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QUADRATIC WEIGHTED GEOMETRIC MEAN IN HERMITIAN UNITAL BANACH $*$–ALGEBRAS

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Abstract. In this paper we introduce the *quadratic weighted geometric mean*

$$x \circledS_\nu y := \left| \left| yx^{-1} \right|^\nu x \right|^2$$

for invertible elements x, y in a Hermitian unital Banach $*$-algebra and real number ν. We show that

$$x \circledS_\nu y = |x|^2 \sharp_\nu |y|^2,$$

where $\sharp_\nu$ is the usual geometric mean and provide some inequalities for this mean under various assumptions for the elements involved.

1. Introduction

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geqslant 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geqslant 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geqslant b$ means that $a - b \geqslant 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [12] (see also [2, Theorem 41.5])

$$a^* a \geqslant 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [11], Tanahashi and Uchiyama [13] proved the following fundamental properties (see also [5]):

(i) If a, $b \in A$, then $a \geqslant 0$, $b \geqslant 0$ imply $a + b \geqslant 0$ and $\alpha \geqslant 0$ implies $\alpha a \geqslant 0$;

(ii) If a, $b \in A$, then $a > 0$, $b \geqslant 0$ imply $a + b > 0$;

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(iii) If $a,\ b \in A$, then either $a \geqslant b > 0$ or $a > b \geqslant 0$ imply $a > 0$;

(iv) If $a > 0$, then $a^{-1} > 0$;

(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leqslant a$ if and only if $cbc \leqslant cac$;

(vi) If $0 < a < 1$, then $1 < a^{-1}$;

(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leqslant a$, then $0 < a^{-1} \leqslant b^{-1}$.

Okayasu [11] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a,\ b \in A$ and $p \in [0,1]$ then $a > b\ \ (a \geqslant b)$ implies that $a^p > b^p\ \ (a^p \geqslant b^p)$.

In order to introduce the real power of a positive element, we need the following facts [2, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i}\int_{\gamma} f(z)(z-a)^{-1}\,dz.$$

It is well known (see for instance [3, pp. 201–204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^{\alpha} := \frac{1}{2\pi i}\int_{\gamma} z^{\alpha}(z-a)^{-1}\,dz,$$

where z^{α} is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^{\alpha} \in A$. Moreover, since z^{α} is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^{\alpha}) = (\sigma(a))^{\alpha} = \{z^{\alpha} : z \in \sigma(a)\} \subset (0,\infty).$$

Following [5], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^{\alpha} \in A$ with $a^{\alpha} > 0$ and $\left(a^2\right)^{1/2} = a$, [13, Lemma 6];

(ix) If $0 < a \in A$ and $\alpha,\ \beta \in \mathbb{R}$, then $a^{\alpha}a^{\beta} = a^{\alpha+\beta}$;

(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^{\alpha})^{-1} = \left(a^{-1}\right)^{\alpha} = a^{-\alpha}$;

(xi) If $0 < a,\ b \in A$, $\alpha,\ \beta \in \mathbb{R}$ and $ab = ba$, then $a^{\alpha} b^{\beta} = b^{\beta} a^{\alpha}$.

We define the following means for $\nu \in [0,1]$, see also [5] for different notations:

$$a \nabla_{\nu} b := (1-\nu) a + \nu b,\ a,\ b \in A \tag{A}$$

the *weighted arithmetic mean* of (a,b),

$$a!_{\nu} b := \left((1-\nu) a^{-1} + \nu b^{-1}\right)^{-1},\ a,\ b > 0 \tag{H}$$

the *weighted harmonic mean* of positive elements (a,b) and

$$a \sharp_{\nu} b := a^{1/2} \left(a^{-1/2} b a^{-1/2}\right)^{\upsilon} a^{1/2} \tag{G}$$

the *weighted geometric mean* of positive elements (a,b). Our notations above are motivated by the classical notations used in operator theory. For simplicity, if $\nu = \frac{1}{2}$, we use the simpler notations $a \nabla b$, $a!b$ and $a \sharp b$. The definition of weighted geometric mean can be extended for any real ν.

In [5], B. Q. Feng proved the following properties of these means in A a Hermitian unital Banach $*$-algebra:

(xii) If $0 < a,\ b \in A$, then $a!b = b!a$ and $a \sharp b = b \sharp a$;

(xiii) If $0 < a,\ b \in A$ and $c \in \operatorname{Inv}(A)$, then

$$c^* (a!b) c = (c^* a c)! (c^* b c) \text{ and } c^* (a \sharp b) c = (c^* a c) \sharp (c^* b c);$$

(xiv) If $0 < a,\ b \in A$ and $\nu \in [0,1]$, then

$$(a!_{\nu} b)^{-1} = \left(a^{-1}\right) \nabla_{\nu} \left(b^{-1}\right) \text{ and } \left(a^{-1}\right) \sharp_{\nu} \left(b^{-1}\right) = (a \sharp_{\nu} b)^{-1}.$$

Utilising the Spectral Mapping Theorem and the Bernoulli inequality for real numbers, B. Q. Feng obtained in [5] the following inequality between the weighted means introduced above:

$$a \nabla_{\nu} b \geqslant a \sharp_{\nu} b \geqslant a!_{\nu} b \tag{HGA}$$

for any $0 < a,\ b \in A$ and $\nu \in [0,1]$.

In [13], Tanahashi and Uchiyama obtained the following identity of interest:

LEMMA 1. *If* $0 < c,\ d$ *and* λ *is a real number, then*

$$(dcd)^{\lambda} = d c^{1/2} \left(c^{1/2} d^2 c^{1/2}\right)^{\lambda - 1} c^{1/2} d. \tag{1.1}$$

We can prove the following fact:

PROPOSITION 1. *For any* $0 < a,\ b \in A$ *we have*

$$b\sharp_{1-\nu}a = a\sharp_\nu b \tag{1.2}$$

for any real number ν.

Proof. We take in (1.1) $d = b^{-1/2}$ and $c = a$ to get

$$\left(b^{-1/2}ab^{-1/2}\right)^{\lambda} = b^{-1/2}a^{1/2}\left(a^{1/2}b^{-1}a^{1/2}\right)^{\lambda-1}a^{1/2}b^{-1/2}.$$

If we multiply both sides of this equality by $b^{1/2}$ we get

$$b^{1/2}\left(b^{-1/2}ab^{-1/2}\right)^{\lambda}b^{1/2} = a^{1/2}\left(a^{1/2}b^{-1}a^{1/2}\right)^{\lambda-1}a^{1/2}. \tag{1.3}$$

Since

$$\left(a^{1/2}b^{-1}a^{1/2}\right)^{\lambda-1} = \left[\left(a^{1/2}b^{-1}a^{1/2}\right)^{-1}\right]^{1-\lambda} = \left(a^{-1/2}ba^{-1/2}\right)^{1-\lambda}$$

then by (1.3) we get

$$a\sharp_{1-\nu}b = b\sharp_\nu a.$$

By swapping in this equality a with b we get the desired result (1.2). □

In this paper we introduce the *quadratic weighted geometric mean* for invertible elements $x,\ y$ in a Hermitian unital Banach $*$-algebra and real number ν. We show that it can be represented in terms of $\sharp_\nu$, which is the usual geometric mean and provide some inequalities for this mean under various assumptions for the elements involved.

2. Quadratic weighted geometric mean

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We observe that if $x \in \text{Inv}(A)$, then $x^* \in \text{Inv}(A)$, which implies that $x^*x \in \text{Inv}(A)$. Therefore by Shirali-Ford theorem we have $x^*x > 0$. If we define the modulus of the element $c \in A$ by $|c| := (c^*c)^{1/2}$ then for $c \in \text{Inv}(A)$ we have $|c|^2 > 0$ and by (viii), $|c| > 0$. If $c > 0$, then by (viii) we have $|c| = c$.

For $x,\ y \in \text{Inv}(A)$ we consider the element

$$d := (x^*)^{-1}y^*yx^{-1} = \left(yx^{-1}\right)^*yx^{-1} = \left|yx^{-1}\right|^2. \tag{2.1}$$

Since $yx^{-1} \in \text{Inv}(A)$ then $d > 0,\ d \in \text{Inv}(A),\ d^{-1} = \left|yx^{-1}\right|^{-2}$, and also

$$d^{-1} = \left((x^*)^{-1}y^*yx^{-1}\right)^{-1} = xy^{-1}\left(y^{-1}\right)^*x^* = \left|\left(y^{-1}\right)^*x^*\right|^2. \tag{2.2}$$

For $\nu \in \mathbb{R}$, by using the property (viii) we get that $d^{\nu} = \left|yx^{-1}\right|^{2\nu} > 0$ and $d^{\nu/2} = \left|yx^{-1}\right|^{\nu} > 0$. Since

$$x^* d^{\nu} x = x^* \left|yx^{-1}\right|^{2\nu} x = \left|\left|yx^{-1}\right|^{\nu} x\right|^2$$

and $\left|yx^{-1}\right|^{\nu} x \in \operatorname{Inv}(A)$, it follows that $x^* d^{\nu} x > 0$.

We introduce the *quadratic weighted mean* of (x,y) with $x,\ y \in \operatorname{Inv}(A)$ and the real weight $\nu \in \mathbb{R}$, as the positive element denoted by $x \circledS_{\nu} y$ and defined by

$$x \circledS_{\nu} y := x^* \left((x^*)^{-1} y^* y x^{-1}\right)^{\nu} x = x^* \left|yx^{-1}\right|^{2\nu} x = \left|\left|yx^{-1}\right|^{\nu} x\right|^2. \tag{S}$$

When $\nu = 1/2$, we denote $x \circledS_{1/2} y$ by $x \circledS y$ and we have

$$x \circledS y = x^* \left((x^*)^{-1} y^* y x^{-1}\right)^{1/2} x = x^* \left|yx^{-1}\right| x = \left|\left|yx^{-1}\right|^{1/2} x\right|^2.$$

We can also introduce the $1/2$-*quadratic weighted mean* of (x,y) with $x,\ y \in \operatorname{Inv}(A)$ and the real weight $\nu \in \mathbb{R}$ by

$$x \circledS_{\nu}^{1/2} y := (x \circledS_{\nu} y)^{1/2} = \left|\left|yx^{-1}\right|^{\nu} x\right|. \tag{1/2-S}$$

Correspondingly, when $\nu = 1/2$ we denote $x \circledS^{1/2} y$ and we have

$$x \circledS^{1/2} y = \left|\left|yx^{-1}\right|^{1/2} x\right|.$$

The following equalities hold:

PROPOSITION 2. *For any $x,\ y \in \operatorname{Inv}(A)$ and $\nu \in \mathbb{R}$ we have*

$$(x \circledS_{\nu} y)^{-1} = (x^*)^{-1} \circledS_{\nu} (y^*)^{-1} \tag{2.3}$$

and

$$\left(x^{-1}\right) \circledS_{\nu} \left(y^{-1}\right) = (x^* \circledS_{\nu} y^*)^{-1}. \tag{2.4}$$

Proof. We observe that for any $x,\ y \in \operatorname{Inv}(A)$ and $\nu \in \mathbb{R}$ we have

$$(x \circledS_{\nu} y)^{-1} = \left(x^* \left((x^*)^{-1} y^* y x^{-1}\right)^{\nu} x\right)^{-1} = x^{-1} \left(xy^{-1} (y^*)^{-1} x^*\right)^{\nu} (x^*)^{-1}$$

and

$$\begin{aligned}
&(x^*)^{-1} \circledS_{\nu} (y^*)^{-1} \\
&= \left((x^*)^{-1}\right)^* \left(\left(\left((x^*)^{-1}\right)^*\right)^{-1} \left((y^*)^{-1}\right)^* (y^*)^{-1} \left((x^*)^{-1}\right)^{-1}\right)^{\nu} (x^*)^{-1} \\
&= x^{-1} \left(xy^{-1} (y^*)^{-1} x^*\right)^{\nu} (x^*)^{-1},
\end{aligned}$$

which proves (2.3).

If we replace in (2.3) x by x^{-1} and y by y^{-1} we get

$$\left(\left(x^{-1}\right)\circledS_\nu\left(y^{-1}\right)\right)^{-1}=x^*\circledS_\nu y^*$$

and by taking the inverse in this equality we get (2.4). □

If we take in (S) $x=a^{1/2}$ and $y=b^{1/2}$ with $a,\ b>0$ then we get

$$a^{1/2}\circledS_\nu b^{1/2}=a\sharp_\nu b$$

for any $\nu\in\mathbb{R}$ that shows that the quadratic weighted mean can be seen as an extension of the weighted geometric mean for positive elements considered in the introduction.

Let $x,\ y\in\operatorname{Inv}(A)$. If we take in the definition of "$\sharp_\nu$" the elements $a=|x|^2>0$ and $b=|y|^2>0$ we also have for real ν

$$|x|^2\sharp_\nu|y|^2=|x|\left(|x|^{-1}|y|^2|x|^{-1}\right)^\upsilon|x|=|x|\left||y|\,|x|^{-1}\right|^{2\upsilon}|x|=\left|\left||y|\,|x|^{-1}\right|^{\upsilon}|x|\right|^2.$$

It is then natural to ask how the positive elements $x\circledS_\nu y$ and $|x|^2\sharp_\nu|y|^2$ do compare, when $x,\ y\in\operatorname{Inv}(A)$ and $\nu\in\mathbb{R}$?

We need the following lemma that provides a slight generalization of Lemma 1.

LEMMA 2. *If* $0<c,\ d\in\operatorname{Inv}(A)$ *and* λ *is a real number, then*

$$(dcd^*)^\lambda=dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{\lambda-1}c^{1/2}d^*. \tag{2.5}$$

Proof. We provide an argument along the lines in the proof of Lemma 7 from [13]. Consider the functions $F(\lambda):=(dcd^*)^\lambda$ and $G(\lambda):=dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{\lambda-1}c^{1/2}d^*$ defined for $\lambda\in\mathbb{R}$. It is obvious that $F(1)=G(1)$.

We have

$$\begin{aligned}G^2\left(\frac{1}{2}\right)&=dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-1/2}c^{1/2}d^*dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-1/2}c^{1/2}d^*\\&=dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-1/2}c^{1/2}|d|^2c^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-1/2}c^{1/2}d^*\\&=dcd^*=F^2\left(\frac{1}{2}\right)\end{aligned}$$

and

$$\begin{aligned}G^{2^2}\left(\frac{1}{2^2}\right)&=\left(dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{\frac{1-2^2}{2^2}}c^{1/2}d^*\right)^{2^2}\\&=dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-\frac{3}{4}}c^{1/2}d^*dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-\frac{3}{4}}c^{1/2}d^*\end{aligned}$$

$$dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-\frac{3}{4}}c^{1/2}d^*dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-\frac{3}{4}}c^{1/2}d^*$$
$$=dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-\frac{3}{4}}c^{1/2}|d|^2c^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-\frac{3}{4}}c^{1/2}d^*$$
$$dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-\frac{3}{4}}c^{1/2}|d|^2c^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-\frac{3}{4}}c^{1/2}d^*$$
$$=dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-\frac{1}{2}}c^{1/2}d^*dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-\frac{1}{2}}c^{1/2}d^*$$
$$=dc^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-\frac{1}{2}}c^{1/2}|d|^2c^{1/2}\left(c^{1/2}|d|^2c^{1/2}\right)^{-\frac{1}{2}}c^{1/2}d^*$$
$$=dcd^*=F^{2^2}\left(\frac{1}{2^2}\right).$$

By induction we can conclude that $G^{2^n}\left(\frac{1}{2^n}\right)=F^{2^n}\left(\frac{1}{2^n}\right)$ for any natural number $n\geqslant 0$. Since for any $a>0$ we have $\left(a^2\right)^{1/2}=a$, [13, Lemma 6], hence $G\left(\frac{1}{2^n}\right)=F\left(\frac{1}{2^n}\right)$ for any natural number $n\geqslant 0$.

Since $F(\lambda)$; $G(\lambda)$ are analytic on the real line $\mathbb{R}$ and $\frac{1}{2^n}\to 0$ for $n\to 0$, we deduce that $F(\lambda)=G(\lambda)$ for any $\lambda\in\mathbb{R}$. □

REMARK 1. The identity (2.5) was proved by. T. Furuta in [6] for positive operator c and invertible operator d in the Banach algebra of all bonded linear operators on a Hilbert space by using the polar decomposition of the invertible operator $dc^{1/2}$.

THEOREM 1. *If $x,\ y\in \mathrm{Inv}(A)$ and λ is a real number, then*

$$x \circledS_\nu y=|x|^2\sharp_\nu |y|^2 \tag{2.6}$$

Proof. If we take $d=(x^*)^{-1}$ and $c=|y|^2>0$ in (2.5), then we get

$$\left((x^*)^{-1}|y|^2x^{-1}\right)^\lambda=(x^*)^{-1}|y|\left(|y|\left|(x^*)^{-1}\right|^2|y|\right)^{\lambda-1}|y|x^{-1}$$
$$=(x^*)^{-1}|y|\left(|y|\left((x^*)^{-1}\right)^*(x^*)^{-1}|y|\right)^{\lambda-1}|y|x^{-1}$$
$$=(x^*)^{-1}|y|\left(|y|x^{-1}(x^*)^{-1}|y|\right)^{\lambda-1}|y|x^{-1}$$
$$=(x^*)^{-1}|y|\left(|y|(x^*x)^{-1}|y|\right)^{\lambda-1}|y|x^{-1}$$
$$=(x^*)^{-1}|y|\left(|y||x|^{-2}|y|\right)^{\lambda-1}|y|x^{-1}.$$

If we multiply this equality at left by x^* and at right by x, we get

$$x^*\left((x^*)^{-1}|y|^2x^{-1}\right)^\lambda x=|y|\left(|y||x|^{-2}|y|\right)^{\lambda-1}|y|=|y|\left(|y|^{-1}|x|^2|y|^{-1}\right)^{1-\lambda}|y|,$$

which means that

$$x \circledS_\nu y = |y|^2 \sharp_{1-\nu} |x|^2 . \tag{2.7}$$

By (1.2) we have for $a = |x|^2 > 0$ and $b = |y|^2$ that

$$|y|^2 \sharp_{1-\nu} |x|^2 = |x|^2 \sharp_\nu |y|^2 . \tag{2.8}$$

Utilising (2.7) and (2.8) we deduce (2.6). □

Now, assume that $f(z)$ is analytic in the right half open plane $\{\operatorname{Re} z > 0\}$ and for the interval $I \subset (0,\infty)$ assume that $f(z) \geqslant 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0,\infty)$$

meaning that $f(u) \geqslant 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A.

LEMMA 3. *Let $f(z)$ and $g(z)$ be analytic in the right half open plane $\{\operatorname{Re} z > 0\}$ and for the interval $I \subset (0,\infty)$ assume that $f(z) \geqslant g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geqslant g(u)$ in the order of A.*

We have the following inequalities between means:

THEOREM 2. *For any $x,\ y \in \operatorname{Inv}(A)$ and $\nu \in [0,1]$ we have*

$$|x|^2 \nabla_\nu |y|^2 \geqslant x \circledS_\nu y \geqslant |x|^2 !_\nu |y|^2 . \tag{2.9}$$

Proof. 1. Follows by the inequality (HGA) and representation (2.6)

2. A direct proof using Lemma 3 is as follows.

For $t > 0$ and $\nu \in [0,1]$ we have the scalar arithmetic mean-geometric mean-harmonic mean inequality

$$1 - \nu + \nu t \geqslant t^\nu \geqslant \left(1 - \nu + \nu t^{-1}\right)^{-1} . \tag{2.10}$$

Consider the functions $f(z) := 1 - \nu + \nu z$, $g(z) := z^\nu$ and $h(z) = \left(1 - \nu + \nu z^{-1}\right)^{-1}$ where z^ν is the principal of the power function. Then $f(z)$, $g(z)$ and $h(z)$ are analytic in the right half open plane $\{\operatorname{Re} z > 0\}$ of the complex plane and by (2.10) we have $f(z) \geqslant g(z) \geqslant h(z)$ for any $z > 0$.

If $0 < u \in \operatorname{Inv}(A)$ and $\nu \in [0,1]$, then by Lemma 3 we get

$$1 - \nu + \nu u \geqslant u^\nu \geqslant \left(1 - \nu + \nu u^{-1}\right)^{-1} .$$

If $x,\ y \in \operatorname{Inv}(A)$, then by taking $u = \left|yx^{-1}\right|^2 \in \operatorname{Inv}(A)$ we get

$$1 - \nu + \nu \left|yx^{-1}\right|^2 \geqslant \left|yx^{-1}\right|^{2\nu} \geqslant \left(1 - \nu + \nu \left|yx^{-1}\right|^{-2}\right)^{-1} \tag{2.11}$$

for any $\nu \in [0,1]$.

If $a > 0$ and $c \in \text{Inv}(A)$ then obviously $c^*ac = \left|a^{1/2}c\right|^2 > 0$. This implies that, if $a \geqslant b > 0$, then $c^*ac \geqslant c^*bc > 0$.

Therefore, if we multiply the inequality (2.11) at left with x^* and at right with x, then we get

$$x^*\left(1-\nu+\nu\left|yx^{-1}\right|^2\right)x \geqslant x^*\left|yx^{-1}\right|^{2\nu}x \geqslant x^*\left(1-\nu+\nu\left|yx^{-1}\right|^{-2}\right)^{-1}x \tag{2.12}$$

for any $\nu \in [0,1]$.

Observe that

$$\begin{aligned} x^*\left(1-\nu+\nu\left|yx^{-1}\right|^2\right)x &= x^*\left(1-\nu+\nu\left(x^*\right)^{-1}y^*yx^{-1}\right)x \\ &= x^*\left(1-\nu+\nu\left(x^*\right)^{-1}y^*yx^{-1}\right)x \\ &= (1-\nu)\left|x\right|^2+\nu\left|y\right|^2 = \left|x\right|^2\nabla_\nu\left|y\right|^2 \end{aligned}$$

and

$$\begin{aligned} x^*\left(1-\nu+\nu\left|yx^{-1}\right|^{-2}\right)^{-1}x &= x^*\left(1-\nu+\nu\left(\left(x^*\right)^{-1}y^*yx^{-1}\right)^{-1}\right)^{-1}x \\ &= x^*\left(1-\nu+\nu xy^{-1}\left(y^*\right)^{-1}x^*\right)^{-1}x \\ &= x^*\left(x\left((1-\nu)x^{-1}\left(x^*\right)^{-1}+\nu y^{-1}\left(y^*\right)^{-1}\right)x^*\right)^{-1}x \\ &= x^*\left(x\left((1-\nu)\left(x^*x\right)^{-1}+\nu\left(y^*y\right)^{-1}\right)x^*\right)^{-1}x \\ &= x^*\left(x^*\right)^{-1}\left((1-\nu)\left(x^*x\right)^{-1}+\nu\left(y^*y\right)^{-1}\right)^{-1}x^{-1}x \\ &= \left((1-\nu)\left|x\right|^{-2}+\nu\left|y\right|^{-2}\right)^{-1} = \left|x\right|^2!_\nu\left|y\right|^2. \end{aligned}$$

Therefore by (2.12) we get the desired result (2.9). □

We can define the weighted means for $\nu \in [0,1]$ and the elements $x,\ y \in \text{Inv}(A)$ and $\nu \in [0,1]$ by

$$x\nabla_\nu^{1/2}y := \left(\left|x\right|^2\nabla_\nu\left|y\right|^2\right)^{1/2} = \left((1-\nu)\left|x\right|^2+\nu\left|y\right|^2\right)^{1/2}$$

and

$$x!_\nu^{1/2}y := \left(\left|x\right|^2!_\nu\left|y\right|^2\right)^{1/2} = \left((1-\nu)\left|x\right|^{-2}+\nu\left|y\right|^{-2}\right)^{-1/2}.$$

COROLLARY 1. *Let A be a Hermitian unital Banach $*$-algebra with continuous involution. Then for any $x,\ y \in \text{Inv}(A)$ and $\nu \in [0,1]$ we have*

$$x\nabla_\nu^{1/2}y \geqslant x Ⓢ_\nu^{1/2}y \geqslant x!_\nu^{1/2}y. \tag{2.13}$$

Proof. It follows by taking the square root in the inequality (2.9) and by using Okayasu's result from the introduction. □

Recall that a C^**-algebra* A is a Banach $*$-algebra such that the norm satisfies the condition

$$\|a^*a\| = \|a\|^2 \text{ for any } a \in A.$$

If a C^*-algebra A has a unit 1, then automatically $\|1\| = 1$.

It is well know that, if A is a C^*-algebra, then (see for instance [10, 2.2.5 Theorem])

$$b \geqslant a \geqslant 0 \text{ implies that } \|b\| \geqslant \|a\|.$$

COROLLARY 2. *Let A be a unital C^*-algebra. Then for any $x,\ y \in \operatorname{Inv}(A)$ and $\nu \in [0,1]$ we have*

$$(1-\nu)\|x\|^2 + \nu\|y\|^2 \geqslant \left\|(1-\nu)|x|^2 + \nu|y|^2\right\| \geqslant \left\|\left|yx^{-1}\right|^{\nu} x\right\|^2. \tag{2.14}$$

3. Refinements and reverses

If X is a linear space and $C \subseteq X$ a convex subset in X, then for any convex function $f : C \to \mathbb{R}$ and any $z_i \in C, r_i \geqslant 0$ for $i \in \{1,\dots,k\}, k \geqslant 2$ with $\sum_{i=1}^k r_i = R_k > 0$ one has the *weighted Jensen's inequality:*

$$\frac{1}{R_k}\sum_{i=1}^k r_i f(z_i) \geqslant f\left(\frac{1}{R_k}\sum_{i=1}^k r_i z_i\right). \tag{J}$$

If $f : C \to \mathbb{R}$ is strictly convex and $r_i > 0$ for $i \in \{1,\dots,k\}$ then the equality case hods in (J) if and only if $z_1 = \ldots = z_n$.

By $\mathscr{P}_n$ we denote the set of all nonnegative n-tuples $(p_1,\dots,p_n)$ with the property that $\sum_{i=1}^n p_i = 1$. Consider the *normalised Jensen functional*

$$\mathscr{J}_n(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \geqslant 0,$$

where $f : C \to \mathbb{R}$ be a convex function on the convex set C and $\mathbf{x} = (x_1,\dots,x_n) \in C^n$ and $\mathbf{p} \in \mathscr{P}_n$.

The following result holds [4]:

LEMMA 4. *If $\mathbf{p},\ \mathbf{q} \in \mathscr{P}_n$, $q_i > 0$ for each $i \in \{1,\dots,n\}$ then*

$$\max_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\}\mathscr{J}_n(f,\mathbf{x},\mathbf{q}) \geqslant \mathscr{J}_n(f,\mathbf{x},\mathbf{p}) \geqslant \min_{1\leqslant i\leqslant n}\left\{\frac{p_i}{q_i}\right\}\mathscr{J}_n(f,\mathbf{x},\mathbf{q})\,(\geqslant 0). \tag{3.1}$$

In the case $n = 2$, if we put $p_1 = 1 - p$, $p_2 = p$, $q_1 = 1 - q$ and $q_2 = q$ with $p \in [0,1]$ and $q \in (0,1)$ then by (3.1) we get

$$
\begin{aligned}
&\max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[(1-q)f(x) + qf(y) - f((1-q)x + qy)] \\
&\geqslant (1-p)f(x) + pf(y) - f((1-p)x + py) \\
&\geqslant \min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[(1-q)f(x) + qf(y) - f((1-q)x + qy)]
\end{aligned} \tag{3.2}
$$

for any $x,\ y \in C$.

If we take $q = \frac{1}{2}$ in (3.2), then we get

$$
\begin{aligned}
&2\max\{t, 1-t\}\left[\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \\
&\geqslant (1-t)f(x) + tf(y) - f((1-t)x + ty) \\
&\geqslant 2\min\{t, 1-t\}\left[\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right)\right]
\end{aligned} \tag{3.3}
$$

for any $x,\ y \in C$ and $t \in [0,1]$.

We consider the scalar weighted arithmetic, geometric and harmonic means defined by $A_\nu(a,b) := (1-\nu)a + \nu b$, $G_\nu(a,b) := a^{1-\nu}b^\nu$ and $H_\nu(a,b) = A_\nu^{-1}\left(a^{-1}, b^{-1}\right)$ where $a,\ b > 0$ and $\nu \in [0,1]$.

If we take the convex function $f : \mathbb{R} \to (0,\infty)$, $f(x) = \exp(\alpha x)$, with $\alpha \neq 0$, then we have from (3.2) that

$$
\begin{aligned}
&\max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[A_q(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_q(a,b))] \\
&\geqslant A_p(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_p(a,b)) \\
&\geqslant \min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[A_q(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_q(a,b))]
\end{aligned} \tag{3.4}
$$

for any $p \in [0,1]$ and $q \in (0,1)$ and any $x,\ y \in \mathbb{R}$.

For $q = \frac{1}{2}$ we have by (3.4) that

$$
\begin{aligned}
&2\max\{p, 1-p\}[A(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A(a,b))] \\
&\geqslant A_p(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_p(a,b)) \\
&\geqslant 2\min\{p, 1-p\}[A(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A(a,b))]
\end{aligned} \tag{3.5}
$$

for any $p \in [0,1]$ and any $x,\ y \in \mathbb{R}$.

If we take $x = \ln a$ and $y = \ln b$ in (3.4), then we get

$$
\begin{aligned}
&\max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}\left[A_q(a^\alpha, b^\alpha) - G_q^\alpha(a,b)\right] \\
&\geqslant A_p(a^\alpha, b^\alpha) - G_p^\alpha(a,b) \\
&\geqslant \min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}\left[A_q(a^\alpha, b^\alpha) - G_q^\alpha(a,b)\right]
\end{aligned} \tag{3.6}
$$

for any $a,\ b>0$, for any $p\in[0,1]$, $q\in(0,1)$ and $\alpha\neq 0$.

For $q=\frac{1}{2}$ we have by (3.6) that

$$\max\{p,1-p\}\left(b^{\frac{\alpha}{2}}-a^{\frac{\alpha}{2}}\right)^2 \geqslant A_p(a^\alpha,b^\alpha)-G_p^\alpha(a,b) \geqslant \min\{p,1-p\}\left(b^{\frac{\alpha}{2}}-a^{\frac{\alpha}{2}}\right)^2 \tag{3.7}$$

for any $a,\ b>0$, for any $p\in[0,1]$ and $\alpha\neq 0$.

For $\alpha=1$ we get from (3.7) that

$$\max\{p,1-p\}\left(\sqrt{b}-\sqrt{a}\right)^2 \geqslant A_p(a,b)-G_p(a,b) \geqslant \min\{p,1-p\}\left(\sqrt{b}-\sqrt{a}\right)^2 \tag{3.8}$$

for any $a,\ b>0$ and for any $p\in[0,1]$, which are the inequalities obtained by Kittaneh and Manasrah in [8] and [9].

For $\alpha=1$ in (3.6) we obtain

$$\begin{aligned}&\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}[A_q(a,b)-G_q(a,b)]\\ &\geqslant A_p(a,b)-G_p(a,b)\\ &\geqslant \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}[A_q(a,b)-G_q(a,b)],\end{aligned} \tag{3.9}$$

for any $a,\ b>0$, for any $p\in[0,1]$, which is the inequality (2.1) from [1] in the particular case $\lambda=1$ in a slightly more general form for the weights $p,\ q$.

We have the following refinement and reverse for the inequality (2.1):

THEOREM 3. *For any $x,\ y\in\operatorname{Inv}(A)$ we have for $p\in[0,1]$ and $q\in(0,1)$ that*

$$\begin{aligned}&\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left(|x|^2\nabla_q|y|^2-x\circledS_q y\right)\\ &\geqslant |x|^2\nabla_p|y|^2-x\circledS_p y\\ &\geqslant \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left(|x|^2\nabla_q|y|^2-x\circledS_q y\right).\end{aligned} \tag{3.10}$$

In particular, we have

$$\begin{aligned}&2\max\{p,1-p\}\left(|x|^2\nabla|y|^2-x\circledS y\right)\\ &\geqslant |x|^2\nabla_p|y|^2-x\circledS_p y\\ &\geqslant 2\min\{p,1-p\}\left(|x|^2\nabla|y|^2-x\circledS y\right),\end{aligned} \tag{3.11}$$

for any $p\in[0,1]$.

Proof. From the inequality (3.9) for $a=1$ and $b=t>0$ we have

$$\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}(1-q+qt-t^q)\geqslant 1-p+pt-t^p \geqslant \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}(1-q+qt-t^q), \tag{3.12}$$

where $p\in[0,1]$ and $q\in(0,1)$.

Consider the functions $f(z):=\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}(1-q+qz-z^q)$, $g(z):=1-p+pz-z^p$ and $h(z)=\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}(1-q+qt-t^q)$ where z^ν, $\nu\in\{p,q\}$, is the principal of the power function. Then $f(z)$, $g(z)$ and $h(z)$ are analytic in the right half open plane $\{\mathrm{Re}\, z>0\}$ of the complex plane and and by (3.12) we have $f(z)\geqslant g(z)\geqslant h(z)$ for any $z>0$.

If $0<u\in \mathrm{Inv}(A)$ and $\nu\in[0,1]$, then by Lemma 3 we get

$$\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}(1-q+qu-u^q)\geqslant 1-p+pu-u^p \geqslant \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}(1-q+qu-u^q), \tag{3.13}$$

where $p\in[0,1]$ and $q\in(0,1)$.

If $x,\ y\in \mathrm{Inv}(A)$, then by taking $u=\left|yx^{-1}\right|^2\in \mathrm{Inv}(A)$ in (3.13) we have

$$\begin{aligned}&\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left(1-q+q\left|yx^{-1}\right|^2-\left(\left|yx^{-1}\right|^2\right)^q\right)\\&\geqslant 1-p+p\left|yx^{-1}\right|^2-\left(\left|yx^{-1}\right|^2\right)^p\\&\geqslant \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left(1-q+q\left|yx^{-1}\right|^2-\left(\left|yx^{-1}\right|^2\right)^q\right),\end{aligned} \tag{3.14}$$

where $p\in[0,1]$ and $q\in(0,1)$.

By multiplying the inequality (3.14) at left with x^* and at right with x we get the desired result (3.10). □

REMARK 2. If $0<a,\ b\in A$, then by taking $x=a^{1/2}$ and $y=b^{1/2}$ in (3.10) and (3.11) we get

$$\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}(a\nabla_q b-a\sharp_q b)\geqslant a\nabla_p b-a\sharp_p b \geqslant \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}(a\nabla_q b-a\sharp_q b), \tag{3.15}$$

for any $p\in[0,1]$ and $q\in(0,1)$.

In particular, for $q=1/2$ we have

$$2\max\{p,1-p\}(a\nabla b-a\sharp b)\geqslant a\nabla_p b-a\sharp_p b \geqslant 2\min\{p,1-p\}(a\nabla b-a\sharp b), \tag{3.16}$$

for any $p\in[0,1]$.

4. Inequalities under boundedness conditions

We consider the function $f_\nu:[0,\infty)\to[0,\infty)$ defined for $\nu\in(0,1)$ by

$$f_\nu(t)=1-\nu+\nu t-t^\nu=A_\nu(1,t)-G_\nu(1,t),$$

where $A_\nu(\cdot,\cdot)$ and $G_\nu(\cdot,\cdot)$ are the scalar arithmetic and geometric means.

The following lemma holds.

LEMMA 5. *For any $t\in[k,K]\subset[0,\infty)$ we have*

$$\max_{t\in[k,K]} f_\nu(x)=\Delta_\nu(k,K):=\begin{cases} A_\nu(1,k)-G_\nu(1,k) \text{ if } K<1,\\[2ex] \max\{A_\nu(1,k)-G_\nu(1,k),A_\nu(1,K)-G_\nu(1,K)\}\\ \text{if } k\leqslant 1\leqslant K,\\[2ex] A_\nu(1,K)-G_\nu(1,K) \text{ if } 1<k\end{cases} \tag{4.1}$$

and

$$\min_{t\in[k,K]} f_\nu(x)=\delta_\nu(k,K):=\begin{cases} A_\nu(1,K)-G_\nu(1,K) \text{ if } K<1,\\[2ex] 0 \text{ if } k\leqslant 1\leqslant K,\\[2ex] A_\nu(1,k)-G_\nu(1,k) \text{ if } 1<K.\end{cases} \tag{4.2}$$

Proof. The function f_ν is differentiable and

$$f_\nu'(t)=\nu\left(1-t^{\nu-1}\right)=\nu\frac{t^{1-\nu}-1}{t^{1-\nu}},\ t>0,$$

which shows that the function f_ν is decreasing on $[0,1]$ and increasing on $[1,\infty)$, $f_\nu(0)=1-\nu$, $f_\nu(1)=0$, $\lim_{t\to\infty} f_\nu(t)=\infty$ and the equation $f_\nu(t)=1-\nu$ for $t>0$ has the unique solution $t_\nu=\nu^{\frac{1}{\nu-1}}>1$.

Therefore, by considering the 3 possible situations for the location of the interval $[k,K]$ and the number 1 we get the desired bounds (4.1) and (4.2). □

REMARK 3. We have the inequalities

$$0\leqslant f_\nu(t)\leqslant 1-\nu \text{ for any } t\in\left[0,\nu^{\frac{1}{\nu-1}}\right]$$

and

$$1-\nu\leqslant f_\nu(t) \text{ for any } t\in\left[\nu^{\frac{1}{\nu-1}},\infty\right).$$

Assume that $x,\ y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that

$$M \geqslant \left|yx^{-1}\right| \geqslant m. \tag{4.3}$$

The inequality (4.3) is equivalent to

$$M^2 \geqslant \left|yx^{-1}\right|^2 = (x^*)^{-1}|y|^2 x^{-1} \geqslant m^2.$$

If we multiply at left with x^* and at right with x we get the equivalent relation

$$M^2|x|^2 \geqslant |y|^2 \geqslant m^2|x|^2. \tag{4.4}$$

We have:

THEOREM 4. *Assume that $x,\ y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that either (4.3), or, equivalently (4.4) is true. Then we have the inequalities*

$$\Delta_\nu\left(m^2, M^2\right)|x|^2 \geqslant |x|^2 \nabla_\nu |y|^2 - x \circledS_\nu y \geqslant \delta_\nu\left(m^2, M^2\right)|x|^2, \tag{4.5}$$

for any $\nu \in [0,1]$, where $\Delta_\nu(\cdot,\cdot)$ and $\delta_\nu(\cdot,\cdot)$ are defined by (4.1) and (4.2), respectively.

Proof. From Lemma 5 we have the double inequality

$$\Delta_\nu(k,K) \geqslant 1 - \nu + \nu t - t^\nu \geqslant \delta_\nu(k,K)$$

for any $x \in [k,K] \subset (0,\infty)$ and $\nu \in [0,1]$.

If $u \in A$ is an element such that $0 < k \leqslant u \leqslant K$, then $\sigma(u) \subset [k,K]$ and by Lemma 3 we have in the order of A that

$$\Delta_\nu(k,K) \geqslant 1 - \nu + \nu u - u^\nu \geqslant \delta_\nu(k,K) \tag{4.6}$$

for any $\nu \in [0,1]$.

If we take $u = \left|yx^{-1}\right|^2$, then by (4.3) we have $0 < m^2 \leqslant u \leqslant M^2$ and by (4.6) we get in the order of A that

$$\Delta_\nu\left(m^2, M^2\right) \geqslant 1 - \nu + \nu\left|yx^{-1}\right|^2 - \left|yx^{-1}\right|^{2\nu} \geqslant \delta_\nu\left(m^2, M^2\right) \tag{4.7}$$

for any $\nu \in [0,1]$.

If we multiply this inequality at left with x^* and at right with x we get

$$\begin{aligned}\Delta_\nu\left(m^2, M^2\right)|x|^2 &\geqslant (1-\nu)|x|^2 + \nu x^*\left|yx^{-1}\right|^2 x - x^*\left|yx^{-1}\right|^{2\nu} x \\ &\geqslant \delta_\nu\left(m^2, M^2\right)|x|^2\end{aligned} \tag{4.8}$$

and since $x^*\left|yx^{-1}\right|^2 x = x^*(x^*)^{-1}|y|^2 x^{-1} x = |y|^2$ and $x^*\left|yx^{-1}\right|^{2\nu} x = x \circledS_\nu y$ we get from (4.8) the desired result (4.5). $\square$

COROLLARY 3. *With the assumptions of Theorem 4 we have*

$$R\times\begin{cases}(1-m)^2|x|^2 \text{ if } M<1,\\ \max\left\{(1-m)^2,(M-1)^2\right\}|x|^2 \text{ if } m\leqslant 1\leqslant M,\\ (M-1)^2|x|^2 \text{ if } 1<m,\end{cases} \tag{4.9}$$

$$\geqslant |x|^2\nabla_\nu|y|^2-x\circledS_\nu y\geqslant r\times\begin{cases}(1-M)^2|x|^2 \text{ if } M<1,\\ 0 \text{ if } m\leqslant 1\leqslant M,\\ (m-1)^2|x|^2 \text{ if } 1<m,\end{cases}$$

where $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ *and* $R=\max\{1-\nu,\nu\}$.

Proof. From the inequality (3.8) we have for $b=t$ and $a=1$ that

$$R\left(\sqrt{t}-1\right)^2\geqslant f_\nu(t)=1-\nu+\nu t-t^\nu\geqslant r\left(\sqrt{t}-1\right)^2$$

for any $t\in[0,1]$.

Then we have

$$\Delta_\nu\left(m^2,M^2\right)\leqslant R\times\begin{cases}(1-m)^2 \text{ if } M<1,\\ \max\left\{(1-m)^2,(M-1)^2\right\} \text{ if } m\leqslant 1\leqslant M,\\ (M-1)^2 \text{ if } 1<m\end{cases}$$

and

$$\delta_\nu\left(m^2,M^2\right)\geqslant r\times\begin{cases}(1-M)^2 \text{ if } M<1,\\ 0 \text{ if } m\leqslant 1\leqslant M,\\ (m-1)^2 \text{ if } 1<m,\end{cases}$$

which by Theorem 4 proves the corollary. □

We observe that, with the assumptions of Theorem 4 and if A is a unital C^*-algebra, then by taking the norm in (4.5), we get

$$\Delta_\nu\left(m^2,M^2\right)\|x\|^2\geqslant\left\||x|^2\nabla_\nu|y|^2-x\circledS_\nu y\right\|\geqslant\delta_\nu\left(m^2,M^2\right)\|x\|^2, \tag{4.10}$$

for any $\nu\in[0,1]$, which, by triangle inequality also implies that

$$\Delta_\nu\left(m^2,M^2\right)\|x\|^2\geqslant\left\|(1-\nu)|x|^2+\nu|y|^2\right\|-\left\|\left|yx^{-1}\right|^\nu x\right\|^2\geqslant 0 \tag{4.11}$$

for any $\nu\in[0,1]$. This provides a reverse for the second inequality in (2.14).

REMARK 4. If $0 < a,\ b \in A$ and there exists the constants $0 < k < K$ such that

$$Ka \geqslant b \geqslant ka > 0, \tag{4.12}$$

then by (4.5) we get

$$\Delta_\nu(k,K)\,a \geqslant a\nabla_\nu b - a\sharp_\nu b \geqslant \delta_\nu(k,K)\,a, \tag{4.13}$$

while by (4.9) we get

$$R \times \begin{cases} \left(1-\sqrt{k}\right)^2 a \text{ if } K < 1, \\ \max\left\{\left(1-\sqrt{k}\right)^2, \left(\sqrt{K}-1\right)^2\right\} a \text{ if } m \leqslant 1 \leqslant M, \\ \left(\sqrt{K}-1\right)^2 a \text{ if } 1 < k, \end{cases} \tag{4.14}$$

$$\geqslant a\nabla_\nu b - a\sharp_\nu b \geqslant r \times \begin{cases} \left(1-\sqrt{K}\right)^2 a \text{ if } K < 1, \\ 0 \text{ if } k \leqslant 1 \leqslant K, \\ \left(\sqrt{k}-1\right)^2 a \text{ if } 1 < k \end{cases}$$

where $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

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