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THE NEW REVERSES OF YOUNG TYPE INEQUALITIES FOR NUMBERS, OPERATORS AND MATRICES

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Abstract. In this note, we give reverses of Young inequality for numbers. Then we establish operator and matrix inequalities corresponding the obtained numerical inequalities.

1. Introduction

The inequality

$$a^{\nu}b^{1-\nu} \leqslant \nu a+(1-\nu)b, \tag{1.1}$$

is the famous Young inequality for non-negative real numbers a,b and $0 \leqslant \nu \leqslant 1$. In (1.1), the equality holds if and only if $a=b$. The inequality (1.1) is also called ν-weighted arithmetic-geometric mean inequality. For the special case, when $\nu=\frac{1}{2}$, we obtain

$$\sqrt{ab} \leqslant \frac{a+b}{2},$$

which is called the fundamental arithmetic-geometric mean inequality.

The quantity $K(t,2)=\frac{(t+1)^2}{4t}$, for $t>0$ is the so called Kantorovich constant and satisfies the following conditions:

(i) $K(1,2)=1$ and $K(t,2)=K(\frac{1}{t},2) \geqslant 1$ for $t>0$.

(ii) $K(t,2)$ is monotone increasing on the interval $[1,\infty)$ and monotone decreasing on the interval $(0,1]$.

In 2011, Zou et. al. [15] proved the following result that is a refined version of (1.1) in terms of Kantorovich constant.

$$\nu a+(1-\nu)b \geqslant K(h,2)^{r}b^{1-\nu}a^{\nu} \tag{1.2}$$

where $r=\min\{\nu,1-\nu\}$ and $h=\frac{b}{a}$.

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X. Hu [7] presented the following inequalities which can be viewed as refinement of the inequality (1.1).

$$\nu^2a^2+(1-\nu)^2b^2 \geqslant \nu^2(a-b)^2+[(\nu a)^\nu b^{1-\nu}]^2, \quad 0\leqslant \nu \leqslant \frac{1}{2}, \tag{1.3}$$

$$\nu^2a^2+(1-\nu)^2b^2 \geqslant (1-\nu)^2(a-b)^2+[a^\nu((1-\nu)b)^{1-\nu}]^2, \quad \frac{1}{2}\leqslant \nu \leqslant 1. \tag{1.4}$$

A. Burqan and M. Khandaqji [4] gave reverses of Young type inequalities as follows:

$$\nu^2a+(1-\nu)^2b \leqslant (1-\nu)^2(\sqrt{a}-\sqrt{b})^2+a^\nu[(1-\nu)^2b]^{1-\nu}, \quad 0\leqslant \nu \leqslant \frac{1}{2}, \tag{1.5}$$

$$\nu^2a+(1-\nu)^2b \leqslant \nu^2(\sqrt{a}-\sqrt{b})^2+\nu^{2\nu}a^\nu b^{1-\nu}, \quad \frac{1}{2}\leqslant \nu \leqslant 1. \tag{1.6}$$

Throughout this paper, let $B(H)$ denote the C^*-algebra of all bounded linear operators on the Hilbert space H, while $B(H)^+$ and $B(H)^{++}$, respectively, denote the class of all positive bounded linear operators and the class of all invertible positive bounded linear operators in $B(H)$. Also, I stands for the identity operator. Let $A,B\in B(H)$. We say A is positive and we write $A\geqslant 0$, if $\langle Ax,x\rangle \geqslant 0$ for every $x\in H$. Moreover, we say $A\geqslant B(B\geqslant A)$ if $A-B\geqslant 0(B-A\geqslant 0)$ respectively. Also, the adjoint of A and the absolute value of A, respectively, are defined by A^* and $|A|=(AA^*)^{\frac{1}{2}}$.

Let $M_n(\mathbb{C})$ be the space of all $n\times n$ matrices with entries in complex field $\mathbb{C}$. The unitarily invariance of the norm $\|.\|$ means that $\|UAV\|=\|A\|$ for all $A\in M_n(\mathbb{C})$ and for all unitary matrices $U,V\in M_n(\mathbb{C})$. An example of a unitary invariant norm is the Hilbert-Schmidt norm defined by

$$\|A\|_2=\sqrt{\sum_{j=1}^{n} s_j^2(A)},$$

where $s_1(A)\geqslant s_2(A)\geqslant \ldots\ldots.. \geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A|=(A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity.

It should be mentioned here that the theory of operator means was started by Ando and was developed in paper [11] by Kubo and Ando.

For $A,B\in B(H)^{++}$ and $0\leqslant \nu \leqslant 1$, the ν-weighted geometric mean $A\sharp_\nu B$, the ν-weighted arithmetic mean $A\nabla_\nu B$ and the ν-weighted harmonic mean $A!_\nu B$, respectively, are defined by

$$A\sharp_\nu B=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu A^{\frac{1}{2}},$$

$$A\nabla_\nu B=(1-\nu)A+\nu B$$

and

$$A!_\nu B=\left((1-\nu)A^{-1}+\nu B^{-1}\right)^{-1}.$$

For the special case, when $\nu = \frac{1}{2}$, we define respectively, by $A\sharp B$, $A\nabla B$ and $A!B$ for brevity. A operator version of (1.1) proved in [5] as follows:

$$A\sharp_\nu B \leqslant A\nabla_\nu B,$$

where $A,B \in B(H)^{++}$ and $0 \leqslant \nu \leqslant 1$. It is well-known that

$$A\sharp B \leqslant H_\nu(A,B) \leqslant A\nabla B, \tag{1.7}$$

for $A,B \in B(H)^{++}$ and $0 \leqslant \nu \leqslant 1$. The inequality (1.7) is called Heinz double operator inequality and $H_\nu(A,B)$ is called Heinz mean and is defined as

$$H_\nu(A,B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu} B}{2}.$$

For more information about Young and Heinz inequalities (see [1, 2, 3, 6, 8, 9, 10, 12, 13, 14]).

2. Main results

2.1. Scalar versions for reverse Young type inequalities

First, we give reverses of the inequalities (1.3), (1.4), (1.5) and (1.6) using Kantorovich constant.

THEOREM 2.1. *Let a,b be two non-negative real numbers and $0 \leqslant \nu \leqslant \frac{1}{2}$, then*

$$(1-\nu)^{2\nu}[(1-2\nu)a+2\nu b]+(1-\nu)^{2-2\nu}a^{2\nu}b^{1-2\nu}K(h,2)^{-r} \geqslant 2(1-\nu)\sqrt{ab}, \tag{2.1}$$

and

$$(1-\nu)^{2-2\nu}[2\nu a+(1-2\nu)b]+(1-\nu)^{2\nu}a^{1-2\nu}b^{2\nu}K(h,2)^{-r} \geqslant 2(1-\nu)\sqrt{ab}, \tag{2.2}$$

where $h = \frac{a}{b}$ and $r = \min\{2\nu, 1-2\nu\}$.

Proof. Letting $0 \leqslant \nu \leqslant \frac{1}{2}$, by inequality (1.2), we get

$$\begin{aligned}
&(1-\nu)^{2\nu}[(1-2\nu)a+2\nu b]+(1-\nu)^{2-2\nu}a^{2\nu}b^{1-2\nu}K(h,2)^{-r}-2(1-\nu)\sqrt{ab}\\
&\geqslant (1-\nu)^{2\nu}[a^{1-2\nu}b^{2\nu}K(h,2)^{r}]+(1-\nu)^{2-2\nu}a^{2\nu}b^{1-2\nu}K(h,2)^{-r}-2(1-\nu)\sqrt{ab}\\
&= \left[(1-\nu)^{\nu}a^{\frac{1-2\nu}{2}}b^{\nu}K(h,2)^{\frac{r}{2}}-(1-\nu)^{1-\nu}a^{\nu}b^{\frac{1-2\nu}{2}}K(h,2)^{-\frac{r}{2}}\right]^2\\
&\geqslant 0,
\end{aligned}$$

which implies that

$$(1-\nu)^{2\nu}[(1-2\nu)a+2\nu b]+(1-\nu)^{2-2\nu}a^{2\nu}b^{1-2\nu}K(h,2)^{-r} \geqslant 2(1-\nu)\sqrt{ab}.$$

Analogously, by employing (1.2), we have

$$\begin{aligned}
&(1-\nu)^{2-2\nu}[2\nu a+(1-2\nu)b]+(1-\nu)^{2\nu}a^{1-2\nu}b^{2\nu}K(h,2)^{-r}-2(1-\nu)\sqrt{ab}\\
&\geqslant (1-\nu)^{2-2\nu}[a^{2\nu}b^{1-2\nu}K(h,2)^{r}]+(1-\nu)^{2\nu}a^{1-2\nu}b^{2\nu}K(h,2)^{-r}-2(1-\nu)\sqrt{ab}\\
&=\left[(1-\nu)^{1-\nu}a^{\nu}b^{\frac{1-2\nu}{2}}K(h,2)^{\frac{r}{2}}-(1-\nu)^{\nu}a^{\frac{1-2\nu}{2}}b^{\nu}K(h,2)^{-\frac{r}{2}}\right]^2\\
&\geqslant 0,
\end{aligned}$$

and so,

$$(1-\nu)^{2-2\nu}[2\nu a+(1-2\nu)b]+(1-\nu)^{2\nu}a^{1-2\nu}b^{2\nu}K(h,2)^{-r}\geqslant 2(1-\nu)\sqrt{ab}.$$

This completes the proof. □

THEOREM 2.2. *Let a,b be two non-negative real numbers and $0\leqslant\nu\leqslant\frac{1}{2}$. Then*

$$(1-\nu)^{2-2\nu}[\nu a+(1-\nu)b]+(1-\nu)^{2\nu}a^{1-\nu}b^{\nu}K(h,2)^{-r}\geqslant 2(1-\nu)\sqrt{ab}, \tag{2.3}$$

and

$$(1-\nu)^{2\nu}[(1-\nu)a+\nu b]+(1-\nu)^{2-2\nu}a^{\nu}b^{1-\nu}K(h,2)^{-r}\geqslant 2(1-\nu)\sqrt{ab}, \tag{2.4}$$

where $h=\frac{a}{b}$ and $r=\min\{\nu,1-\nu\}$.

Proof. For $0\leqslant\nu\leqslant\frac{1}{2}$, by inequality (1.2), we find that

$$\begin{aligned}
&(1-\nu)^{2-2\nu}[\nu a+(1-\nu)b]+(1-\nu)^{2\nu}a^{1-\nu}b^{\nu}K(h,2)^{-r}-2(1-\nu)\sqrt{ab}\\
&\geqslant (1-\nu)^{2-2\nu}a^{\nu}b^{1-\nu}K(h,2)^{r}+(1-\nu)^{2\nu}a^{1-\nu}b^{\nu}K(h,2)^{-r}-2(1-\nu)\sqrt{ab}\\
&=\left[(1-\nu)^{\nu}a^{\frac{1-\nu}{2}}b^{\frac{\nu}{2}}K(h,2)^{-\frac{r}{2}}-(1-\nu)^{1-\nu}a^{\frac{\nu}{2}}b^{\frac{1-2\nu}{2}}K(h,2)^{\frac{r}{2}}\right]^2\\
&\geqslant 0,
\end{aligned}$$

i.e.,

$$(1-\nu)^{2-2\nu}[\nu a+(1-\nu)b]+(1-\nu)^{2\nu}a^{1-\nu}b^{\nu}K(h,2)^{-r}\geqslant 2(1-\nu)\sqrt{ab}.$$

Again by inequality (1.2), it follows that

$$\begin{aligned}
&(1-\nu)^{2\nu}[(1-\nu)a+\nu b]+(1-\nu)^{2-2\nu}a^{\nu}b^{1-\nu}K(h,2)^{-r}-2(1-\nu)\sqrt{ab}\\
&\geqslant (1-\nu)^{2\nu}a^{1-\nu}b^{\nu}K(h,2)^{r}+(1-\nu)^{2-2\nu}a^{\nu}b^{1-\nu}K(h,2)^{-r}-2(1-\nu)\sqrt{ab}\\
&=\left[(1-\nu)^{\nu}a^{\frac{1-\nu}{2}}b^{\frac{\nu}{2}}K(h,2)^{\frac{r}{2}}-(1-\nu)^{1-\nu}a^{\frac{\nu}{2}}b^{\frac{1-\nu}{2}}K^{-\frac{r}{2}}\right]^2\\
&\geqslant 0,
\end{aligned}$$

and so (2.4) is proved. This completes the proof. □

The following results are immediate consequences of Theorem 2.1 and Theorem 2.2.

COROLLARY 2.3. *Let a,b be two non-negative real numbers and $0 \leqslant \nu \leqslant \frac{1}{2}$, then*

$$(1-\nu)^{2\nu}\left(\frac{a+b}{2}\right)+(1-\nu)^{2-2\nu}\left(\frac{a^{2\nu}b^{1-2\nu}+a^{1-2\nu}b^{2\nu}}{2}\right)K(h,2)^{-r} \geqslant 2(1-\nu)\sqrt{ab}, \tag{2.5}$$

and

$$(1-\nu)^{2-2\nu}\left(\frac{a+b}{2}\right)+(1-\nu)^{2\nu}\left(\frac{a^{1-2\nu}b^{2\nu}+a^{2\nu}b^{1-2\nu}}{2}\right)K(h,2)^{-r} \geqslant 2(1-\nu)\sqrt{ab}, \tag{2.6}$$

where $h=\frac{a}{b}$ and $r=\min\{2\nu,1-2\nu\}$.

COROLLARY 2.4. *Let a,b be two non-negative real numbers and $0 \leqslant \nu \leqslant \frac{1}{2}$, then*

$$(1-\nu)^{2-2\nu}\left(\frac{a+b}{2}\right)+(1-\nu)^{2\nu}\left(\frac{a^{\nu}b^{1-\nu}+a^{1-\nu}b^{\nu}}{2}\right)K(h,2)^{-r} \geqslant 2(1-\nu)\sqrt{ab}, \tag{2.7}$$

and

$$(1-\nu)^{2\nu}\left(\frac{a+b}{2}\right)+(1-\nu)^{2-2\nu}\left(\frac{a^{1-\nu}b^{\nu}+a^{\nu}b^{1-\nu}}{2}\right)K(h,2)^{-r} \geqslant 2(1-\nu)\sqrt{ab}, \tag{2.8}$$

where $h=\frac{a}{b}$ and $r=\min\{\nu,1-\nu\}$.

2.2. Operator versions for reverse Young type inequalities

In this section, we present operator versions based on Theorem 2.1 and Theorem 2.2.

Techniques are based on the monotonicity property of operator functions stated in following form:

Let $X \in B(H)^{++}$ with $Sp(X)$ and f and g be continuous real-valued functions so that $f(t) \geqslant g(t)$ on $Sp(X)$, then $f(X) \geqslant g(X)$.

THEOREM 2.5. *Suppose that $A,B \in B(H)^{++}$ satisfy $0 < mI \leqslant A,B \leqslant MI$ where M,m are positive real numbers such that $m<M$. Then for $0 \leqslant \nu \leqslant \frac{1}{2}$*

$$(1-\nu)^{2\nu}(A\nabla B)+(1-\nu)^{2-2\nu}H_{2\nu}(A,B)K(h,2)^{-r} \geqslant 2(1-\nu)(A\sharp B) \tag{2.9}$$

and

$$(1-\nu)^{2-2\nu}(A\nabla B)+(1-\nu)^{2\nu}H_{2\nu}(A,B)K(h,2)^{-r} \geqslant 2(1-\nu)(A\sharp B), \tag{2.10}$$

where $h=\frac{M}{m}$ and $r=\min\{2\nu,1-2\nu\}$.

Proof. For $0 \leqslant \nu \leqslant \frac{1}{2}$, by inequality (2.1), it follows that

$$(1-\nu)^{2\nu}[(1-2\nu)+2\nu x]+(1-\nu)^{2-2\nu}x^{1-2\nu}K(x,2)^{-r} \geqslant 2(1-\nu)\sqrt{x}, \tag{2.11}$$

for $x>0$. Considering $X=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, the property $0<mI \leqslant A,B \leqslant MI$ ensure us that

$$0<h^{'}I \leqslant X \leqslant hI$$

and therefore $Sp(X) \subset [h^{'},h] \subset (1,+\infty)$. Setting $X=A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \geqslant 0$ in (2.11) and using above monotonicity principle for operator functions yield the following inequality:

$$\begin{aligned}&(1-\nu)^{2\nu}\left[(1-2\nu)I+2\nu(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right]\\ &\quad+(1-\nu)^{2-2\nu}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{1-2\nu}K(h,2)^{-r}\\ &\geqslant 2(1-\nu)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}.\end{aligned} \tag{2.12}$$

Finally, multiplying both sides (2.12) by $A^{\frac{1}{2}}$, we obtain

$$(1-\nu)^{2\nu}[(1-2\nu)A+2\nu B]+(1-\nu)^{2-2\nu}(A\sharp_{1-2\nu}B)K(h,2)^{-r} \geqslant 2(1-\nu)(A\sharp B).$$

Replacing A and B by B and A (respectively) in the inequality above, we get

$$(1-\nu)^{2\nu}[(1-2\nu)B+2\nu A]+(1-\nu)^{2-2\nu}(A\sharp_{2\nu}B)K(h,2)^{-r} \geqslant 2(1-\nu)(A\sharp B).$$

Summing two latter inequalities, we have

$$(1-\nu)^{2\nu}(A\nabla B)+(1-\nu)^{2-2\nu}H_{2\nu}(A,B)K(h,2)^{-r} \geqslant 2(1-\nu)(A\sharp B).$$

The inequality (2.10) prove in similar way. This completes the proof. □

THEOREM 2.6. *With the same assumptions as in Theorem 2.5. Then the following inequalities hold:*

$$(1-\nu)^{2-2\nu}(A\nabla B)+(1-\nu)^{2\nu}H_\nu(A,B)K(h,2)^{-r} \geqslant 2(1-\nu)(A\sharp B) \tag{2.13}$$

and

$$(1-\nu)^{2\nu}(A\nabla B)+(1-\nu)^{2-2\nu}H_\nu(A,B)K(h,2)^{-r} \geqslant 2(1-\nu)(A\sharp B), \tag{2.14}$$

where, $h=\frac{M}{m}$ *and* $r=\min\{\nu,1-\nu\}$.

Proof. The proof follows by utilizing of (2.3) and (2.4) and by applying a similar manner as in Theorem 2.5. □

2.3. Matrix versions for reverse Young type inequalities

In this section, we give some matrix versions based on Corollary 2.3 and Corollary 2.4.

The famous spectral theorem says that every positive semidefinite matrix is unitarily diagonalizable, so for positive semidefinite matrices A and B there are unitary matrices U_1 and U_2 such that $A = U_1 D U_1^*$ and $B = U_2 E U_2^*$ where

$$D = diag(\lambda_1, \dots, \lambda_n)$$

and

$$E = diag(\mu_1, \dots, \mu_n)$$

$(\lambda_i, \mu_i \geqslant 0)$ for $0 \leqslant i \leqslant n$.

THEOREM 2.7. *Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive definite and $0 \leqslant \nu \leqslant \frac{1}{2}$, then*

$$\left\| (1-\nu)^{2\nu} \frac{AX + XB}{2} + (1-\nu)^{2-2\nu} \frac{A^{2\nu} X B^{1-2\nu} + A^{1-2\nu} X B^{2\nu}}{2} K(h,2)^{-r} \right\|_2^2$$
$$\geqslant (2(1-\nu))^2 \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2, \tag{2.15}$$

and

$$\left\| (1-\nu)^{2-2\nu} \frac{AX + XB}{2} + (1-\nu)^{2\nu} \frac{A^{2\nu} X B^{1-2\nu} + A^{1-2\nu} X B^{2\nu}}{2} K(h,2)^{-r} \right\|_2^2$$
$$\geqslant (2(1-\nu))^2 \left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2, \tag{2.16}$$

where $h = \frac{M}{m}$ and $r = \min\{2\nu, 1-2\nu\}$.

Proof. For our computations, let $Y = U_1^* X U_2 = [y_{ij}]$ $(1 \leqslant i, j \leqslant n)$. Then, we have

$$A^{\frac{1}{2}} X B^{\frac{1}{2}} = U_1 \left(\lambda_i^{\frac{1}{2}} \mu_j^{\frac{1}{2}} y_{ij} \right) U_2^*,$$

and so

$$\left\| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right\|_2^2 = \left\| U_1 (\lambda_i^{\frac{1}{2}} \mu_j^{\frac{1}{2}} y_{ij}) U_2^* \right\|_2^2 = \sum_{i,j=1}^{n} \left(\lambda_i^{\frac{1}{2}} \mu_j^{\frac{1}{2}} \right)^2 |y_{ij}|^2.$$

Similarly, one can show

$$
\begin{aligned}
&(1-\nu)^{2\nu}\frac{AX+XB}{2}+(1-\nu)^{2-2\nu}\frac{A^{2\nu}XB^{1-2\nu}+A^{1-2\nu}XB^{2\nu}}{2}K^{-r}(h,2)\\
&=U_1\Bigg[(1-\nu)^{2\nu}\frac{D(U_1^*XU_2)+(U_1^*XU_2)E}{2}\\
&\quad+(1-\nu)^{2-2\nu}\frac{D^{2\nu}(U_1^*XU_2)E^{1-2\nu}+D^{1-2\nu}(U_1^*XU_2)E^{2\nu}}{2}K^{-r}(h,2)\Bigg]U_2^*\\
&=U_1\left[(1-\nu)^{2\nu}\frac{DY+YE}{2}+(1-\nu)^{2-2\nu}\frac{D^{2\nu}YE^{1-2\nu}+D^{1-2\nu}YE^{2\nu}}{2}K^{-r}(h,2)\right]U_2^*.
\end{aligned}
$$

From the inequality (2.5) and the unitarily invariant property of $\|.\|_2^2$, we have

$$
\begin{aligned}
&(2(1-\nu))^2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2\\
&=\sum_{i,j}^{n}\left(2(1-\nu)\sqrt{\lambda_i\mu_j}\right)^2|y_{ij}|^2\\
&\leqslant\sum_{i,j}^{n}\left((1-\nu)^{2\nu}\frac{\lambda_i+\mu_j}{2}+K^{-r}(1-\nu)^{2-2\nu}\left(\frac{\lambda_i^{2\nu}\mu_j^{1-2\nu}+\lambda_i^{1-2\nu}\mu_j^{2\nu}}{2}\right)\right)^2|y_{ij}|^2\\
&=\left\|(1-\nu)^{2\nu}\frac{AX+XB}{2}+(1-\nu)^{2-2\nu}\frac{A^{2\nu}XB^{1-2\nu}+A^{1-2\nu}XB^{2\nu}}{2}K(h,2)^{-r}\right\|_2^2.
\end{aligned}
$$

The inequality (2.16) can be proven in a similar method, we omit its details. □

THEOREM 2.8. *With the assumptions of Theorem 2.7, we have the following estimates:*

$$
\begin{aligned}
&\left\|(1-\nu)^{2-2\nu}\frac{AX+XB}{2}+(1-\nu)^{2\nu}\frac{A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}}{2}K(h,2)^{-r}\right\|_2^2\\
&\geqslant(2(1-\nu))^2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2, \qquad (2.17)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|(1-\nu)^{2\nu}\frac{AX+XB}{2}+(1-\nu)^{2-2\nu}\frac{A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}}{2}K(h,2)^{-r}\right\|_2^2\\
&\geqslant(2(1-\nu))^2\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2, \qquad (2.18)
\end{aligned}
$$

where $h=\frac{M}{m}$ *and* $r=\min\{\nu,1-\nu\}$.

Proof. According to the inequalities (2.7) and (2.8) and by a process similar to the proof of Theorem 2.7, we can deduce the desired inequalities. □

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