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NORMS OF HYPONORMAL WEIGHTED COMPOSITION OPERATORS ON THE HARDY AND WEIGHTED BERGMAN SPACES

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Abstract. In this paper, first we find norms of hyponormal weighted composition operators $C_{\psi,\varphi}$, when φ has a Denjoy-Wolff point on the unit circle. Then for φ which is analytic self-map of $\mathbb{D}$ with a fixed point in $\mathbb{D}$, we investigate norms of hyponormal weighted composition operators $C_{\psi,\varphi}$.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The algebra $A(\mathbb{D})$ consists of all continuous functions on the closure of $\mathbb{D}$ that are analytic on $\mathbb{D}$. The Hilbert spaces of primary interest to us will be the Hardy space H^2 and the weighted Bergman spaces A^2_α. For f analytic on $\mathbb{D}$, we denote by $\hat{f}(n)$ the nth coefficient of the Maclaurin series of f. The Hardy space H^2 is the collection of all such functions f for which

$$\|f\|_1^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty.$$

For $\alpha > -1$, the weighted Bergman space A^2_α consists of all analytic f on $\mathbb{D}$ such that

$$\|f\|_{\alpha+2}^2 = \int_{\mathbb{D}} |f(z)|^2 (\alpha+1)(1-|z|^2)^\alpha dA(z) < \infty,$$

where dA is normalized area measure on $\mathbb{D}$. Throughout this paper, let $\gamma = 1$ for H^2 and $\gamma = \alpha+2$ for A^2_α. We know that both the weighted Bergman space and the Hardy space are reproducing kernel Hilbert spaces, when the reproducing kernel for evaluation at w is given by $K_w(z) = (1-\overline{w}z)^{-\gamma}$ for $z, w \in \mathbb{D}$. Also the norm of K_w is $(1-|w|^2)^{-\gamma/2}$. We write H^∞ for the space of bounded analytic functions on $\mathbb{D}$, with supremum norm $\|f\|_\infty$.

Let φ be an analytic self-map of $\mathbb{D}$. If H is a Hilbert space of analytic functions on $\mathbb{D}$, the composition operator C_φ on H is defined by the rule $C_\varphi(f) = f\circ\varphi$. Moreover, for an analytic function ψ on $\mathbb{D}$ and an analytic self-map φ of $\mathbb{D}$, we define the weighted composition operator $C_{\psi,\varphi}$ on H by $C_{\psi,\varphi} f = \psi(f\circ\varphi)$ for all $f \in H$. Such weighted composition operators are clearly bounded on H^2 and A^2_α when ψ

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is bounded on $\mathbb{D}$, but the boundedness of ψ on $\mathbb{D}$ is not necessary for $C_{\psi,\varphi}$ to be bounded. Moreover, for a bounded composition operator $C_{\psi,\varphi}$ on H^2 or A^2_α, it is well known and easy to see that

$$C^*_{\psi,\varphi}(K_w) = \overline{\psi(w)}K_{\varphi(w)}. \tag{1}$$

Let P denote the projection of $L^2(\partial\mathbb{D})$ onto H^2. For each $b \in L^\infty(\partial\mathbb{D})$, the Toeplitz operator T_b acts on H^2 by $T_b(f) = P(bf)$. Also suppose that P_α is the projection of $L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α. For each function $w \in L^\infty(\mathbb{D})$, the Toeplitz operator T_w on A^2_α is defined by $T_w(f) = P_\alpha(wf)$. Since P and P_α are bounded, the Toeplitz operators are bounded on H^2 and A^2_α.

A linear-fractional self-map of $\mathbb{D}$ is a map of the form

$$\varphi(z) = \frac{az+b}{cz+d}, \tag{2}$$

with $ad - bc \neq 0$, for which $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. We denote the set of those maps by LFT($\mathbb{D}$). It is well known that the automorphisms of $\mathbb{D}$, denoted Aut($\mathbb{D}$), are the maps in LFT($\mathbb{D}$) that take $\mathbb{D}$ onto itself, and that they are of the form $\varphi(z) = \lambda(a-z)/(1-\overline{a}z)$, where $|\lambda| = 1$ and $|a| < 1$ (see [6]). The automorphisms will be classified into three disjoint types:

- Elliptic if one fixed point is in the disk and the other is in the complement of the closed disk.
- Hyperbolic if both fixed points are on the unit circle.
- Parabolic if there is one fixed point on the unit circle of multiplicity 2.

Suppose $\varphi \in$ LFT($\mathbb{D}$) is as in Equation (2). It is well known that the adjoint of C_φ acting on H^2 and A^2_α is given by $C^*_\varphi = T_g C_\sigma T^*_h$, where $\sigma(z) := (\overline{a}z - \overline{c})/(-\overline{b}z + \overline{d})$ is a self-map of $\mathbb{D}$, $g(z) := (-\overline{b}z + \overline{d})^{-\gamma}$ and $h(z) := (cz+d)^\gamma$. Note that g and h are in H^∞ (see [9] and [19]). If $\varphi(\zeta) = \eta$ for $\zeta, \eta \in \partial\mathbb{D}$, then $\sigma(\eta) = \zeta$. We know that φ is an automorphism if and only if σ is, and in this case $\sigma = \varphi^{-1}$. The map σ is called the Krein adjoint of φ. We will refer to g and h as the Cowen auxiliary functions for φ. From now on, unless otherwise stated, we assume that σ, h and g are given as above.

We say that φ has a finite angular derivative at $\zeta \in \partial\mathbb{D}$ if the nontangential limit $\varphi(\zeta)$ exists, has modulus 1, and $\varphi'(\zeta) = \angle\lim_{z\to\zeta} \frac{\varphi(z)-\varphi(\zeta)}{z-\zeta}$ exists and finite.

Let φ_0 be the identity map and φ_n denote the n-th iterate of φ. It is well known that if φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map on the disk into itself, then there is a point c in $\overline{\mathbb{D}}$ so that the iterates φ_n of φ converge to c uniformly on compact subsets of $\overline{\mathbb{D}}$. The point c is called the Denjoy-Wolff point of φ. The Denjoy-Wolff point c is the unique fixed point of φ in $\overline{\mathbb{D}}$ such that $|\varphi'(c)| \leqslant 1$.

We say that an operator A on a Hilbert space H is hyponormal if $A^*A - AA^* \geqslant 0$, or equivalently if $\|A^*f\| \leqslant \|Af\|$ for all $f \in H$. Recall that an operator T on a Hilbert space H is said to be normal if $TT^* = T^*T$ on H. Also T is unitary if $TT^* = T^*T = I$. The normal composition operators on A^2_α and H^2 have symbol $\varphi(z) = az$, where $|a| \leqslant 1$ (see [12, Theorem 8.2]). Also, it is easy to see that only the rotation maps $\varphi(z) = \zeta z$, $|\zeta| = 1$, induce unitary composition operators C_φ on H^2 and A^2_α. The normal and unitary weighted composition operators on H^2 and A^2_α were investigated in [4] (the work of that paper extends to A^2_α though it was written as if H^2 was the only possible setting). Cowen in [9, Theorem 5] provided a complete characterization of hyponormal composition operators C_φ on H^2 in the case where φ is linear-fractional (an analogue of [9, Theorem 5] for A^2_α has not been obtained yet). After that Zorboska [28] investigated the hyponormal composition operators on the weighted Hardy spaces. As far as we know, there is not a general characterization of hyponormal composition operators. Recently in [10], Cowen et al. investigated the situation where $C^*_{\psi,\varphi}$ is hyponormal.

We know that for an analytic self-map φ of $\mathbb{D}$, if $\varphi(0)=0$, then $\|C_\varphi\|_\gamma = 1$ (see [12, Corollary 3.7] and [25, Lemma 2.3]). When $\varphi(0)\neq 0$, it is quite difficult to determine the norm of C_φ exactly (see [3], [9], [15], [16], [17], [18], [19], [24]).

In this paper, we consider norms of hyponormal weighted composition operators $C_{\psi,\varphi}$. First we find the spectral radii of $C_{\psi,\varphi}$, when φ has a Denjoy-Wolff point in $\partial\mathbb{D}$; we use that information to find norms of hyponormal weighted composition operators $C_{\psi,\varphi}$. Then for φ with $\varphi(0)=0$, we investigate the norms of hyponormal weighted composition operators $C_{\psi,\varphi}$, when there is an integer n such that $\{e^{i\theta} : |\varphi_n(e^{i\theta})| = 1\}$ has at most one element. Next, we find a necessary and sufficient condition for $C_{\psi,\varphi}$ to be hyponormal when φ has a fixed point $p\in\mathbb{D}$ and the modulus of each element of essential spectrum of $C_{\psi,\varphi}$ is not more than $|\psi(p)|$, furthermore, we find the norm of $C_{\psi,\varphi}$. Finally, if $\varphi(p)=p$ for $p\in\mathbb{D}$ and $\{e^{i\theta} : |\varphi_n(e^{i\theta})|=1\}$ has only one element, we find an upper bound and a lower bound for the norms of hyponormal weighted composition operators $C_{\psi,\varphi}$.

2. Norms

Through this paper, the spectrum of T, the essential spectrum of T, the approximate point spectrum of T and the point spectrum of T are denoted by $\sigma_\gamma(T)$, $\sigma_{e,\gamma}(T)$, $\sigma_{ap,\gamma}(T)$ and $\sigma_{p,\gamma}(T)$, respectively, for H^2 and A^2_α. Also the spectral radius of T and the essential spectral radius of T are denoted by $r_\gamma(T)$ and $r_{e,\gamma}(T)$, respectively. Moreover, we denote by $\|T\|_\gamma$ and $\|T\|_{e,\gamma}$ the norm of the operator T and the essential norm of the operator T, respectively, on H^2 and A^2_α.

In [1, Lemma 4.1], Bourdon proved the next lemma for H^2. This result holds on A^2_α, which we record as the next lemma. The idea of the proof of the following lemma is exactly the same as [1, Lemma 4.1], and therefore is omitted.

LEMMA 2.1. *Suppose that φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map of the unit disk into itself with Denjoy-Wolff point ζ. Assume that $\psi\in H^\infty$ extends to be continuous on $\mathbb{D}\cup\{\zeta\}$ (if $\zeta\in\partial\mathbb{D}$). Suppose that $C_{\psi,\varphi}$ is considered as an operator on A^2_α. If λ is an eigenvalue of $C_{\psi,\varphi}$, then $|\lambda|\leqslant|\psi(\zeta)|r_{\alpha+2}(C_\varphi)$. If $\psi(\zeta)=0$ and φ and ψ are nonconstant, then $C_{\psi,\varphi}$ has no eigenvalues.*

We know that if an operator T is hyponormal, then $\|T\| = r(T)$. In this section, we investigate normsand spectral radii of weighted composition operators. The essential spectral radius of C_φ on H^2 was found in [5, Lemma 5.2]. Now we use another proof in order to obtain the essential spectral radius of C_φ on H^2 and A^2_α.

LEMMA 2.2. *Suppose that φ is an analytic map of the unit disk into itself with Denjoy-Wolff point $\zeta \in \partial\mathbb{D}$. Then the essential spectral radius of C_φ on H^2 or A^2_α is $\varphi'(\zeta)^{-\gamma/2}$.*

Proof. By using the general version of the Chain Rule given in [27, Chapter 4, Exercise 10, p. 74], we see that $\varphi_n'(\zeta) = \varphi'(\zeta)^n$. By [21, Theorem 5.2], $r_{e,\gamma}(C_\varphi) = \lim_{n\to\infty} \|C_{\varphi_n}\|_{e,\gamma}^{1/n} \geqslant \lim_{n\to\infty}\left(\frac{1}{|\varphi_n'(\zeta)|^{\gamma/2}}\right)^{1/n} = \varphi'(\zeta)^{-\gamma/2}$. Also by [12, Theorem 3.9] and [19, Theorem 6], $\varphi'(\zeta)^{-\gamma/2} = r_\gamma(C_\varphi) \geqslant r_{e,\gamma}(C_\varphi)$, so the result follows. □

Suppose that φ is an analytic self-map of $\mathbb{D}$ and α is a complex number of modulus 1. Since $\mathrm{Re}\left(\frac{\alpha+\varphi}{\alpha-\varphi}\right)$ is a positive harmonic function on $\mathbb{D}$, there exists a finite positive Borel measure μ_α on $\partial\mathbb{D}$ such that $\frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2} = \mathrm{Re}\left(\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)}\right) = \int_{\partial\mathbb{D}} P_z d\mu_\alpha$ for each $z \in \mathbb{D}$, where $P_z(e^{i\theta}) = (1-|z|^2)/|e^{i\theta}-z|^2$ is the Poisson kernel at z. The measures μ_α are called the Clark measures of φ. There is a unique pair of measures μ_α^{ac} and μ_α^s such that $\mu_\alpha = \mu_\alpha^{ac} + \mu_\alpha^s$, where μ_α^{ac} and μ_α^s are the absolutely continuous and singular parts with respect to Lebesgue measure, respectively. In particular, if φ is a linear-fractional non-automorphism such that $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$, then $\mu_\alpha^s = 0$ when $\alpha \neq \eta$ and $\mu_\eta^s = |\varphi'(\zeta)|^{-1}\delta_\zeta$, where δ_ζ is the unit point mass at ζ. We write $E(\varphi)$ for the closure in $\partial\mathbb{D}$ of the union of the closed supports of μ_α^s as α ranges over the unit circle. We know that $F(\varphi) \subseteq E(\varphi)$ (see [21, p. 2919]). For information about the Clark measures, see [21].

In the next theorem, the set of points where the range of φ meets $\partial\mathbb{D}$ is

$$\{\zeta \in \partial\mathbb{D} : \varphi(\zeta) \in \partial\mathbb{D}\}.$$

THEOREM 2.3. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that $\varphi \in A(\mathbb{D})$ and the set of points where the range of φ meets $\partial\mathbb{D}$ is finite. Assume that there are a positive integer n and $\zeta \in \partial\mathbb{D}$ such that $E(\varphi_n) = \{\zeta\}$, where ζ is the Denjoy-Wolff point of φ. Let $\psi \in H^\infty$ be continuous at ζ. If $C_{\psi,\varphi}$ is hyponormal, then*

$$\|C_{\psi,\varphi}\|_\gamma = |\psi(\zeta)|\varphi'(\zeta)^{-\gamma/2}.$$

Proof. Since ζ is the Denjoy-Wolff point of φ, $F(\varphi_n) = \{\zeta\}$. Also we have $\varphi_n'(\zeta) = \varphi'(\zeta)^n$. Since $\sigma_{e,\gamma}(C_{\varphi_n})$ is a compact set, by Lemma 2.2, there is $\lambda \in \sigma_{e,\gamma}(C_{\varphi_n})$ such that $|\lambda| = r_{e,\gamma}(C_{\varphi_n}) = \varphi'(\zeta)^{-n\gamma/2}$. Since $\varphi(\zeta) = \zeta$, [20, Corollary 2.2] and [13, Proposition 2.3] imply that $\sigma_{e,\gamma}(C^n_{\psi,\varphi}) = \sigma_{e,\gamma}(T_{\psi\cdot\psi\circ\varphi\ldots\psi\circ\varphi_{n-1}}C_{\varphi_n}) = \psi(\zeta)^n\sigma_{e,\gamma}(C_{\varphi_n})$. We may now apply Lemma 2.1 and [1, Lemma 4.1] to observe that $|\mu| \leqslant |\psi(\zeta)|^n r_\gamma(C_{\varphi_n})$ for each μ in $\sigma_{p,\gamma}(C^n_{\psi,\varphi})$. Then by [19, Theorem 6] and [12, Theorem 3.9], we have for each $\mu \in \sigma_{p,\gamma}(C^n_{\psi,\varphi})$, $|\mu| \leqslant |\psi(\zeta)|^n\varphi'(\zeta)^{-n\gamma/2}$. By [7, Proposition 6.7, p. 210] and [7, Proposition 4.4, p. 359],

$$\partial\sigma_\gamma(C^n_{\psi,\varphi}) \subseteq \sigma_{ap,\gamma}(C^n_{\psi,\varphi}) \subseteq \sigma_{p,\gamma}(C^n_{\psi,\varphi}) \cup \sigma_{e,\gamma}(C^n_{\psi,\varphi}).$$

Therefore, we can easily see that $r_\gamma(C^n_{\psi,\varphi}) = |\psi(\zeta)|^n\varphi'(\zeta)^{-n\gamma/2}$. We have

$$r_\gamma(C_{\psi,\varphi}) = \lim_{k\to\infty}\|C^{nk}_{\psi,\varphi}\|_\gamma^{1/(nk)} = \left(r_\gamma(C^n_{\psi,\varphi})\right)^{1/n} = |\psi(\zeta)|\varphi'(\zeta)^{-\gamma/2}. \quad \square$$

A map $\varphi \in \mathrm{LFT}(\mathbb{D})$ is called parabolic if it has a fixed point $\zeta \in \partial\mathbb{D}$ of multiplicity 2. The map $\tau(z) := (1+\overline{\zeta}z)/(1-\overline{\zeta}z)$ takes the unit disk onto the right half-plane Π and sends ζ to ∞. Therefore, $\phi := \tau \circ \varphi \circ \tau^{-1}$ is a self-map of Π which fixes only ∞, and so must be the mapping of translation by some number t, where necessarily $\mathrm{Re}t \geqslant 0$. Note that $\mathrm{Re}t = 0$ if and only if $\varphi \in \mathrm{Aut}(\mathbb{D})$. When the number t is strictly positive, we call φ a positive parabolic non-automorphism. Among the linear-fractional self-maps of $\mathbb{D}$ fixing $\zeta \in \partial\mathbb{D}$, the parabolic ones are characterized by $\varphi'(\zeta) = 1$.

In [11, Theorem 21], Cowen et al. showed that a hyponormal weighted composition operator on H^2 with a composition symbol φ as a positive parabolic non-automorphism is automatically normal. In the following proposition, for φ which is a parabolic non-automorphism, we find a necessary condition for $C_{\psi,\varphi}$ to be hyponormal on H^2 and A^2_α.

COROLLARY 2.4. *Let φ be a parabolic non-automorphism with a fixed point $\zeta \in \partial\mathbb{D}$. Suppose that $\psi \in H^\infty$ is continuous at ζ. If $C_{\psi,\varphi}$ is hyponormal on H^2 or A^2_α, then for each $w \in \mathbb{D}$,*

$$|\psi(\zeta)| \geqslant |\psi(w)| \left(\frac{1-|w|^2}{1-|\varphi(w)|^2} \right)^{\gamma/2}. \tag{3}$$

Proof. Since $C_{\psi,\varphi}$ is hyponormal, by [8, Proposition 4.6 p. 47] and Theorem 2.3, $\|C_{\psi,\varphi}\|_\gamma = |\psi(\zeta)|$. By Equation (1), we have

$$\|C_{\psi,\varphi}\|_\gamma = \|C^*_{\psi,\varphi}\|_\gamma \geqslant \|C^*_{\psi,\varphi}(K_w/\|K_w\|_\gamma)\|_\gamma = |\psi(w)| \left(\frac{1-|w|^2}{1-|\varphi(w)|^2} \right)^{\gamma/2},$$

as desired. □

EXAMPLE 2.5. Let φ be a parabolic non-automorphism with fixed point 1. Assume that $\psi_1(z) = \frac{2-z}{4}$ and $\psi_2(z) = -3z^2+2z+3$. Setting $w=0$ and $\zeta = 1$ in Equation (3), we get $C_{\psi_1,\varphi}$ and $C_{\psi_2,\varphi}$ are not hyponormal.

PROPOSITION 2.6. *Let φ be an analytic self-map of $\mathbb{D}$ with $\varphi(0) = 0$ and $\psi \in H^\infty$. Suppose there is a positive integer n that $\{e^{i\theta} : |\varphi_n(e^{i\theta})| = 1\} = \emptyset$. If $C_{\psi,\varphi}$ is hyponormal on H^2 or A^2_α, then $\|C_{\psi,\varphi}\|_\gamma = |\psi(0)|$.*

Proof. Since $\overline{\varphi_n(\mathbb{D})} \subseteq \mathbb{D}$, by [12, p. 129], $C^n_{\psi,\varphi} = C_{\psi\cdot\psi\circ\varphi\ldots\psi\circ\varphi_{n-1},\varphi_n}$ is compact. Therefore, $\sigma_{e,\gamma}(C^n_{\psi,\varphi}) = \{0\}$ and so by the Spectral Mapping Theorem, $\sigma_{e,\gamma}(C_{\psi,\varphi}) = \{0\}$. Schwarz's Lemma implies that 0 is the Denjoy-Wolff point. Invoking Lemma 2.1 and [1, Lemma 4.1], for each $\lambda \in \sigma_{p,\gamma}(C_{\psi,\varphi})$, $|\lambda| \leqslant |\psi(0)| r_\gamma(C_\varphi)$. Hence by [19, Theorem 6] and [12, Theorem 3.9], for each $\lambda \in \sigma_{p,\gamma}(C_{\psi,\varphi})$, $|\lambda| \leqslant |\psi(0)|$. By [7, Proposition 6.7, p. 210] and [7, Proposition 4.4, p. 359], $\partial\sigma_\gamma(C_{\psi,\varphi}) \subseteq \sigma_{e,\gamma}(C_{\psi,\varphi}) \cup \sigma_{p,\gamma}(C_{\psi,\varphi})$. Hence $r_\gamma(C_{\psi,\varphi}) \leqslant |\psi(0)|$. We have

$$\|C_{\psi,\varphi}\|_\gamma = \|C^*_{\psi,\varphi}\|_\gamma \geqslant \|C^*_{\psi,\varphi}K_0\|_\gamma = |\psi(0)| \|K_{\varphi(0)}\|_\gamma = |\psi(0)|.$$

Since $C_{\psi,\varphi}$ is hyponormal, $\|C_{\psi,\varphi}\|_\gamma = |\psi(0)|$. □

PROPOSITION 2.7. *Let φ be analytic on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\varphi(0) = 0$. Assume that there is an integer n such that $\{e^{i\theta} : |\varphi_n(e^{i\theta})| = 1\}$ has only one element ζ which is a fixed point of φ and $\zeta \in F(\varphi)$. Suppose that $\varphi \in A(\mathbb{D})$ and $\psi \in H^\infty$ is continuous at ζ. If $C_{\psi,\varphi}$ is hyponormal on H^2 or A^2_α, then*

$$\frac{|\psi(\zeta)|}{|\varphi'(\zeta)|^{\gamma/2}} \leqslant \|C_{\psi,\varphi}\|_\gamma \leqslant max\{|\psi(\zeta)|, |\psi(0)|\}.$$

Proof. Assume that $\varphi(0) = 0$ and $\varphi(\zeta) = \zeta$, where $\zeta \in \partial\mathbb{D}$. By [20, Corollary 2.2] and [13, Proposition 2.3], $\sigma_{e,\gamma}(C^n_{\psi,\varphi}) = \sigma_{e,\gamma}(C_{\psi\cdot\psi\circ\varphi\ldots\psi\circ\varphi_{n-1},\varphi_n}) = \psi(\zeta)^n\sigma_{e,\gamma}(C_{\varphi_n})$. By [12, Corollary 3.7] and [25, Lemma 2.3], we see that $r_{e,\gamma}(C^n_{\psi,\varphi}) \leqslant |\psi(\zeta)|^n r_{e,\gamma}(C_{\varphi_n}) \leqslant |\psi(\zeta)|^n\|C_{\varphi_n}\|_\gamma \leqslant |\psi(\zeta)|^n$. Hence by the Spectral Mapping Theorem, $r_{e,\gamma}(C_{\psi,\varphi}) \leqslant |\psi(\zeta)|$. As we saw in the proof of Proposition 2.6, [12, Theorem 3.9], [19, Theorem 6], Lemma 2.1 and [1, Lemma 4.1] imply that for each $\lambda \in \sigma_{p,\gamma}(C_{\psi,\varphi})$, $|\lambda| \leqslant |\psi(0)|$. Also by [7, Proposition 6.7, p. 210] and [7, Proposition 4.4, p. 359], $\partial\sigma_\gamma(C_{\psi,\varphi}) \subseteq \sigma_{e,\gamma}(C_{\psi,\varphi}) \cup \sigma_{p,\gamma}(C_{\psi,\varphi})$. Hence $r_\gamma(C_{\psi,\varphi}) \leqslant \max\{|\psi(\zeta)|, |\psi(0)|\}$. By Equation (1), we see that for each $r < 1$,

$$\|C_{\psi,\varphi}\|^2_\gamma = \|C^*_{\psi,\varphi}\|^2_\gamma \geqslant \left\|C^*_{\psi,\varphi}\frac{K_{r\zeta}}{\|K_{r\zeta}\|_\gamma}\right\|^2_\gamma = |\psi(r\zeta)|^2\left(\frac{1-|r\zeta|^2}{1-|\varphi(r\zeta)|^2}\right)^\gamma.$$

Then by the Julia-Caratheodory Theorem (see [12, Theorem 2.44, p. 51]),

$$\|C_{\psi,\varphi}\|^2_\gamma \geqslant \lim_{r\to 1}|\psi(r\zeta)|^2\left(\frac{1+|r\zeta|}{1+|\varphi(r\zeta)|}\right)^\gamma\left(\frac{1-|r\zeta|}{1-|\varphi(r\zeta)|}\right)^\gamma = |\psi(\zeta)|^2\frac{1}{|\varphi'(\zeta)|^\gamma}.$$

Since $\|C_{\psi,\varphi}\|_\gamma = r_\gamma(C_{\psi,\varphi})$, the result follows. □

Let H be a Hilbert space of functions analytic on the unit disk. If the monomials $1, z, z^2, \ldots$ are an orthogonal set of non-zero vectors with dense span in H, then H is called a weighted Hardy space. We will assume that the norm satisfies the normalization $\|1\| = 1$. The weight sequence for a weighted Hardy space H is defined to be $\beta(n) = \|z^n\|$. The weighted Hardy space with weight sequence $\beta(n)$ will be denoted $H^2(\beta)$. The norm on $H^2(\beta)$ is given by $\left\|\sum_{j=0}^\infty a_j z^j\right\|^2 = \sum_{j=0}^\infty |a_j|^2\beta(j)^2$. We know that H^2 and A^2_α are weighted Hardy spaces (see [12]).

Suppose that φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map of the unit disk into itself with $\varphi(0) = 0$. In the following proposition, we find a necessary and sufficient condition for $C_{\psi,\varphi}$ to be hyponormal.

PROPOSITION 2.8. *Suppose that φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map of the unit disk into itself with $\varphi(0) = 0$. Assume that $\psi \in H^\infty$ and for each $\lambda \in \sigma_{e,\gamma}(C_{\psi,\varphi})$, $|\lambda| \leqslant |\psi(0)|$. The weighted composition operator $C_{\psi,\varphi}$ is hyponormal on H^2 or A^2_α if and only if ψ is constant and C_φ is hyponormal; moreover, in this case $\|C_{\psi,\varphi}\|_\gamma = |\psi(0)|$.*

Proof. Suppose that $C_{\psi,\varphi}$ is hyponormal. By [1, Lemma 4.1], Lemma 2.1, [12, Theorem 3.9] and [19, Theorem 6], for each $\lambda \in \sigma_{p,\gamma}(C_{\psi,\varphi})$, $|\lambda| \leqslant |\psi(0)|$. Furthermore, [7, Proposition 6.7, p. 210] and [7, Proposition 4.4, p. 359] imply that

$r_\gamma(C_{\psi,\varphi}) \leqslant |\psi(0)|$. Since $C_{\psi,\varphi}$ is hyponormal, $|\psi(0)| \geqslant \|C_{\psi,\varphi}\|_\gamma \geqslant \|C_{\psi,\varphi}1\|_\gamma = \|\psi\|_\gamma$. Since H^2 and A^2_α are weighted Hardy spaces, it is easy to see that $\|\psi\|_\gamma \geqslant |\psi(0)|$. Hence ψ is constant and $\|C_{\psi,\varphi}\|_\gamma = r_\gamma(C_{\psi,\varphi}) = |\psi(0)|$.

Conversely, it is obvious. □

Note that one direction of the statment of Proposition 2.8 is that the hyponormality of $C_{\psi,\varphi}$ implies the hyponormality of C_φ. If we assume that $C_{\psi,\varphi}$ is hyponormal under the conditions of Proposition 2.8 and φ is a linear-fractional, Cowen's characterization (see [9, Theorem 5]) yields $\varphi(z) = z/(uz+v)$, where $v > 1$ and $|u| = v-1$ or $\varphi(z) = \lambda z$ with $|\lambda| < 1$. Therefore, by Proposition 2.8, $C_{\psi,\varphi}$ is hyponormal if and only if ψ is constant and either $\varphi(z) = z/(uz+v)$, where $v > 1$ and $|u| = v-1$ or $\varphi(z) = \lambda z$ with $|\lambda| < 1$. If φ is not a linear-fractional, Proposition 2.8 does not provide much useful information, since the hyponormality of C_φ is nearly as mysterious as the hyponormality of $C_{\psi,\varphi}$.

By the definition of hyponormality, we can see that if a hyponormal operator is unitarily equivalent to another operator, then that other operator is also hyponormal; we use this fact in the proof of the following two theorems.

Suppose that φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map of the unit disk into itself. In the following theorem, we see that if $C_{\psi,\varphi}$ is hyponormal, when $\varphi(p) = p$ for some $p \in \mathbb{D}$ and $r_{e,\gamma}(C_{\psi,\varphi}) \leqslant |\psi(p)|$, then the function ψ has a simple linear-fractional form that is the same as what Bourdon et al. found in [4, Theorem 10]. Furthermore, in the next theorem, we find a necessary and sufficient condition for $C_{\psi,\varphi}$ to be hyponormal.

THEOREM 2.9. *Suppose that φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is an analytic map of the unit disk into itself with $\varphi(p) = p$, where $p \in \mathbb{D}$. Assume that $\psi \in H^\infty$. Suppose that for each $\lambda \in \sigma_{e,\gamma}(C_{\psi,\varphi})$, $|\lambda| \leqslant |\psi(p)|$. The weighted composition operator $C_{\psi,\varphi}$ is hyponormal on H^2 or A^2_α if and only if $\psi = \psi(p)\frac{K_p}{K_p\circ\varphi}$ and $C_{\alpha_p\circ\varphi\circ\alpha_p}$ is hyponormal, where $\alpha_p(z) = (p-z)/(1-\overline{p}z)$; moreover, in this case $\|C_{\psi,\varphi}\|_\gamma = |\psi(p)|$.*

Proof. Let $C_{\psi,\varphi}$ be hyponormal. Suppose that $\psi_p = K_p/\|K_p\|_\gamma$. By [4, Theorem 6] and [22, Corollary 3.6], we have

$$H := C^*_{\psi_p,\alpha_p} C_{\psi,\varphi} C_{\psi_p,\alpha_p} \tag{4}$$

is hyponormal. Let σ be the Krein adjoint of α_p and g and h be the Cowen auxiliary functions for α_p. Since $T^*_h T^*_{\psi_p} = (T_{\psi_p h})^* = T_{1/\|K_p\|_\gamma}$ and $C^*_{\alpha_p} = T_g C_{\alpha_p} T^*_h$, by [13, Remark 2.1(a)], we see that

$$\begin{aligned} H &= T_g C_{\alpha_p} T^*_h T^*_{\psi_p} T_\psi C_\varphi T_{\psi_p} C_{\alpha_p} \\ &= \frac{1}{\|K_p\|_\gamma} T_g T_{\psi\circ\alpha_p} T_{\psi_p\circ\varphi\circ\alpha_p} C_{\alpha_p} C_\varphi C_{\alpha_p} \\ &= C_{q,\alpha_p\circ\varphi\circ\alpha_p}, \end{aligned} \tag{5}$$

where $q = (g\cdot\psi\circ\alpha_p\cdot\psi_p\circ\varphi\circ\alpha_p)/\|K_p\|_\gamma$ (see the proof of [4, Theorem 10]). Since H is unitarily equivalent to $C_{\psi,\varphi}$, $\sigma_{e,\gamma}(C_{\psi,\varphi}) = \sigma_{e,\gamma}(H)$. Since $q(0) = \psi(p)$, for each $\lambda \in \sigma_{e,\gamma}(C_{q,\alpha_p\circ\varphi\circ\alpha_p})$, $|\lambda| \leqslant |q(0)|$. We may apply Proposition 2.8 to conclude that

$\|C_{q,\alpha_p\circ\varphi\circ\alpha_p}\|_\gamma = r_\gamma(C_{q,\alpha_p\circ\varphi\circ\alpha_p}) = |q(0)| = |\psi(p)|$. Since $r_\gamma(C_{q,\alpha_p\circ\varphi\circ\alpha_p}) = r_\gamma(C_{\psi,\varphi})$ and $C_{\psi,\varphi}$ is hyponormal, $\|C_{\psi,\varphi}\|_\gamma = r_\gamma(C_{\psi,\varphi}) = |\psi(p)|$, as desired. Also Proposition 2.8 implies that q is constant. Since $q \equiv \psi(p)$, $g\cdot\psi\circ\alpha_p\cdot\psi_p\circ\varphi\circ\alpha_p \equiv \|K_p\|_\gamma\psi(p)$ on $\mathbb{D}$. Then $g\circ\alpha_p\cdot\psi\cdot\psi_p\circ\varphi \equiv \|K_p\|_\gamma\psi(p)$ on $\mathbb{D}$. It follows that $\psi = \|K_p\|_\gamma\psi(p)/(g\circ\alpha_p\cdot\psi_p\circ\varphi)$. Observe $g\circ\alpha_p = \|K_p\|_\gamma^2/K_p$ and $\psi_p\circ\varphi = K_p\circ\varphi/\|K_p\|_\gamma$. Hence $\psi = \psi(p)\frac{K_p}{K_p\circ\varphi}$.

Conversely, let $\psi = \psi(p)\frac{K_p}{K_p\circ\varphi}$ and $C_{\alpha_p\circ\varphi\circ\alpha_p}$ be hyponormal. Again let $H := C^*_{\psi_p,\alpha_p}C_{\psi,\varphi}C_{\psi_p,\alpha_p}$. As we saw in the proof of the previous direction, $H = C_{q,\alpha_p\circ\varphi\circ\alpha_p}$, where $q = (g\cdot\psi\circ\alpha_p\cdot\psi_p\circ\varphi\circ\alpha_p)/\|K_p\|_\gamma$. It is easy to see that $\alpha_p\circ\varphi\circ\alpha_p$ fixes 0. Since $C_{\alpha_p\circ\varphi\circ\alpha_p}$ is hyponormal, if we show that q is constant, then H is hyponormal and so $C_{\psi,\varphi}$ is hyponormal. One might see that q is constant as follows. It is not hard to see that q is constant if and only if $q\circ\alpha_p$ is constant. We obtain $q\circ\alpha_p = \frac{1}{\|K_p\|_\gamma}g\circ\alpha_p\cdot\psi\cdot\psi_p\circ\varphi = \frac{1}{\|K_p\|_\gamma}\cdot\frac{\|K_p\|_\gamma^2}{K_p}\cdot\psi(p)\frac{K_p}{K_p\circ\varphi}\frac{K_p\circ\varphi}{\|K_p\|_\gamma} = \psi(p)$ (note that $g\circ\alpha_p$ and $\psi_p\circ\varphi$ were obtained in the proof of the preceding direction). □

Note that by the similar idea which was stated in the second and third sentences of the proof of Proposition 2.8, the condition $|\lambda| \leqslant |\psi(p)|$ is equivalent to $\sigma_\gamma(C_{\psi,\varphi})$ being a subset of the closed disk with center 0 and radius $|\psi(p)|$. Although the spectrum and essential spectrum of most weighted composition operators $C_{\psi,\varphi}$ have not been found, we apply some useful results to determine the essential spectrum of certain weighted composition operators, particularly in the case where φ is a linear-fractional. If there is an integer $N > 0$ such that $\varphi_N(\overline{\mathbb{D}}) \subseteq \mathbb{D}$, then $C_{\psi,\varphi}$ is power compact (see [12, p. 129]) and so $\sigma_{e,\gamma}(C_{\psi,\varphi}) = \{0\}$ and obviously the condition $|\lambda| \leqslant |\psi(p)|$ in Theorem 2.9 holds. Now suppose that there is a positive integer n such that $\{e^{i\theta} : |\varphi_n(e^{i\theta})| = 1\}$ has only one element ζ which is a fixed point of φ. If φ and ψ satisfy the conditions of [13, Proposition 2.3] and [20, Corollary 2.2], $\sigma_{e,\gamma}(C^n_{\psi,\varphi}) = \sigma_{e,\gamma}(T_{\psi\cdot\psi\circ\varphi\ldots\psi\circ\varphi_{n-1}}C_{\varphi_n}) = \psi(\zeta)^n\sigma_{e,\gamma}(C_{\varphi_n})$. Hence if we know $\sigma_{e,\gamma}(C_{\varphi_n})$, then by the Spectral Mapping Theorem, we have information about $\sigma_{e,\gamma}(C_{\psi,\varphi})$. In particular, if φ is a linear-fractional non-automorphism with fixed points $p\in\mathbb{D}$ and $\zeta\in\partial\mathbb{D}$, then $\sigma_{e,\gamma}(C_{\psi,\varphi}) = \psi(\zeta)\sigma_{e,\gamma}(C_\varphi)$. Since we know $\sigma_{e,\gamma}(C_\varphi)$, we have $\sigma_{e,\gamma}(C_{\psi,\varphi})$ (see [2, Theorem 2.8 (ii)] and [19, Theorem 11]) and so we can investigate the condition $|\lambda| \leqslant |\psi(p)|$ in Theorem 2.9. In Example 2.13, we find a hyponormal weighted composition operator such that the condition $|\lambda| \leqslant |\psi(p)|$ does not hold.

Now we give an example which shows that Theorem 2.9 is particulary useful and can identify new classes of hyponormal weighted composition operators whose norms can be calculated.

EXAMPLE 2.10. Let $\varphi(z) = 1/(3-2z)$ that $\varphi(1/2) = 1/2$ and $\varphi(1) = 1$. Suppose that $\psi = 3\frac{K_{1/2}}{K_{1/2}\circ\varphi}$. By [2, Theorem 2.8 (ii)] and [20, Corollary 2.2], $\sigma_{e,1}(C_{\psi,\varphi}) = \psi(1)\sigma_{e,1}(C_\varphi) = \{z : |z| \leqslant \frac{|\psi(1)|}{\sqrt{\varphi'(1)}}\} = \{z : |z| \leqslant 3/\sqrt{2}\}$. We have $\psi(1/2) = 3$ and $\frac{3}{\sqrt{2}} \leqslant 3$. After calculation, we obtain $\alpha_{1/2}\circ\varphi\circ\alpha_{1/2} = \frac{z}{z+2}$. We infer from [9, Theorem 5] that $C_{\frac{z}{z+2}}$ is hyponormal. Then by Theorem 2.9, $C_{\psi,\varphi}$ is hyponormal on H^2 and

$\|C_{\psi,\varphi}\|_1 = 3$.

COROLLARY 2.11. *Suppose that φ is analytic in a neighborhood of the closed unit disk. Let there exist $p \in \mathbb{D}$ such that $\varphi(p) = p$. Assume that there is a positive integer n such that $\{e^{i\theta} : |\varphi_n(e^{i\theta})| = 1\}$ has only one element ζ which is a fixed point of φ and $\zeta \in F(\varphi)$. Suppose that $\psi \in H^\infty$ is continuous at ζ and $|\psi(\zeta)| \leqslant |\psi(p)|$. The weighted composition operator $C_{\psi,\varphi}$ is hyponormal on H^2 if and only if $\psi = \psi(p)\frac{K_p}{K_p\circ\varphi}$ and $C_{\alpha_p\circ\varphi\circ\alpha_p}$ is hyponormal, where $\alpha_p(z) = (p-z)/(1-\overline{p}z)$; moreover, in this case $\|C_{\psi,\varphi}\|_1 = |\psi(p)|$.*

Proof. By [20, Corollary 2.2], $\sigma_{e,1}(C^n_{\psi,\varphi}) = \sigma_{e,1}(T_{\psi\cdot\psi\circ\varphi\ldots\psi\circ\varphi_{n-1}}C_{\varphi_n}) = \psi(\zeta)^n\sigma_{e,1}(C_{\varphi_n})$. Also [12, Exercise 3.2.5] and the general version of the Chain Rule given in [27, Chapter 4, Exercise 10, p. 74] imply that $|\lambda| \leqslant |\psi(\zeta)|^n r_{e,1}(C_{\varphi_n}) \leqslant |\psi(\zeta)|^n|\varphi'(\zeta)|^{-n/2} \leqslant |\psi(\zeta)|^n$ for each $\lambda \in \sigma_{e,1}(C^n_{\psi,\varphi})$. By the Spectral Mapping Theorem, we can see that for each $t \in \sigma_{e,1}(C_{\psi,\varphi})$, $|t| \leqslant |\psi(\zeta)|$. The result follows from Theorem 2.9. □

As we saw in the proof of Corollary 2.11, sometimes for investigating the condition $|\lambda| \leqslant |\psi(p)|$ in Theorem 2.9, knowing essential norms of composition operators and weighted composition operators might be useful because $r_{e,\gamma}(C_{\psi,\varphi}) \leqslant \|C_{\psi,\varphi}\|_{e,\gamma}$; for more information about the essential norm of composition operators and weighted composition operators see [23] and [26].

THEOREM 2.12. *Let $\varphi \in A(\mathbb{D})$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Assume p and ζ satisfy the hypotheses of the second and third sentences of Corollary 2.11. Suppose $\psi \in H^\infty$ is continuous at ζ. If $C_{\psi,\varphi}$ is hyponormal on H^2 or A^2_α, then*

$$\frac{\mu}{|\varphi'(\zeta)|^{\gamma/2}} \leqslant \|C_{\psi,\varphi}\|_\gamma \leqslant \max\{\mu, |\psi(p)|\},$$

where $\mu = |\psi(\zeta)K_p(\alpha_p(\zeta))K_p(\zeta)|/\|K_p\|^2_\gamma$ with $\alpha_p(z) = (p-z)/(1-\bar{p}z)$.

Proof. Suppose that $C_{\psi,\varphi}$ is hyponormal. Let H be as in Equation (5). We see that the map $\alpha_p\circ\varphi\circ\alpha_p$ fixes $\alpha_p(\zeta)$. Since $\alpha_p^{-1} = \alpha_p$, by [6, Corollary 7.6, p. 99], $(\alpha_p\circ\varphi\circ\alpha_p)'(\alpha_p(\zeta)) = \varphi'(\zeta)$. By Proposition 2.7, we have

$$\frac{|q(\alpha_p(\zeta))|}{|\varphi'(\zeta)|^{\gamma/2}} \leqslant \|C_{q,\alpha_p\circ\varphi\circ\alpha_p}\|_\gamma \leqslant \max\{|q(\alpha_p(\zeta))|, |q(0)|\}, \tag{6}$$

where $q = (g\cdot\psi\circ\alpha_p\cdot\psi_p\circ\varphi\circ\alpha_p)/\|K_p\|_\gamma$ with g the Cowen auxiliary function for α_p and $\psi_p = K_p/\|K_p\|_\gamma$. We have $q(0) = \psi(p)$. Moreover,

$$q(\alpha_p(\zeta)) = \frac{\psi(\zeta)K_p(\alpha_p(\zeta))K_p(\zeta)}{\|K_p\|^2_\gamma}.$$

Since H is unitarily equivalent to $C_{\psi,\varphi}$, $r_\gamma(H) = r_\gamma(C_{\psi,\varphi})$. Since $C_{\psi,\varphi}$ and H are hyponormal, by Equations (5) and (6), the result follows. □

EXAMPLE 2.13. For $0 < s < 1$, let $\Phi(z) = sz/(1-(1-s)z)$, $\Psi(z) = 1/(1-(1-s)z)$. We know that $\Phi(0) = 0$, $\Phi(1) = 1$ and by [14, Example 3.6], $C_{\Psi,\Phi}$ is hyponormal on H^2. Suppose that $C_{\psi,\varphi} = C^*_{\psi_{1/2},\alpha_{1/2}} C_{\Psi,\Phi} C_{\psi_{1/2},\alpha_{1/2}}$, where $\psi_{1/2}$ and $\alpha_{1/2}$ were defined in the proof and statement of Theorem 2.9. Equation (5) implies that $\psi = (g\cdot\Psi\circ\alpha_{1/2}\cdot\psi_{1/2}\circ\Phi\circ\alpha_{1/2})/\|K_{1/2}\|_1$ and $\varphi = \alpha_{1/2}\circ\Phi\circ\alpha_{1/2}$, where g is the Cowen auxiliary function for $\alpha_{1/2}$. Since $C_{\Psi,\Phi}$ is hyponormal on H^2, $C_{\psi,\varphi}$ is also hyponormal on H^2. We can see that $\varphi(1/2) = 1/2$ and $\varphi(-1) = -1$. After calculation, we see that $\psi(1/2) = 1$, $\mu = 1/s$ and $\varphi'(-1) = 1/s$, where μ was defined in the statement of Theorem 2.12. Then

$$\left(\frac{1}{s}\right)^{1/2} \leqslant \|C_{\psi,\varphi}\|_1 \leqslant \frac{1}{s}.$$

Note that $C_{\psi,\varphi}$ in Example 2.13 does not satisfy one of the hypotheses of Theorem 2.9. Because by [20, Corollary 2.2] and [2, Theorem 2.8 (ii)], $\sigma_{e,1}(C_{\psi,\varphi}) = \psi(-1)\sigma_{e,1}(C_\varphi) = \{z : |z| \leqslant \frac{|\psi(-1)|}{\sqrt{\varphi'(-1)}}\} = \{z : |z| \leqslant |\psi(-1)|\sqrt{s}\}$. After calculation, we have $g(-1) = 2/3$, $(\Psi\circ\alpha_{1/2})(-1) = 1/s$ and $(\psi_{1/2}\circ\Phi\circ\alpha_{1/2})(-1) = \sqrt{3}$ and so $\psi(-1) = 1/s$. It follows that $\sigma_{e,1}(C_{\psi,\varphi}) = \{z : |z| \leqslant \frac{1}{\sqrt{s}}\}$. Since $\psi(1/2) = 1$, $\frac{1}{\sqrt{s}} > |\psi(1/2)|$. It shows that the condition $|\lambda| \leqslant |\psi(1/2)|$ does not hold in the statement of Theorem 2.9.

REMARK 2.14. Let φ be an analytic self-map of $\mathbb{D}$ with $\varphi(p) = p$ and $\psi \in H^\infty$. Assume that there is a positive integer n such that $\{e^{i\theta} : |\varphi_n(e^{i\theta})| = 1\} = \emptyset$. As we saw in the proof of Theorem 2.12, $C_{\psi,\varphi}$ is unitarily equivalent to H. Proposition 2.6 shows that $\|C_{\psi,\varphi}\|_\gamma = |q(0)|$. Since $q(0) = \psi(p)$ (as we stated it in the proof of Theorem 2.12), $\|C_{\psi,\varphi}\|_\gamma = |\psi(p)|$.

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