac{g_k}{Lip(g_k)}\in B_{Y^{\#}}$; $k=1,\ldots,n$. Then $\mu\in W(B_{Y^{\#}})$. For $\sigma_1,\ldots,\sigma_m\in\mathbb{R}$, $x'_1,\ldots,x'_m,x''_1,\ldots,x''_m\in X$, we conclude from Proposition 4.3 that

$$\begin{aligned}
&\left[\sum_{j=1}^{m}|\sigma_j|^{\frac{s}{1-\theta}}\left[\sum_{k=1}^{n}\left|\langle g_k,Tx'_j\rangle_{(Y^{\#},Y)}-\langle g_k,Tx''_j\rangle_{(Y^{\#},Y)}\right|^{s}d_Y(Tx'_j,Tx''_j)^{\frac{\theta s}{1-\theta}}\right]^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}}\\
&=\left[\sum_{j=1}^{m}|\sigma_j|^{\frac{s}{1-\theta}}\left[\int_{B_{Y^{\#}}}\left|\langle g,Tx'_j\rangle_{(Y^{\#},Y)}-\langle g,Tx''_j\rangle_{(Y^{\#},Y)}\right|^{s}\right.\right.\\
&\qquad\left.\left.\times d_Y(Tx'_j,Tx''_j)^{\frac{\theta s}{1-\theta}}d\mu(g)\right]^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}}\left\|(g_k)_{k=1}^{n}\right|\ell_s(Y^{\#})\right\|^{1-\theta}\\
&\leqslant \mathfrak{m}^{L}_{(s;q,\theta)}(\sigma,Tx',Tx'')\cdot\left\|(g_k)_{k=1}^{n}\right|\ell_s(Y^{\#})\right\|^{1-\theta}\\
&\leqslant m^{L,\theta}_{(s;q)}(T)\cdot\delta^{L}_{q\theta}(\sigma,x',x'')\cdot\left\|(g_k)_{k=1}^{n}\right|\ell_s(Y^{\#})\right\|^{1-\theta}.
\end{aligned}$$

To show the converse, observe that (4.4) means

$$\left[\sum_{j=1}^{m}|\sigma_j|^{\frac{s}{1-\theta}}\left[\int_{B_{Y^{\#}}}\left|\langle g,Tx'_j\rangle_{(Y^{\#},Y)}-\langle g,Tx''_j\rangle_{(Y^{\#},Y)}\right|^{s}\cdot d_Y(Tx'_j,Tx''_j)^{\frac{\theta s}{1-\theta}}d\mu(g)\right]^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}} \leqslant C\cdot\delta^{L}_{q\theta}(\sigma,x',x'') \tag{4.5}$$

for each discrete probability measure μ on $B_{Y^{\#}}$ and $\sigma_1,\ldots,\sigma_m\in\mathbb{R}$; $x'_1,\ldots,x'_m,x''_1,\ldots,x''_m\in X$. Since the class of all finitely supported probability measures on $B_{Y^{\#}}$ is dense in the class of every probability measure on $B_{Y^{\#}}$ for the weak star topology defined by

$C(B_{Y^{\#}})$, it follows that (4.5) satisfied for every probability measure μ on $B_{Y^{\#}}$ and $\sigma_1,\dots,\sigma_m \in \mathbb{R}$, $x'_1,\dots,x'_m,x''_1,\dots,x''_m \in X$. Taking the supremum over $\mu \in W(B_{Y^{\#}})$ on the left side of (4.5) and using Proposition 4.3, we have

$$\mathfrak{m}^{L,\theta}_{(s;q)}(\sigma,Tx',Tx'') \leqslant C\cdot\delta^L_{q,\theta}(\sigma,x',x''). \quad \square$$

PROPOSITION 4.8. *Let $0<q\leqslant r\leqslant t\leqslant\infty$ and $0\leqslant\theta<1$. If S from Y to Z is a Lipschitz $(t;s,\theta)$-mixing map and T from X to Y is a Lipschitz $(s;q,\theta)$-mixing map, then $S\circ T$ from X to Z is a Lipschitz $(t;q,\theta)$–mixing map. Moreover*

$$\mathbf{m}^{L,\theta}_{(t;q)}(S\circ T) \leqslant \mathbf{m}^{L,\theta}_{(t;s)}(S)\cdot\mathbf{m}^{L,\theta}_{(s;q)}(T).$$

Proof. From Definition 4.1, we have

$$\begin{aligned}\mathfrak{m}^{L,\theta}_{(t;q)}\big(\sigma,(S\circ T)x',(S\circ T)x''\big) &= \inf\left\|\tau\Big|\ell_{\frac{t^*(q)}{1-\theta}}\right\|\delta^L_{t\theta}\Big(\frac{\sigma}{\tau},(S\circ T)x',(S\circ T)x''\Big)\\ &= \inf_{\tau_1\cdot\tau_2}\left\|\tau_1\cdot\tau_2\Big|\ell_{\frac{t^*(q)}{1-\theta}}\right\|\delta^L_{t\theta}\left(\frac{\sigma}{\tau_1\cdot\tau_2},(S\circ T)x',(S\circ T)x''\right).\end{aligned}$$

Let $\sigma'=\frac{\sigma}{\tau_1}$. Since $\frac{1}{t^*(s)}+\frac{1}{s^*(q)}=\frac{1}{t^*(q)}$ with the Hölder inequality give us

$$\begin{aligned}\mathfrak{m}^{L,\theta}_{(t;q)}\big(\sigma,(S\circ T)x',(S\circ T)x''\big) &\leqslant \inf_{\tau_1\cdot\tau_2}\left\|\tau_1\Big|\ell_{\frac{s^*(q)}{1-\theta}}\right\|\cdot\left\|\tau_2\Big|\ell_{\frac{t^*(s)}{1-\theta}}\right\|\delta^L_{t,\theta}\left(\frac{\sigma'}{\tau_2},S(Tx'),S(Tx'')\right)\\ &= \inf_{\tau_1}\left\|\tau_1\Big|\ell_{\frac{s^*(q)}{1-\theta}}\right\|\cdot\inf_{\tau_2}\left\|\tau_2\Big|\ell_{\frac{t^*(s)}{1-\theta}}\right\|\delta^L_{t\theta}\left(\frac{\sigma'}{\tau_2},S(Tx'),S(Tx'')\right).\\ &= \inf_{\tau_1}\left\|\tau_1\Big|\ell_{\frac{s^*(q)}{1-\theta}}\right\|\cdot\mathfrak{m}^{L,\theta}_{(t;s)}\big(\sigma',S(Tx'),S(Tx'')\big)\\ &\leqslant \mathbf{m}^{L,\theta}_{(t;s)}(S)\cdot\inf_{\tau_1}\left\|\tau_1\Big|\ell_{\frac{s^*(q)}{1-\theta}}\right\|\cdot\delta^L_{s\theta}(\sigma',Tx',Tx'')\\ &\leqslant \mathbf{m}^{L,\theta}_{(t;s)}(S)\cdot\mathfrak{m}^{L,\theta}_{(s;q)}(\sigma,Tx',Tx'')\\ &\leqslant \mathbf{m}^{L,\theta}_{(t;s)}(S)\cdot\mathbf{m}^{L,\theta}_{(s;q)}(T)\cdot\delta^L_{q\theta}(\sigma,x',x'').\end{aligned}$$

Finally, it follows from Definition 4.4 that

$$\mathbf{m}^{L,\theta}_{(t;q)}(S\circ T) \leqslant \mathbf{m}^{L,\theta}_{(t;s)}(S)\cdot\mathbf{m}^{L,\theta}_{(s;q)}(T). \quad \square$$

Now we finish this section by presenting the relationship between the Lipschitz $(s;q,\theta)$-mixing map and its Lipschitz dual. We start by recalling the definitions of the (s,q)-type linear operators. The absolute moments

$$c_{sq} = \left(\int_{\mathbb{R}}|l|^q\,d\mu_s(l)\right)^{\frac{1}{p}} = 2\cdot\left[\frac{\Gamma\left(\frac{s-q}{s}\right)\cdot\Gamma\left(\frac{1+q}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}\right]^{\frac{1}{q}}$$

exist for $0<q<s<2$. Let $0<q<s<2$. For arbitrary finite sequence $x=(x_k)_{k=1}^n \subset E$, we put

$$t_{(s,q)}(x) = c_{sq}^{-1} \cdot \left(\int_{\mathbb{R}^n} \left\| \sum_{k=1}^n t_k \cdot x_k \right\|^q d\mu_s^n(t) \right)^{\frac{1}{q}},$$

where $t=(t_1,\ldots,t_n)\in\mathbb{R}^n$ and μ_s^n stand for the n-fold product of s-stable laws μ_s.

An operator $u: E \longrightarrow F$ between Banach spaces is called (s,q)-type if there exists a constant C such that

$$t_{(s,q)}(ux) \leqslant C \, \|x|\ell_s(E)\| \tag{4.6}$$

for arbitrary finite sequences $x=(x_j)_j$ in E; $k=1,\ldots,n$ and $n\in\mathbb{N}$. We put $\mathbf{T}_{(s,q)}(T) = \inf C$.

The class of these operators is represented by $\mathfrak{T}_{(s,q)}$. The following proposition can be proved as in [14, Corollary 4.3].

PROPOSITION 4.9. *Let $0<q<s<2$ and $0\leqslant\theta<1$. If the Lipschitz dual operator $T^\#: Y^\# \longrightarrow X^\#$ of the map $T\in Lip(X,Y)$ is of (s,q)-type, then T is a Lipschitz $(s;q,\theta)$-mixing map. Moreover*

$$\mathbf{m}_{(s;q)}^{L,\theta}(T) \leqslant Lip(T)^\theta \cdot \mathbf{T}_{(s,q)}(T^\#)^{1-\theta}.$$

5. A quotient Lipschitz theorem

We generalize the concept of quotients Lipschitz ideals between arbitrary metric spaces and Banach spaces presented by the third author in [13, Sec.4.5], to quotients Lipschitz ideals between arbitrary metric spaces. As an example, we show that the concept of $(s;q,\theta)$-mixing maps provides a Lipschitz extension of quotient theorem. We start by recalling the necessary theory of *Lipschitz operator ideals* (see [3, 13]) and then introducing the main definition.

DEFINITION 5.1. A *Lipschitz operator ideal* $\mathscr{I}_{Lip}$ is a subclass of Lip such that for every pointed metric space X and every Banach space E the components

$$\mathscr{I}_{Lip}(X,E) := Lip(X,E) \cap \mathscr{I}_{Lip}$$

satisfy:

(i) $\mathscr{I}_{Lip}(X,E)$ is a linear subspace of $Lip(X,E)$.

(ii) $vg \in \mathscr{I}_{Lip}(X,E)$ for $v\in E$ and $g\in X^\#$.

(iii) The ideal property: if $S\in Lip(Y,X)$, $T\in\mathscr{I}_{Lip}(X,E)$ and $w\in\mathscr{L}(E,F)$, then the composition $w\circ T\circ S$ is in $\mathscr{I}_{Lip}(Y,F)$.

A Lipschitz operator ideal $\mathscr{I}_{Lip}$ is a normed (Banach) Lipschitz operator ideal if there is $\|.\|_{\mathscr{I}_{Lip}}: \mathscr{I}_{Lip} \longrightarrow [0,+\infty[$ that satisfies

(i') For every pointed metric space X and every Banach space E, the pair $(\mathscr{I}_{Lip}(X,E), \|.\|_{\mathscr{I}_{Lip}})$ is a normed (Banach) space and $Lip(T) \leqslant \|T\|_{\mathscr{I}_{Lip}}$ for all $T \in \mathscr{I}_{Lip}(X,E)$.

(ii') $\|Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathscr{I}_{Lip}} = 1$.

(iii') If $S \in Lip(Y,X)$, $T \in \mathscr{I}_{Lip}(X,E)$ and $w \in \mathscr{L}(E,F)$, then

$$\|w \circ T \circ S\|_{\mathscr{I}_{Lip}} \leqslant Lip(S)\,\|T\|_{\mathscr{I}_{Lip}}\,\|w\|.$$

We say that a Lipschitz operator Banach ideal $\mathscr{I}_{Lip}$ is **strong** if, $S \in Lip(Y,X)$, $T \in \mathscr{I}_{Lip}(X,E)$ and $R \in Lip(E,F)$, then the composition $R \circ T \circ S$ belongs to $\mathscr{I}_{Lip}(Y,F)$ and

$$\|R \circ T \circ S\|_{\mathscr{I}_{Lip}} \leqslant Lip(S)\,\|T\|_{\mathscr{I}_{Lip}}\,Lip(R).$$

Suppose that, for every pair of metric spaces X and Y, the class

$$\mathscr{L} := \bigcup_{X,Y} \mathscr{L}(X,Y)$$

stands for all Lipschitz maps acting between arbitrary metric spaces X and Y. The Lipschitz operator ideal between arbitrary metric spaces X and Y as follows.

DEFINITION 5.2. Suppose that, for every pair of metric spaces X and Y, we are given a subset $\mathscr{I}_{Lip}(X,Y)$ of $\mathscr{L}(X,Y)$. The class

$$\mathscr{I}_{Lip} := \bigcup_{X,Y} \mathscr{I}_{Lip}(X,Y)$$

is said to be a Lipschitz operator ideal, if the following conditions are satisfied:

(**i**) If $Y = F$ is a Banach space, then $ge \in \mathscr{I}_{Lip}(X,F)$ for $g \in X^{\#}$ and $e \in F$.

(**ii**) $BTA \in \mathscr{I}_{Lip}(X_0,Y_0)$ for $A \in \mathscr{L}(X_0,X)$, $T \in \mathscr{I}_{Lip}(X,Y)$, and $B \in \mathscr{L}(Y,Y_0)$.

Condition (**i**) implies that $\mathscr{I}_{Lip}$ contains nonzero Lipschitz operators.

DEFINITION 5.3. Let $\mathscr{I}_{Lip}$ be a Lipschitz operator ideal between arbitrary metric spaces. A Lipschitz map $T \in Lip(X,Y)$ belongs to the quotient ${\mathscr{I}_{Lip}}^{-1} \circ \mathscr{I}_{Lip}$ if $S \circ T \in \mathscr{I}_{Lip}(X,Z)$ for all $S \in \mathscr{I}_{Lip}(Y,Z)$, where Z is an arbitrary pointed metric space.

It is not difficult to prove that

PROPOSITION 5.4. *${\mathscr{I}_{Lip}}^{-1} \circ \mathscr{I}_{Lip}$ is a Lipschitz operator ideal between arbitrary metric spaces.*

The proof of the next proposition is similar to [13, Proposition 30] and will be omitted.

PROPOSITION 5.5. *Let $\mathscr{I}_{Lip}$ be a strong Lipschitz operator Banach ideal. Then, ${\mathscr{I}_{Lip}}^{-1} \circ \mathscr{I}_{Lip}$ is a Lipschitz operator Banach ideal. Moreover, we have*

$$\|T\|_{\mathscr{I}_{Lip}^{-1} \circ \mathscr{I}_{Lip}} := \sup\left\{ \|S \circ T\|_{\mathscr{I}_{Lip}} : S \in \mathscr{I}_{Lip}(E,F), \|S\|_{\mathscr{I}_{Lip}} \leqslant 1 \right\}$$

where F is an arbitrary Banach space.

Now we are ready to present the main results of this section. In order to prove this theorem we need the following preliminary results for (q,θ)-absolutely Lipschitz mapping below was first given by Achour et al. in [2].

DEFINITION 5.6. Let $1 \leqslant q < \infty$ and $0 \leqslant \theta < 1$. Let X and Y be pointed metric spaces. A map $T \in \mathrm{Lip}(X,Y)$ is called (q,θ)-absolutely Lipschitz, if there exists a constant $K \geqslant 0$, a pointed metric space G and a Lipschitz operator $S \in \Pi_q^L(X,G)$ such that

$$d(T(x),T(x')) \leqslant K \cdot d(S(x),S(x'))^{1-\theta} \cdot d(x,x')^{\theta}$$

for all $x,x' \in X$. In this case $\pi_{q,\theta}^L(T)$ denotes the infimum of all $K\pi_q^L(S)^{1-\theta}$, where S differs over all Lipschitz p-summing operators defined on X that fulfill the above condition.

The space of all (q,θ)-absolutely Lipschitz mappings between pointed metric spaces X and Y is denoted by $\Pi_{q,\theta}^L(X,Y)$. Recall that the be pointed metric space X is isometrically embedded in $\AE(X)$ via the mapping $\delta_X : X \longrightarrow \AE(X)$ given by $\delta_X(x) = m_{x0}$ (see [19]). Let μ be a Borel probability measure on $B_{X^{\#}}$. Consider the canonical inclusion $i : \AE(X) \longrightarrow C(B_{X^{\#}})$, given by $i(\sum_{j=1}^{n} \lambda_j m_{x_j x'_j}) := \sum_{j=1}^{n} \lambda_j \langle m_{x_j x'_j}, \cdot \rangle$. Following [2, Section 4], on $i(\AE(X))$

$$\|i(m)\|_{q,\theta} := \inf\left\{ \sum_{j=1}^{n} |\lambda_j| \, d(x_j,x'_j)^{\theta} \left(\int_{B_{X^{\#}}} |f(x_j) - f(x'_j)|^p \, d\mu(f) \right)^{\frac{1-\theta}{q}} \right\} \tag{5.1}$$

where the infimum is taken over all representations of m of the form $m = \sum_{j=1}^{n} \lambda_j m_{x_j x'_j}$. Consider on $i \circ \delta_X(X)$ the pseudo-metric induced by $\|\cdot\|_{q,\theta}$:

$$d_{q,\theta}(i \circ \delta_X(x), i \circ \delta_X(x')) := \left\| i \circ \delta_X(x) - i \circ \delta_X(x') \right\|_{q,\theta}$$

and the relation of equivalence $\mathscr{R}$ given by

$$i \circ \delta_X(x) : \mathscr{R} : i \circ \delta_X(x') \Leftrightarrow d_{q,\theta}(i \circ \delta_X(x), i \circ \delta_X(x')) = 0.$$

We put $X_{q,\theta}^{\mu} := \frac{i \circ \delta_X(X)}{\mathscr{R}}$ and let $\phi : i(\delta_X(X)) \longrightarrow X_{q,\theta}^{\mu}$ be the projection. The following results is due to Achour et al. in [2, Theorem 2.4 and Theorem 3.1].

THEOREM 5.7. *Let* $1 \leqslant q < \infty$, $0 \leqslant \theta < 1$ *and* $T \in \mathrm{Lip}(X,Y)$. *The following statements are equivalent.*

(i) $T \in \Pi_{q,\theta}^L(X,Y)$.

(ii) There is a constant $C \geqslant 0$ *and a regular Borel probability measure* μ *on* $B_{X^{\#}}$ *such that*

$$d(T(x),T(x')) \leqslant C \left(\int_{B_{X^{\#}}} \left(|f(x) - f(x')|^{1-\theta} d(x,x')^{\theta} \right)^{\frac{q}{1-\theta}} d\mu(f) \right)^{\frac{1-\theta}{q}}$$

for all $x,x' \in X$.

(iii) *There is a constant* $C \geqslant 0$ *such that for all* $(x_i)_{i=1}^n, (x_i')_{i=1}^n$ *in* X *and all* $(a_i)_{i=1}^n \subset \mathbb{R}^+$ *we have*

$$\left(\sum_{i=1}^n a_i d(T(x_i), T(x_i'))^{\frac{q}{1-\theta}}\right)^{\frac{1-\theta}{q}} \leqslant C \sup_{f \in B_{X^\#}} \left(\sum_{i=1}^n a_i \left(|f(x_i) - f(x_i')|^{1-\theta} d(x_i, x_i')^\theta\right)^{\frac{q}{1-\theta}}\right)^{\frac{1-\theta}{q}}.$$

(iv) *There exists a regular Borel probability measure* μ *on* $B_{X^\#}$ *and a Lipschitz operator* $v : X_{q,\theta}^\mu \to Y$ *such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ \Big\downarrow \delta_X & & \Big\uparrow v \\ \delta_X(X) & \xrightarrow{\quad \phi \circ i \quad} & X_{q,\theta}^\mu \end{array}$$

Furthermore, the infimum of the constants $C \geqslant 0$ *in* (2) *and* (3) *is* $\pi_{q,\theta}^L(T)$.

THEOREM 5.8. *Let* $1 < q \leqslant s \leqslant \infty$, $0 \leqslant \theta < 1$, $C \geqslant 0$ *and let* T *a Lipschitz mapping from* X *to* Y. *The following are equivalent:*

a) *For any pointed metric space* Z *and any* (s,θ)*-absolutely Lipschitz is mapping* $S : Y \longrightarrow Z$, *the composition* $S \circ T$ *is* (q,θ)*-absolutely Lipschitz mapping and* $\pi_{q,\theta}^L(S \circ T) \leqslant C \pi_{s,\theta}^L(S)$.

b) *There is a constant* $C \geqslant 0$ *such that for any probability measure* $\nu \in W(B_{Y^\#})$ *there exists* $\mu \in W(B_{X^\#})$ *such that for every* x', x'' *in* X.

$$\inf \left\{ \sum_{j=1}^n \left[\int_{B_{Y^\#}} |\lambda_j|^{\frac{s}{1-\theta}} \left| \left\langle g, y_j' \right\rangle_{(Y^\#,Y)} - \left\langle g, y_j'' \right\rangle_{(Y^\#,Y)} \right|^s d_Y(y_j', y_j'')^{\frac{\theta s}{1-\theta}} d\nu(g) \right]^{\frac{1-\theta}{s}} : m_{Tx'Tx''} = \sum_{j=1}^n \lambda_j m_{y_j' y_j''} \right\}$$
$$\leqslant C \cdot \inf \left\{ \sum_{j=1}^n \left[\int_{B_{X^\#}} |\lambda_j'|^{\frac{q}{1-\theta}} \left| \left\langle f, x_j' \right\rangle_{(X^\#,X)} - \left\langle f, x_j'' \right\rangle_{(X^\#,X)} \right|^q d_X(x_j', x_j'')^{\frac{\theta q}{1-\theta}} \, d\mu(f) \right]^{\frac{1-\theta}{q}} : m_{x'x''} = \sum_{j=1}^n \lambda_j' m_{x_j' x_j''} \right\} \tag{5.2}$$

c) T *is* (s,q,θ)*-mixing Lipschitz with* $\mathbf{m}_{(s;q)}^{L,\theta}(T) \leqslant C$.

Furthermore, the infimum of the constants C *in either (a), (b) or (c) is* $\mathbf{m}_{(s;q)}^{L,\theta}(T)$.

Proof. (a)$\Longrightarrow$(b) Let ν be a probability measure on $B_{Y^\#}$. By using (iv) in Theorem 5.7 we get a (s,θ)-absolutely Lipschitz operator $J_\nu^\theta = \phi \circ i \circ \delta_Y : Y \longrightarrow Y_{q,\theta}^\nu$ with $\pi_{s,\theta}^L(J_\nu^\theta) \leqslant 1$. Hence, from (a), the composition $J_\nu^\theta \circ T$ is (q,θ)-absolutely Lipschitz map with

$$\pi_{q,\theta}^L(J_\nu^\theta \circ T) \leqslant C.\pi_{s,\theta}^L(J_\nu^\theta) \leqslant C.$$

By the Lipschitz Pietsch Domination Theorem (ii) in Theorem 5.7, there is a probability measure μ on $B_{X^\#}$ such that for all x', x'' in X,

$$\begin{aligned}&\left\|J_\nu^\theta(Tx') - J_\nu^\theta(Tx'')\right\|_{s,\theta}\\ &\leqslant \pi_{q,\theta}^L(J_\nu^\theta \circ T)\left[\int_{B_{X^\#}} \left(\left|f(x') - f(x'')\right|^{1-\theta} d_X(x',x'')^\theta\right)^{\frac{q}{1-\theta}} d\mu(f)\right]^{\frac{1-\theta}{q}}\end{aligned}$$

i.e.

$$\begin{aligned}d_{s,\theta}(i\circ\delta_Y(Tx'), i\circ\delta_X(Tx'')) &= \|i(m_{Tx'Tx''})\|_{s,\theta}\\ &\leqslant C\left[\int_{B_{X^\#}} \left(\left|f(x') - f(x'')\right|^{1-\theta} d_X(x',x'')^\theta\right)^{\frac{q}{1-\theta}} d\mu(f)\right]^{\frac{1-\theta}{q}}.\end{aligned}$$

Therefore, by (5.1), we get

$$\begin{aligned}&\|i(m_{Tx'Tx''})\|_{s,\theta}\\ &= \inf\left\{\begin{array}{c}\sum_{j=1}^n |\lambda_j|\, d_Y(y'_j,y''_j)^\theta \left[\int_{B_{Y^\#}} \left|\left\langle g, y'_j\right\rangle_{(Y^\#,Y)} - \left\langle g, y''_j\right\rangle_{(Y^\#,Y)}\right|^s d\nu(g)\right]^{\frac{1-\theta}{s}}\\ : m_{Tx'Tx''} = \sum_{j=1}^n \lambda_j m_{y'_j y''_j}\end{array}\right\}.\end{aligned}$$

We conclude that

$$\begin{aligned}&\inf\left\{\begin{array}{c}\sum_{j=1}^n \left[\int_{B_{Y^\#}} |\lambda_j|^{\frac{s}{1-\theta}} \left|\left\langle g, y'_j\right\rangle_{(Y^\#,Y)} - \left\langle g, y''_j\right\rangle_{(Y^\#,Y)}\right|^s d_Y(y'_j,y''_j)^{\frac{\theta s}{1-\theta}} d\nu(g)\right]^{\frac{1-\theta}{s}}\\ : m_{Tx'Tx''} = \sum_{j=1}^n \lambda_j m_{y'_j y''_j}\end{array}\right\}\\ &\leqslant C\left[\int_{B_{X^\#}} \left(\left|f(x') - f(x'')\right|^{1-\theta} d_X(x',x'')^\theta\right)^{\frac{q}{1-\theta}} d\mu(f)\right]^{\frac{1-\theta}{q}}. \qquad (5.3)\end{aligned}$$

Let $m_{x'x''} = \sum_{j=1}^n \lambda'_j m_{x'_j x''_j}$, so the second part of inequality (5.3) becomes as follows

$$C\Big[\int_{B_{X^\#}} \left(\left|f(x')-f(x'')\right|^{1-\theta} d_X(x',x'')^{\theta}\right)^{\frac{q}{1-\theta}} d\mu(f)\Big]^{\frac{1-\theta}{q}}$$

$$= C\Big[\int_{B_{X^\#}} \left(|\langle f, m_{x'x''}\rangle|^{1-\theta} \|m_{x'x''}\|^{\theta}\right)^{\frac{q}{1-\theta}} d\mu(f)\Big]^{\frac{1-\theta}{q}}$$

$$\overset{(*)}{=} C\Big[\int_{B_{X^\#}} \left(\left|\left\langle f, \sum_{j=1}^{n} \lambda_j' m_{x_j',x_j''})\right\rangle\right|^{1-\theta} \left\|\sum_{j=1}^{n} \lambda_j' m_{x_j'x_j''}\right\|^{\theta}\right)^{\frac{q}{1-\theta}} d\mu(f)\Big]^{\frac{1-\theta}{q}}.$$

So, by the triangular inequality

$$(*) \leqslant C\Big[\int_{B_{X^\#}} \Big[(\sum_{j=1}^{n} |\lambda_j'|)^{1-\theta} \left|\left\langle f, m_{x_j'x_j''})\right\rangle\right|^{1-\theta} (\sum_{j=1}^{n} |\lambda_j'|)^{\theta} \left\|m_{x_j'x_j''}\right\|^{\theta}\Big]^{\frac{q}{1-\theta}} d\mu(f)\Big]^{\frac{1-\theta}{q}}$$

$$= C\Big[\int_{B_{X^\#}} \left(\sum_{j=1}^{n} |\lambda_j'| \left|\left\langle f, m_{x_j'x_j''})\right\rangle\right|^{1-\theta} \left\|m_{x_j'x_j''}\right\|^{\theta}\right)^{\frac{q}{1-\theta}} d\mu(f)\Big]^{\frac{1-\theta}{q}}$$

$$\leqslant C\sum_{j=1}^{n}\Big[\int_{B_{X^\#}} |\lambda_j'|^{\frac{q}{1-\theta}} \left|\left\langle f, m_{x_j'x_j''})\right\rangle\right|^{q} \left\|m_{x_j'x_j''}\right\|^{\frac{\theta q}{1-\theta}} d\mu(f)\Big]^{\frac{1-\theta}{q}}$$

$$= C\sum_{j=1}^{n}\Big[\int_{B_{X^\#}} |\lambda_j'|^{\frac{q}{1-\theta}} \left|\langle f, x_j'\rangle_{(X^\#,X)} - \langle f, x_j''\rangle_{(X^\#,X)}\right|^{q} d_X(x_j',x_j'')^{\frac{\theta q}{1-\theta}} d\mu(f)\Big]^{\frac{1-\theta}{q}}.$$

Taking the infimum over all representations of $m_{x'x''}$, we get

$$(*) \leqslant C\cdot\inf\left\{\begin{array}{c}\sum_{j=1}^{n}\Big[\int_{B_{X^\#}} \left|\lambda_j'\right|^{\frac{q}{1-\theta}} \left|\left\langle f, x_j'\right\rangle_{(X^\#,X)} - \left\langle f, x_j''\right\rangle_{(X^\#,X)}\right|^{q} d_X(x_j',x_j'')^{\frac{\theta q}{1-\theta}} d\mu(f)\Big]^{\frac{1-\theta}{q}} \\ : m_{x'x''} = \sum_{j=1}^{n} \lambda_j' m_{x_j'x_j''}\end{array}\right\}.$$

By (5.3) and $(*)$ we have the condition (b).

(b)$\Longrightarrow$(c) Let σ in $\mathbb{R}$ and x',x'' in X. Therefore, for all $\varepsilon > 0$, there exists a representation of $m_{xx'}$ and $m_{Tx'Tx''}$ such that

$$\left(\sum_{j=1}^{m}\Big[\int_{B_{Y^\#}} |\sigma_j|^{\frac{s}{1-\theta}} \left|\langle g, Tx_j'\rangle_{(Y^\#,Y)} - \langle g, Tx_j''\rangle_{(Y^\#,Y)}\right|^{s} d_Y(Tx_j',Tx_j'')^{\frac{\theta s}{1-\theta}} d\nu(g)\Big]^{\frac{q}{s}}\right)^{\frac{1-\theta}{q}}$$

$$< C\cdot\Big[\sum_{j=1}^{m}\int_{B_{X^\#}} |\sigma_j|^{\frac{q}{1-\theta}} \left|\langle f, x_j'\rangle_{(X^\#,X)} - \langle f, x_j''\rangle_{(X^\#,X)}\right|^{q} d_X(x_j',x_j'')^{\frac{\theta q}{1-\theta}} d\mu(f)\Big]^{\frac{1-\theta}{q}} + \varepsilon$$

$$\leqslant C\cdot \delta_{q,\theta}^{L}(\sigma,x',x'') + \varepsilon. \tag{5.4}$$

Letting $\varepsilon \longrightarrow 0$ and taking the supremum over all $\nu \in W(B_{Y^\#})$ on the left side of (5.4), we get

$$\mathfrak{m}^L_{(s;q,\theta)}(\sigma, Tx', Tx'') \leqslant C \cdot \delta^L_{q,\theta}(\sigma, x', x'').$$

(c)$\Longrightarrow$(a) Assume that T is a Lipschitz $(s;q,\theta)$-mixing map. Then, by Proposition 4.3, we have

$$\left[\sum_{j=1}^{m}\left[\int_{B_{Y^\#}} |\sigma_j|^{\frac{s}{1-\theta}} \left|g(Tx'_j) - g(Tx''_j)\right|^s d_Y(Tx'_j, Tx''_j)^{\frac{\theta s}{1-\theta}} d\nu(g)\right]^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}}$$
$$\leqslant \mathbf{m}^{L,\theta}_{(s;q)}(T) \cdot \delta^L_{q,\theta}(\sigma, x', x''). \tag{5.5}$$

Now, let $S : Y \longrightarrow Z$ be a (s,θ)-absolutely Lipschitz mapping. By the Lipschitz Pietsch Domination theorem (ii) in Theorem 5.7, there is a probability measure ν on $B_{Y^\#}$ such that for all y', y'' in Y,

$$d_Z(Sy', Sy'') \leqslant \pi^L_{s,\theta}(S)\left[\int_{B_{Y^\#}} \left(\left|g(y') - g(y'')\right|^{1-\theta} d_Y(y', y'')^{\theta}\right)^{\frac{s}{1-\theta}} d\nu(g)\right]^{\frac{1-\theta}{s}}.$$

Then

$$\left[\sum_{j=1}^{m} |\sigma_j|^{\frac{q}{1-\theta}} d_Z(Sy'_j, Sy''_j)^{\frac{q}{1-\theta}}\right]^{\frac{1-\theta}{q}}$$
$$\leqslant \pi^{L,\theta}_s(S)\left[\sum_{j=1}^{m} |\sigma_j|^{\frac{s}{1-\theta}} \left(\int_{B_{Y^\#}} \left(\left|g(y'_j) - g(y''_j)\right|^{1-\theta} d_Y(y'_j, y''_j)^{\theta}\right)^{\frac{s}{1-\theta}} d\nu(g)\right)^{\frac{q}{s}}\right]^{\frac{1-\theta}{q}} \tag{5.6}$$

for all $\sigma_1, \ldots, \sigma_m \in \mathbb{R}$, $x'_1, \ldots, x'_m, x''_1, \ldots, x''_m \in X$ and $m \in \mathbb{N}$. Then, from the inequalities (5.6) and (5.5), we have

$$\left[\sum_{j=1}^{m} |\sigma_j|^{\frac{q}{1-\theta}} d_Z\left(S \circ T(x'_j), S \circ T(x''_j)\right)^{\frac{q}{1-\theta}}\right]^{\frac{1-\theta}{q}} \leqslant \pi^{L,\theta}_s(S) \cdot \mathbf{m}^{L,\theta}_{(s;q)}(T) \cdot \delta^L_{q\theta}(\sigma, x', x''). \tag{5.7}$$

Hence $S \circ T$ is (q,θ)-absolutely Lipschitz map with

$$\pi^L_{q,\theta}(S \circ T) \leqslant \pi^L_{s,\theta}(S) \cdot \mathbf{m}^{L,\theta}_{(s;q)}(T). \tag{5.8}$$

□

As immediate consequence of the above theorem, the following corollary holds.

COROLLARY 5.9. *Let* $1 \leqslant q < \infty$ *and* $0 \leqslant \theta < 1$. *Then*

$$\mathbf{M}_{(s;q)}^{L,\theta}(X,Y) = \Pi_{s,\theta}^{L}(Y,Z)^{-1} \circ \Pi_{q,\theta}^{L}(X,Z)).$$

As (q,θ)-absolutely Lipschitz mappings is a strong Banach Lipschitz ideal [2, Remark 2.7], the above corollary and the Proposition 5.5 given

PROPOSITION 5.10. *Let* $1 \leqslant q,s \leqslant \infty$ *and* $0 \leqslant \theta < 1$.

1. $\left(\mathbf{M}_{(s;q)}^{L,\theta}(X,E), \mathbf{m}_{(s;q)}^{L,\theta}(\cdot)\right)$ *is a strong Lipschitz operator Banach ideal.*
2. *For any* $T \in \mathrm{Lip}(X,Y)$, $\mathbf{m}_{(s;q)}^{L,\theta}(T) = Lip(T)$ *whenever* $s \leqslant p$ *and* $\mathbf{m}_{(\infty;q)}^{L,\theta}(T) = \pi_{q,\theta}^{L}(T)$, *so only the case* $1 \leqslant q < s < \infty$ *gives something new.*

The main relationship between the Lipschitz (s,q)-mixing and the Lipschitz (s,q,θ)-mixing mappings is the following.

PROPOSITION 5.11. *Every Lipschitz* (s,q)*-mixing map* $T : X \longrightarrow Y$ *is Lipschitz* (s,q,θ)*-mixing and satisfies*

$$\mathbf{m}_{(s;q)}^{L,\theta}(S) \leqslant Lip(S)^{\theta}.\mathbf{m}_{(s;q)}^{L}(S)^{1-\theta}.$$

Proof. Let $T \in \mathbf{M}_{(s;q)}^{L}(X,Y)$ and $0 \leqslant \theta < 1$. Consider $S \in \Pi_{s,\theta}^{L}(Y,Z)$, then by Definition 5.6 there exists an $S_0 \in \Pi_{s}^{L}(Y,Z_0)$ satisfying

$$d(S(y),S(y')) \leqslant Kd(S_0(y),S_0(y'))^{1-\theta}d(y,y')^{\theta} \tag{5.9}$$

for all $y,y' \in Y$ and

$$\pi_{s,\theta}^{L}(S) = \inf\left\{K.\pi_{s}^{L}(S_0)^{1-\theta}\right\} \tag{5.10}$$

We show that $S \circ T \in \Pi_{q,\theta}^{L}(X,Z)$. For all $x,x' \in X$, then from (5.9) we have

$$\begin{aligned}
&d(S\circ T(x),S\circ T(x')) \\
&\leqslant Kd\left(S_0\circ T(x),S_0\circ T(x')\right)^{1-\theta}d(T(x),T(x'))^{\theta} \\
&\leqslant K.Lip(T)^{\theta}.d(R(x),R(x'))^{1-\theta}d(x,x'))^{\theta},
\end{aligned}$$

where $R = S_0 \circ T$.

Since $T \in \mathbf{M}_{(s;q)}^{L}(X,Y)$ it follows that $R = S_0 \circ T \in \Pi_{q}^{L}(X,Z)$ (see [5]) and

$$\pi_{q}^{L}(R) \leqslant \mathbf{m}_{(s;q)}^{L}(T).\pi_{s}^{L}(S_0).$$

Again, by the Definition 5.6, $S \circ T$ is (q,θ)-absolutely Lipschitz and

$$\begin{aligned}
\pi_{q,\theta}^{L}(S\circ T) &\leqslant K.Lip(T)^{\theta}.\pi_{q}^{L}(R)^{1-\theta} \\
&\leqslant Lip(T)^{\theta}.\mathbf{m}_{(s;q)}^{L}(T)^{1-\theta}.(K.\pi_{s}^{L}(S_0)^{1-\theta}).
\end{aligned}$$

Taking the infimum on both sides over all $S_0 \in \Pi_s^L(Y, Z_0)$ and $K \geqslant 0$ satisfying (5.9), we get

$$\pi_{q,\theta}^L(S \circ T) \leqslant Lip(T)^\theta . \mathbf{m}_{(s;q)}^L(T)^{1-\theta} . \pi_{s,\theta}^L(S).$$

Then by (1) in Theorem 5.8 we have $T \in \mathbf{M}_{(s;q)}^{L,\theta}(X,Y)$ with

$$\mathbf{m}_{(s;q)}^{L,\theta}(T) \leqslant Lip(T)^\theta . \mathbf{m}_{(s;q)}^L(T)^{1-\theta}. \quad \square$$

In [6] vector valued molecules were naturally considered. An E-valued molecule on X is a finitely supported map $m : X \longrightarrow E$ such that $\sum_{x \in X} m(x) = 0$. The vector space of all E-valued molecules on X is designated by $\mathscr{M}(X,E)$. Let $1 \leqslant r < \infty$ such that $r^* = \frac{p^*}{1-\theta}$, for a molecule $m \in \mathscr{M}(X,E)$, the (p,θ)-Chevet-Saphar norm is given by

$$cs_{p,\theta}(m) = \inf\left\{ \left\| (\lambda_j \left\| v_j \right\|)_{j=1}^n \right\|_r \delta_{p^*,\theta}^{Lip}\left((\lambda_j^{-1}, x_j, x_j')_{j=1}^n \right) : m = \sum_{j=1}^n v_j m_{x_j x_j'}, \lambda_j > 0 \right\}.$$

We denote by $CS_{p,\theta}(X,E)$ the space $\mathscr{M}(X,E)$ endowed with the norm $cs_{p,\theta}$. By [2, Theorem 3.3] the spaces $CS_{p,\theta}(X,E)^*$ and $\Pi_{p^*,\theta}^L(X,E^*)$ are isometrically isomorphic via the canonical pairing,

$$\langle m, S \rangle = \sum_{j=1}^n \left\langle v_j, S(x_j) - S(x_j') \right\rangle. \tag{5.11}$$

Following [6], recall that for any Banach space G, the Lipschitz mapping $S : X \to Y$ induces a well-defined linear mapping $S_G : \mathscr{M}(X,G) \to \mathscr{M}(Y,G)$ given by

$$S_G\left(\sum_{j=1}^n v_j m_{x_j x_j'}\right) = \sum_{j=1}^n v_j m_{S(x_j) S(x_j')}.$$

The following theorem can be proved as in [5].

THEOREM 5.12. *Let $1 \leqslant p < \infty$, $0 \leqslant \theta < 1$ and $S \in \mathrm{Lip}(X,Y)$. The following statements are equivalent.*

1. *S is Lipschitz (s,p,θ)-mixing map.*

2. *For every Banach space G, the operator*

$$S_G : CS_{p^*,\theta}(X,G) \to CS_{s^*,\theta}(Y,G)$$

is continuous and $\left\| S_G : CS_{p^*,\theta}(X,G) \to CS_{s^*,\theta}(Y,G) \right\| \leqslant \mathbf{m}_{(s;p)}^{L,\theta}(S)$.

Proof. The necessity condition, suppose that S is a Lipschitz (s,p,θ)-mixing map. Let $\varphi \in \mathscr{CS}_{s^*,\theta}(Y,G)^*$ with $\|\varphi\| \leqslant 1$. Since $\mathscr{CS}_{s^*,\theta}(Y,G)^* \equiv \Pi^L_{s,\theta}(Y,G^*)$ by [2, Theorem 3.3], we can identify φ with a map $L_\varphi \in \Pi^L_{s,\theta}(Y,G^*)$ with $\pi^L_{s,\theta}(L_\varphi) = \|\varphi\| \leqslant 1$. Let $\mathbf{m}$ be a $G-$valued molecule on X, say $\mathbf{m} = \sum_{j=1}^{m} v_j \mathbf{m}_{x'_j x''_j}$ with $x'_j, x''_j \in X$ and $v_j \in G$. Then $S_G(\mathbf{m}) = \sum_{j=1}^{m} v_j \mathbf{m}_{Sx'_j Sx''_j}$. The pairing formula defined in (5.11), the Hölder inequality $(\frac{1}{r^*} + \frac{1-\theta}{p} = 1)$ and Theorem 5.8 naturally come together to give us

$$\langle \varphi, S_G(\mathbf{m})\rangle = \sum_{j=1}^{m} \left\langle L_\varphi(Sx'_j) - L_\varphi(Sx''_j), v_j \right\rangle = \left\langle L_\varphi \circ S, \mathbf{m} \right\rangle.$$

Hence

$$|\langle \varphi, S_G(\mathbf{m})\rangle| \leqslant \mathbf{m}^{L,\theta}_{(s;p)}(S) \cdot \delta^L_{p,\theta}\left(\frac{1}{\sigma}, x', x''\right) \cdot \left\| \sigma \cdot \|v\| \right|_{\ell_{r^*}} \Big\|. \tag{5.12}$$

Taking the infimum over all representations of $\mathbf{m}$ and $\sigma \subset \mathbb{R}$ on the right side of (5.12) and the supremum over all such φ on the left side of (5.12), we have

$$\sup_{\varphi \in B_{\mathscr{CS}_{s^*,\theta}(Y,G)^*}} |\langle \varphi, S_G(\mathbf{m})\rangle| \leqslant \mathbf{m}^{L,\theta}_{(s;p)}(S) \cdot cs_{p^*,\theta}(\mathbf{m}). \tag{5.13}$$

Then $cs_{s^*,\theta}(S_G(\mathbf{m})) \leqslant \mathbf{m}^{L,\theta}_{(s;p)}(S) \cdot cs_{p^*,\theta}(\mathbf{m})$ and $\|S_G\| \leqslant \mathbf{m}^{L,\theta}_{(s;p)}(S)$.

The sufficient condition, suppose that $S_G : CS_{p^*,\theta}(X,G) \to CS_{s^*,\theta}(Y,G)$ is a bounded linear operator. Let $T : Y \to G^*$ be a (s,θ)-absolutely Lipschitz operator. It suffices to show that $T \circ S \in \Pi^L_{p,\theta}(X,G^*)$. Assume $\mathbf{m}$ is an $G-$valued molecule on X, say $\mathbf{m} = \sum_{j=1}^{m} v_j \mathbf{m}_{x'_j x''_j}$ with $x'_j, x''_j \in X$ and $v_j \in G$. Then

$$\langle T \circ S, \mathbf{m}\rangle = \sum_{j=1}^{m} \left\langle v_j, (T\circ S)x'_j - (T \circ S)x''_j \right\rangle = \left\langle T, \sum_{j=1}^{m} v_j \mathbf{m}_{Sx'_j Sx''_j} \right\rangle = \langle T, S_G(\mathbf{m})\rangle.$$

By the duality between the (s,θ)-absolutely norm and the (s^*,Θ)-Chevet-Saphar norm, together with the boundedness of S_G,

$$\begin{aligned} |\langle T \circ S, \mathbf{m}\rangle| &= |\langle T, S_G(\mathbf{m})\rangle| \\ &\leqslant \pi^L_{s,\theta}(T).cs_{s^*,\theta}(S_G(\mathbf{m})) \\ &\leqslant \pi^L_{s,\theta}(T).\|S_G\|.cs_{p^*,\theta}(\mathbf{m}). \end{aligned}$$

Therefore, from the duality between $cs_{p^*,\theta}(.)$ and $\pi^L_{p,\theta}(.)$ after taking the supremum over all molecules $\mathbf{m}$ with $cs_{p^*,\theta}(.) \leqslant 1$ on both sides above, we obtain

$$\pi^L_{p,\theta}(T \circ S) \leqslant \|S_G\|.\pi^L_{s,\theta}(T).$$

By Theorem 5.8 we get that S is Lipschitz (s,p,θ)-mixing map with

$$\mathbf{m}^{L,\theta}_{(s;p)}(S) \leqslant \|S_G\|. \quad \square$$

REFERENCES

[1] D. ACHOUR, E. DAHIA, P. RUEDA AND E. A. SÁNCHEZ PÉREZ, *Factorization of strongly* (p,σ)*-continuous multilinear operators*, Linear and Multilinear Algebra **62**, No. 12 (2014) 1649–1670.

[2] D. ACHOUR, P. RUEDA AND R. YAHI, (p,σ)*-absolutely Lipschitz operators*, Ann. Funct. Anal. **8** (1) (2017), 38–50.

[3] D. ACHOUR, P. RUEDA, E. A. SÁNCHEZ PÉREZ AND R. YAHID, *Lipschitz operator ideals and the approximation property*, J. Math. Anal. Appl. **436** (2016), 217–236.

[4] R. F. ARENS AND J. EELS JR., *On embedding uniform and topological spaces*, Pacific J. Math **6** (1956), 397–403.

[5] J. A. CHÁVEZ-DOMÍNGUEZ, *Lipschitz* (q,p)*-mixing operators*, Proc. Amer. Math. Soc. **140** (2012), 3101–3115.

[6] J. A. CHÁVEZ-DOMÍNGUEZ, *Duality for Lipschitz* p*-summing operators*, J. Funct. Anal. **261** (2011), 387–407, Lipschitz (q,p)-mixing operators, Proc. Amer. Math. Soc. **140** (2012), 3101–3115.

[7] E. DAHIA, *On the tensorial representation of multi-linear ideals*, Ph. D. thesis, University of Mohamed Boudiaf, M'sila (Algeria), 2014.

[8] E. DAHIA, D. ACHOUR AND E. A. SANCHEZ-PEREZ, *Absolutely continuous multilinear operators*, J. Math. Anal. Appl. **397** (2013) 205–224.

[9] H. JARCHOW, U. MATTER, *Interpolative constructions for operator ideals*, Note. Mat. (1) **8** (1988), 45–56.

[10] J. A. LOPEZ MOLINA, AND E. A. SÁNCHEZ PÉREZ, *Ideales de operadores absolutamente continuos*, Rev. Real Acad. Ciencias Exactas, Pisicas Y Naturales, Madrid 87 (1993), 349–378.

[11] U. MATTER, *Absolute continuous operators and super-reflexivity*, Math. Nachr **130** (1987), 193–216.

[12] A. PIETSCH, *Operator Ideals*, Deutsch, Verlag Wiss, Berlin, 1978; North-Holland, Amsterdam-London-New York-Tokyo, 1980.

[13] M. A. S. SALEH, *Nonlinear operator ideals (part I)*, `arXiv:1507.00861 [math AF]`.

[14] M. A. S. SALEH, *New types of Lipschitz summing maps between metric spaces*, Math. Nachr **290** (2017), 1347–1373.

[15] E. A. SÁNCHEZ PÉREZ, *Products and quotients of* (p,σ)*-absolutely continuous operator ideals*, Tr. J. of Mathematics **20** (1996) 323–334.

[16] E. A. SÁNCHEZ PÉREZ, *Ideals de operadores absolutamente continuos y normas tensoriales asociadas*, PhD Thesis, Universidad politécnica de Valencia, Spain, 1997.

[17] I. SAWASHIMA, *Methods of Lipschitz duals*, Lecture Notes Ec. Math. Sust. **419**, Springer Verlag (1975), pp. 247–259.

[18] C. A. S. SOARES, *Aplicacoes multilineares e polinômios misto somantes*, PhD Thesis, Universidade Estadual de Campinas (UNICAMP), 1998.

[19] N. WEAVER, *Lipschitz Algebras*, World Scientific Publishing Co. Singapore (1999).

(Received December 26, 2016)

Dahmane Achour
Laboratoire d'Analyse Fonctionnelle et Géométrie des Espaces
University of M'sila
28000 M'sila, Algeria
e-mail: dachourdz@yahoo.fr

Elhadj Dahia
Laboratoire d'Analyse Fonctionnelle et Géométrie des Espaces
University of M'sila
28000 M'sila, Algeria
e-mail: hajdahia@gmail.com

M. A. S. Saleh
Department of Mathematics and Computer Applications
College of Science, Al-Nahrain University
Baghdad, Iraq
e-mail: mas@sc.nahrainuniv.edu.iq